Applied Mathematical Sciences, Vol. 1, 2007, no. 33, 1631 - 1650

Observer Design and Admissible Disturbances: A Discrete Disturbed System

M. Rachik [1], S. Saadi [2], Y. Rahhou, O. El Kahlaoui

Université Hassan II-Mohammédia
Faculté des Sciences Ben M'sik
Département de Mathématiques, BP. 7955
Sidi Othmane, Casablanca, Maroc

Abstract

In this paper, we are interested in discrete linear systems subject to disturbances. The initial state being supposed unknown, the observer constitutes a traditional estimator. In this work, and for an improvement of the Luenberger observer, we establish that by an adequate choice of the initial state observer one can act on the error which had with the estimate, in the presence of disturbances. The results obtained are illustrated through a numerical simulation.

Keywords: Observers, discrete linear systems, admissible disturbances, mathematical programming

1 Introduction

Every time, that one of the parameters of a system is partially or completely unknown, it becomes necessary to carry out the estimate of the state of this last. It is in this context that D. Luenberger introduced and developed, for the first time the observers theory [13], [14]. Fascinated by this new approach, the scientists adapted it to different types of systems: discrete systems [5], [15], [16]; stochastic systems [3]; time varying systems [1], [8] and hereditary systems [7]; and disturbed systems [11], [12], [17], [20].

[1] rachik@math.net
[2] smahansaadi@hotmail.com

in spite of the important role played by the observers in the systems theory, they remain of limited interest for a great class of systems. Indeed, since the observer is by definition an asymptotic estimator of the state of origin, means that information which it provides are only reliable after a long length of time . Thus, the observers are not very interesting for the systems at short duration of evolution, or for systems concerned with epidemiology, environment or space navigation, where a delayed estimate of the state of the system can have disastrous consequences.

To contribute to the resolution of this problem, we take interest in this paper, in improving the performances of a discrete linear disturbed system, described by the difference equation

$$\begin{cases} x_{i+1} = Ax_i + Bu_i + Dd_i, \quad \forall i \geq 0 \\ x_0 \end{cases} \tag{1}$$

with the corresponding disturbed output signal

$$y_i = Cx_i + E\overline{d}_i$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^q$, $d_i \in \mathbb{R}^r$ and $\overline{d}_i \in \mathbb{R}^{r'}$ are respectively, the state vector, the control vector, the output vector and the disturbance vectors. Without loss of generality, we can assume that $\overline{d}_i = d_i$ (r'=r). If not, we replace d_i and $\overline{d}_i$ by the new disturbance vector $\widetilde{d}_i = \begin{pmatrix} d_i \\ \overline{d}_i \end{pmatrix}$ and the matrices D and E, respectively by the matrices $\widetilde{D} = \begin{pmatrix} D & 0 \end{pmatrix}$ and $\widetilde{E} = \begin{pmatrix} 0 & E \end{pmatrix}$.
The disturbances $(d_i)_{i\geq 0}$ that are liable to affect the system are supposed finite age, that means $(d_i)_{i\geq 0} \in \mathcal{D}$ where $\mathcal{D} = \{(d_i)_{i\geq 0} : d_i \in \mathbb{R}^r, \ \ and \ \ d_i = 0, \ \ \forall i \geq I\}$ with I a positive integer that indicate the age of disturbances. While A, B, C, D and E are constant matrices of appropriate dimensions.
Our main objective, in this paper is to construct an observer of the system (1) described by the equation

$$\begin{cases} z_{i+1} = Fz_i + Pu_i + Hy_i, \quad \forall i \geq 0 \\ z_0 \end{cases} \tag{2}$$

where $z_i \in \mathbb{R}^p$ is the state observer, F, P and H are matrices of suitable dimensions, such that the estimate state converges towards Tx_i (where $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$) with an assigned rate of convergence. More precisely, for given a threshold of tolerance $\alpha = (\alpha)_{i\geq 0}$, and while supposing that the unknown state belongs to a convex polyhedron $\mathcal{P}$ we seek to verify the condition that we call $(\alpha, \mathcal{P})$-condition

$$||Tx_i - z_i|| \leq \alpha_i \ ; \quad \forall i \geq 0 \ \ and \ \ \forall x \in \mathcal{P} \tag{3}$$

For $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, where $(x_i)_{i\geq 0}$ is the solution of the equation (1) corresponding to the initialization $x_0 = x$.
In all the sequel of this paper, we will call an estimator (2) that verifies the $(\alpha, \mathcal{P})$-condition (3), an $(\alpha, \mathcal{P})$-observer.
In recent work [4], Rachik et al have presented a study of $(\alpha, \mathcal{P})$-observers for non disturbed linear systems ($d_i = 0, \ \ \forall i \geq 0$). they have characterized a class $\mathcal{M}$ of initial states observer such that the estimator (2) with $z_0 \in \mathcal{M}$ is an $(\alpha, \mathcal{P})$-observer for the system (1) with ($d_i = 0, \ \ \forall i \geq 0$).

As following this work, and taking into account the presence of disturbances that result from the natural interaction which exists between a system and its environment, we will be interested in this paper in the determination of the couples $(z_0, (d_i)_{i\geq 0})$ for which the estimator (2) is an $(\alpha, \mathcal{P})$-observer for the perturbed system (1). For that, we will fix z_0 in an appropriate class, and for this z_0 we will characterize the set of all disturbances $(d_i)_{i\geq 0} \in \mathcal{D}$ such that the $(\alpha, \mathcal{P})$-condition is checked. In other words we are interested in determining the set

$$\mathcal{D}_{z_0}(\alpha, \mathcal{P}) = \{\delta = (d_0, d_1, ..., d_{I-1}) \in \mathbb{R}^{rI} / ||Tx_i - z_i|| \leq \alpha_i, \quad \forall i \geq 1, \ \ \forall x_0 \in \mathcal{P}\},$$

with adequate choice of z_0 This paper is organized as follows. In a first step, using the hypothesis on the geometry of $\mathcal{P}$, we show the existence of all a class of $(\alpha, \mathcal{P})$-observers. In a second step, we give a theoretical as well as algorithmic characterization of the set $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$.
finally, to illustrate the obtained results, a numerical simulation is given.

2 preliminary results

The observer (2) only uses known variables u and y, d being non measured. the whole of all its matrices have to be properly defined, the objective of this section is to show that the observer of Luenberger constitutes a good asymptotic estimator of the system (1) and this for all disturbances $(d_i)_{i\geq 0} \in \mathcal{D}$.

Proposition 2.1 *Equation (2) specifies an observer of the system (1) if the following conditions hold ,*

1. $TA - FT = HC$

2. $P = TB$

3. F *is asymptotically stable*

Proof

Let $e_i = z_i - Tx_i$ be the observer error , then for all $i \geq 0$ we have,

$$\begin{aligned} e_{i+1} &= z_{i+1} - Tx_{i+1} \\ &= Fz_i + Pu_i + Hy_i - TAx_i - TBu_i - TDd_i \\ &= Fe_i + (FT - TA + HC)x_i + (P - TB)u_i + (HE - TD)d_i \end{aligned}$$

the conditions 1 and 2 yields

$$e_{i+1} = Fe_i + (HE - TD)d_i$$

Which implies

$$e_i = F^i e_0 + \sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1}, \quad \forall i \geq 1 \tag{4}$$

As the disturbances are of finite age I , (i.e)

$$d_j = 0 \quad \forall j \geq I$$

then for $i \geq I + 1$ we have

$$e_i = F^i e_0 + \sum_{j=1}^{I} F^{i-j}(HE - TD)d_{j-1} + \underbrace{\sum_{j=I+1}^{i} F^{i-j}(HE - TD)d_{j-1}}_{=0} \tag{5}$$

Which implies ,

$$e_i = F^{i-I-1}[F^{I+1}e_0 + \sum_{j=1}^{I} F^{I+1-j}(HE - TD)d_{j-1}] \tag{6}$$

We deduce from condition 3 that

$$\lim_{i \longrightarrow +\infty} F^i = 0 \quad so \quad \lim_{i \longrightarrow +\infty} e_i = 0.$$

■

The design of state observers for disturbed systems has been treated also in some previous papers, in particular in [6],[9],[10], [12], [18] and [19].

3 improvement of the observer error

3.1 Problem Formulation

For the system (1), we are interested in determining Tx_i based on the measured output y_i and control signal u_i. But, because of the presence of disturbances, and the fact of not knowing x_0, we can not determinate the state Tx_i exactly. therefore, we use the observer (2) to estimate it, by imposing on the error a tolerance threshold. The problem being addressed in this paper can be formulated as follows
Let $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ be a real and positive sequence, decreasing to 0 such that the sequence $(\frac{\alpha_i}{\alpha_{i+1}})_{i\geq 0}$ is decreasing. (For example $\alpha_i = \frac{1}{i+1}, \quad i \geq 0$; $\alpha_i = \frac{1}{(i+1)^s}, \quad s \in [1, +\infty[$; and $\alpha_i = \rho^i, \quad \rho < 1.$)

Given a convex and compact polyhedron $\mathcal{P}$ of $\mathbb{R}^n$ containing x_0 we aim to determine among the class of finite age disturbances that are liable to affect the system, those for which the observer error converges to 0 with assigned speed α. More precisely we are concerned with the characterization of the set

$$\mathcal{D}_{z_0}(\alpha, \mathcal{P}) = \{\delta = (d_0, d_1, ..., d_{I-1}) \in \mathbb{R}^{rI} / ||Tx_i - z_i|| \leq \alpha_i, \quad \forall i \geq 1 \ \ \forall x_0 \in \mathcal{P}\}.$$

for an adequate choice of the initial state observer z_0.
That means, if $\delta \in \mathcal{D}_{z_0}(\alpha, \mathcal{P})$ the observer (2) with initial state z_0 is an $(\alpha, \mathcal{P})$-observer of the system (1) affected by the vector disturbance δ.

3.2 Admissible set $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$

We start this section with some technical results which will be used in the sequel. Let us define, for $z_0 \in \mathbb{R}^p$ the functionals

$$\begin{aligned} \psi_{x_0} : \mathbb{R}^{rI} &\longrightarrow \mathbb{R}^p \\ \delta = \begin{pmatrix} d_0 \\ \vdots \\ d_{I-1} \end{pmatrix} &\longmapsto F^{I+1}(z_0 - Tx_0) + \sum_{j=1}^{I} F^{I+1-j}(HE - TD) d_{j-1} \end{aligned}$$

and

$$\begin{aligned} \psi_{i,x_0} : \mathbb{R}^{rI} &\longrightarrow \mathbb{R}^p \\ \delta = \begin{pmatrix} d_0 \\ \vdots \\ d_{I-1} \end{pmatrix} &\longmapsto F^{i}(z_0 - Tx_0) + \sum_{j=1}^{i} F^{i-j}(HE - TD) d_{j-1}. \end{aligned}$$

Then we deduce from (6)

$$\begin{aligned}\mathcal{D}_{z_0}(\alpha,\mathcal{P}) &= \{\delta=(d_0,d_1,...,d_{I-1})\in\mathbb{R}^{rI}/||e_i||\le\alpha_i,\quad \forall i\ge 1,\quad \forall x_0\in\mathcal{P}\}.\\ &= \{\delta=(d_0,d_1,...,d_{I-1})\in\mathbb{R}^{rI}/||\psi_{i,x_0}(\delta)||\le\alpha_i,\quad \forall i\ge 1,\ \ \forall x_0\in\mathcal{P}\}.\\ &= \mathcal{G}\bigcap\mathcal{H}.\end{aligned}$$

Where

$$\mathcal{G}=\{\delta=(d_0,d_1,...,d_{I-1})\in\mathbb{R}^{rI}/||\psi_{i,x_0}(\delta)||\le\alpha_i,\quad \forall i\in\{1,\dots,I\}\ \ and\ \forall x_0\in\mathcal{P}\}.$$

$$\begin{aligned}\mathcal{H} &= \{\delta=(d_0,d_1,...,d_{I-1})\in\mathbb{R}^{rI}/||F^{i-I-1}\psi_{x_0}(\delta)||\le\alpha_i,\quad \forall i\ge I+1\ \ and\ \forall x_0\in\mathcal{P}\},\\ &= \{\delta=(d_0,d_1,...,d_{I-1})\in\mathbb{R}^{rI}/||F^{k}\psi_{x_0}(\delta)||\le\alpha_{k+I+1},\quad \forall k\ge 0\ \ and\ \forall x_0\in\mathcal{P}\}.\end{aligned}$$

In the following proposition we will show that the knowledge of all the set $\mathcal{P}$ is not necessary to define $\mathcal{D}_{z_0}(\alpha,\mathcal{P})$ but only its vertices. So let us define the set
$vert(\mathcal{P})=\{v_1,v_2,...,v_s\}$ where $v_k,\quad k=1,2,...,s$ are the vertices of $\mathcal{P}$.

Proposition 3.1 *We have*

(i) $\mathcal{D}_{z_0}(\alpha,\mathcal{P})=\bigcap_{k=1}^{s}\mathcal{D}_{z_0,v_k}(\alpha,\mathcal{P})$
where $\mathcal{D}_{z_0,v_k}(\alpha,\mathcal{P})=\{\delta\in\mathbb{R}^{rI}/||\psi_{i,v_k}(\delta)||\le\alpha_i,\quad \forall i\ge 1\}$

(ii) $\mathcal{G}=\bigcap_{k=1}^{s}\mathcal{G}_{v_k}$ *and* $\mathcal{H}=\bigcap_{k=1}^{s}\mathcal{H}_{v_k}$
where $\mathcal{G}_{v_k}=\{\delta\in\mathbb{R}^{rI}/||\psi_{i,v_k}(\delta)||\le\alpha_i,\quad 1\le i\le I\}$

and $\mathcal{H}_{v_k}=\{\delta\in\mathbb{R}^{rI}/||F^{i}\psi_{v_k}(\delta)||\le\alpha_{i+I+1},\quad \forall i\ge 0\}$

with $v_k\in vert(\mathcal{P}),\quad 1\le k\le s$
We have also,

$$\mathcal{D}_{z_0,v_k}(\alpha,\mathcal{P})=\mathcal{G}_{v_k}\cap\mathcal{H}_{v_k},\quad for\ \ all\ \ k=1,2,...,s$$

Proof.
(i) It's clear that $\mathcal{D}_{z_0}(\alpha,\mathcal{P})\subset\bigcap_{k=1}^{s}\mathcal{D}_{z_0,v_k}(\alpha,\mathcal{P})$

reciprocally, let $\delta \in \bigcap_{k=1}^{s} \mathcal{D}_{z_0,v_k}(\alpha,\mathcal{P})$.

For all $x_0 \in \mathcal{P}$, $\exists \beta_k \geq 0 \;\; (1 \leq k \leq s)$ such that $\sum_{k=1}^{s} \beta_k = 1 \;\; and \;\; x_0 = \sum_{k=1}^{s} \beta_k v_k$.

Then

$$\begin{aligned} ||e_i|| &= ||F^i z_0 - F^i T \sum_{k=1}^{s} \beta_k v_k + \sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1}|| \\ &\leq \sum_{k=1}^{s} \beta_k ||F^i z_0 - F^i T v_k + \sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1}|| \\ &\leq \sum_{k=1}^{s} \beta_k \alpha_i = \alpha_i \end{aligned}$$

(ii) Same proof as (i). ■

Remark 3.1 *Before trying to characterize the set $\mathcal{D}_{z_0}(\alpha,\mathcal{P})$, it is natural to justify that this set is not reduced to zero ($\delta = 0$ corresponds to the case where the system is not disturbed). In the following theorem we show that under some conditions, there exists z_0 initials state observer such that $\mathcal{D}_{z_0}(\alpha,\mathcal{P})$ contains a ball centered on $\delta = 0$.*

Theorem 3.1 *We suppose that the following conditions hold*

(i) *there exists $\gamma > 0$ such that $\frac{||F^i||}{\alpha_{i+I+1}} \leq \gamma$, for every $i \geq 0$.*

(ii) *$diam\mathcal{P} < \frac{1}{2\gamma ||T||}$, where $diam\mathcal{P} = \max_{v_k \in vert(\mathcal{P})} ||v_i - v_j||$.*

Then

$$0 \in int\mathcal{D}_{z_0}(\alpha,\mathcal{P}), \quad for \;\; every \quad z_0 \in B(Tv_j, \frac{1}{2\gamma} - ||T|| diam\mathcal{P}).$$

where v_j is a vertex of $\mathcal{P}$, B(a,r) is the ball of radius r, centered on a, and $intE$ is the interior of the set E.

Proof.
The proof will be made in two steps.

Step 1: proof that $0 \in int\mathcal{G}$.

For that we will show that $0 \in int\mathcal{G}_{v_k}$ for all $k = 1, 2, ..., s$

Let us define the functionals

$$\begin{array}{rcl} \varphi_i : \mathbb{R}^{rI} & \longrightarrow & \mathbb{R}^p \\ \delta = \begin{pmatrix} d_0 \\ \vdots \\ d_{I-1} \end{pmatrix} & \longmapsto & \displaystyle\sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1} \end{array}$$

$(\varphi_i)_{0\leq i\leq I}$ are continuous functions, particulary at point 0, consequently, for every integer $i \in \{1, ..., I\}$ there exists $\rho_i > 0$ such that

$$\forall \delta \in B(0, \rho_i), \quad ||\varphi_i(\delta)|| \leq \frac{\alpha_I}{2}.$$

Let $\rho = \min\limits_{0\leq i\leq I} \rho_i$, then

$\forall \delta \in B(0, \rho), \quad ||\varphi_i(\delta)|| \leq \frac{\alpha_I}{2} \leq \frac{\alpha_i}{2}, \quad \forall i \in \{0, 1, ..., I\}.$

For $z_0 \in B(Tv_j, \frac{1}{2\gamma} - ||T||diam(\mathcal{P}))$ and $\delta \in B(0, \rho)$, we have for every $i \in \{0, ..., I\}$ and every $k \in \{1, ..., s\}$.

If $k \neq j$,

$$\begin{array}{rcl} ||\psi_{i,v_k}(\delta)|| & = & ||F^i(z_0 - Tv_k) - \varphi_i(\delta)|| \\ & \leq & ||F^i(z_0 - Tv_j)|| + ||F^i(Tv_k - Tv_j)|| + ||\varphi_i(\delta)|| \\ & \leq & ||F^i||(||z_0 - Tv_j|| + ||T||||v_k - v_j||) + \frac{\alpha_i}{2} \\ & \leq & \gamma\alpha_i(\frac{1}{2\gamma} - ||T||diam(\mathcal{P}) + ||T||diam(\mathcal{P})) + \frac{\alpha_i}{2} \leq \alpha_i. \end{array}$$

If $k = j$,

$$\begin{array}{rcl} ||\psi_{i,v_k}(\delta)|| & = & ||F^i(z_0 - Tv_k) - \varphi_i(\delta)|| \\ & \leq & ||F^i(z_0 - Tv_j)|| + ||\varphi_i(\delta)|| \\ & \leq & ||F^i||||z_0 - Tv_j|| + \frac{\alpha_i}{2} \\ & \leq & \gamma\alpha_i(\frac{1}{2\gamma} - ||T||diam(\mathcal{P})) + \frac{\alpha_i}{2} \leq \alpha_i \end{array}$$

Step 2: proof that $0 \in int\mathcal{H}$

For that we will show that $0 \in int\mathcal{H}_{v_k}$ for all $k = 1, 2, ..., s$
Let us define the functional

$$\begin{array}{rcl} \varphi : \mathbb{R}^{rI} & \longrightarrow & \mathbb{R}^p \\ \delta = \begin{pmatrix} d_0 \\ \vdots \\ d_{I-1} \end{pmatrix} & \longmapsto & \sum_{j=1}^{I} F^{I-j}(HE - TD)d_{j-1} \end{array}$$

φ is continuous function at point 0, consequently, there exists $\beta > 0$ such that

$$\forall \delta \in B(0, \beta), \quad ||\varphi(\delta)|| \leq \frac{1}{2\gamma}.$$

For $z_0 \in B(Tv_j, \frac{1}{2\gamma} - ||T||diam(\mathcal{P}))$ and $\delta \in B(0, \beta)$ we have for every $i \geq 0$ and every $k \in \{1, ..., s\}$.

If $k \neq j$

$$\begin{aligned} ||F^i\psi_{v_k}(\delta)|| &= ||F^{i+I}(z_0 - Tv_k) - F^i\varphi(\delta)|| \\ &\leq ||F^{i+I}(z_0 - Tv_j)|| + ||F^i(Tv_k - Tv_j)|| + \gamma\alpha_{i+I+1}||\varphi(\delta)|| \\ &\leq ||F^{i+I}||(||z_0 - Tv_j|| + ||T||||v_k - v_j||) + \tfrac{\alpha_{i+I+1}}{2} \\ &\leq \gamma\alpha_{i+I+1}(\tfrac{1}{2\gamma} - ||T||diam(\mathcal{P}) + ||T||diam(\mathcal{P})) + \tfrac{\alpha_{i+I+1}}{2} \leq \alpha_{i+I+1}. \end{aligned}$$

If $k = j$

$$\begin{aligned} ||F^i\psi_{v_k}(\delta)|| &= ||F^{i+I}(z_0 - Tv_k) - F^i\varphi(\delta)|| \\ &\leq ||F^{i+I}(z_0 - Tv_j)|| + \gamma\alpha_{i+I+1}||\varphi(\delta)|| \\ &\leq ||F^{i+I}||||z_0 - Tv_j|| + \tfrac{\alpha_{i+I+1}}{2} \\ &\leq \gamma\alpha_{i+I+1}(\tfrac{1}{2\gamma} - ||T||diam(\mathcal{P})) + \tfrac{\alpha_{i+I+1}}{2} \leq \alpha_{i+I+1}. \end{aligned}$$

We deduce from step 1 and step 2 that $0 \in int\mathcal{G} \bigcap int\mathcal{H}$ and consequently $0 \in int\mathcal{D}_{z_0}(\alpha, \mathcal{P})$. ∎

In the following proposition we will give more properties of the set $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$.

Proposition 3.2 **(i)** *$\mathcal{D}_{z_0}(\alpha, \mathcal{P})$ is closed and convex set.*

(ii) *If it exists $\gamma > 0$ such that $\frac{||F^i||}{\alpha_{i+I+1}} \leq \gamma$, for all $i \geq 0$ and if $diam\mathcal{P} < \frac{1}{2\gamma||T||}$. Then,*

$$0 \in int\mathcal{D}_{z_0}(\alpha, \mathcal{P}), \quad for \;\; every \quad z_0 \in conv(\bigcup_{k=1}^{s} B(Tv_k, \beta)),$$

where $convE$ is the convex hull of E, and $\beta = \frac{1}{2\gamma} - ||T||diam(\mathcal{P})$.

Proof.
(i) One can easily verifies that $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$ is convex set.
Since

$$\mathcal{G}_{v_k} = \bigcap_{i \in \{0,\ldots,I\}} \psi_{i,v_k}^{-1}(\{x \in \mathbb{R}^p : ||x|| \leq \alpha_i\})$$

and $(\psi_{i,v_k})_{i\in\{0,\ldots,I\}}$ are continuous, the set $\mathcal{G}_{v_k}$ is closed because the sets $\{x \in \mathbb{R}^p : ||x|| \leq \alpha_i\}$ are closed. We deduce then that $\mathcal{G} = \bigcap_{k=1}^{s} \mathcal{G}_{v_k}$ is also closed.
On the other hand, $\mathcal{H}_{v_k} = \psi_{v_k}^{-1}(\mathcal{S})$, where $\mathcal{S}$ is the closed set given by $\mathcal{S} = \{x \in \mathbb{R}^p : ||F^i x|| \leq \alpha_{i+I+1}, \quad \forall i \geq 0\}$. The continuity of ψ_{v_k} implies that $\mathcal{H}_{v_k}$ is closed. And consequently $\mathcal{H} = \bigcap_{k=1}^{s} \mathcal{H}_{v_k}$ is closed.
Finally we conclude that so is $\mathcal{D}_{z_0}(\alpha, \mathcal{P}) = \mathcal{G} \cap \mathcal{H}$.
(ii)Let $z_0 \in conv(\bigcup_{k=1}^{s} B(Tv_k, \beta))$, then there exists $\lambda_i \in [0, 1]$ *and* $w_i \in \bigcup_{k=1}^{s} B(Tv_k, \beta)$ such that $z_0 = \sum_{i=1}^{l} \lambda_i w_i$ *and* $\sum_{i=1}^{l} \lambda_i = 1$
From theorem 3.1 we deduce that $0 \in \mathcal{D}_{w_k}(\alpha, \mathcal{P})$, i.e, $\exists \rho_k > 0 : B(0, \rho_k) \subset \mathcal{D}_{w_k}(\alpha)$.
Let $\rho = \min_{1 \leq k \leq l} \rho_k$, then for every $\delta \in B(0, \rho)$, every $i \geq 1$ and every $x_0 \in \mathcal{P}$, we have

$$\begin{aligned}\| F^i(z_0 - x_0) + \sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1} \| &= \| F^i(\sum_{k=1}^{l} \lambda_k w_k - x_0) \\ &\quad + \sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1} \| \\ &= \| \sum_{k=1}^{l} \lambda_k (F^i(w_k - x_0) \\ &\quad + \sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1}) \| \\ &\leq \sum_{k=1}^{l} \lambda_k \| F^i(w_k - x_0) \\ &\quad + \sum_{j=1}^{i} F^{i-j}(HE - TD)d_{j-1} \| \\ &\leq \sum_{k=1}^{l} \lambda_k \alpha_i = \alpha_i.\end{aligned}$$

Which implies that $\delta \in \mathcal{D}_{z_0}(\alpha, \mathcal{P})$. ■

Remarks 3.1 **(i)** *The condition diam*$\mathcal{P} < \frac{1}{2\gamma||T||}$ *is not very restrictive, indeed one can consider instead of* T *the operator* ϵT*, where* ϵ *is positive real that can be chosen as little as need.*

(ii) *It is obvious that the set* $\mathcal{G}$ *can be completely obtained by solving a finite number of functional inequalities. However, the set* $\mathcal{H}$ *is defined by an infinite number of inequations, and so it can be hardly obtained. As in Rachik et al.[4] we will give a sufficient condition that able the characterization of the set* $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$ *by finite number of inequalities.*

4 Characterization of the sets $\mathcal{H}_v$

In order to improve the structure of the sets $\mathcal{H}_v$, $v \in vert(\mathcal{P})$ we introduce the following sets,

$$\mathcal{S} = \{\xi \in \mathbb{R}^p : \|F^i \xi\| \leq \alpha_{i+I+1}, \quad \forall i \geq 0\}$$

$$\mathcal{S}_k = \{\xi \in \mathbb{R}^p : \|F^i \xi\| \leq \alpha_{i+I+1}, \quad \forall i \in \{0, 1, ..., k\}\}, \;\; k \geq 0$$

and

$$\mathcal{H}_{k,v} = \{\delta \in \mathbb{R}^{rI} : \|F^i \psi_v(\delta)\| \leq \alpha_{i+I+1}, \quad \forall i \in \{0, 1, ..., k\}\}, \;\; k \geq 0$$

where v is a vertex of the polyhedron $\mathcal{P}$.

Definition 4.1 *The set E ($E = \mathcal{S}$ or $\mathcal{H}_v$) is said to be finitely accessible if there exists an integer k such that $E = E_k$ ($E_k = \mathcal{S}_k$ or $\mathcal{H}_{k,v}$). We denote by k^* the smallest such integer.*

Remarks 4.1 **(i)** *For integers i and j such that $i \geq j$, we have*

$$\mathcal{S} \subset \mathcal{S}_i \subset \mathcal{S}_j \quad \textit{and} \quad \mathcal{H}_v \subset \mathcal{H}_{i,v} \subset \mathcal{H}_{j,v}.$$

(ii) *As*

$$\mathcal{H}_{i,v} = \psi_v^{-1}(\mathcal{S}_i) \quad \textit{and} \quad \mathcal{H}_v = \psi_v^{-1}(\mathcal{S}).$$

Then,

$$\mathcal{S} \textit{ is finitely accessible} \Rightarrow \mathcal{H}_v \textit{ is finitely accessible.}$$

Proposition 4.1 *The set $\mathcal{S}$ is finitely accessible if and only if $\mathcal{S}_{k+1} = \mathcal{S}_k$ for some integer k.*

Proof.
($\Rightarrow$) If $\mathcal{S}$ is finitely accessible then $\mathcal{S}_{k+1} = \mathcal{S}_k$ for all $k \geq k^*$.
($\Leftarrow$) Conversely, suppose that it exists k such that $\mathcal{S}_{k+1} = \mathcal{S}_k$ which is equivalent to $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ (Remark 4.1 (i)).
Let $\xi \in \mathcal{S}_k$, then $\xi \in \mathcal{S}_{k+1}$ which implies for $0 \leq i \leq k$

$$\| F^i(\frac{\alpha_{k+I+1}}{\alpha_{k+I+2}}F\xi) \| = \frac{\alpha_{k+I+1}}{\alpha_{k+I+2}} \| F^{i+1}\xi \| \leq \frac{\alpha_{k+I+1}}{\alpha_{k+I+2}}\alpha_{i+I+2}$$

As the sequel $(\frac{\alpha_j}{\alpha_{j+1}})_{j\geq 0}$ is decreasing then $\frac{\alpha_{k+I+1}}{\alpha_{k+I+2}} \leq \frac{\alpha_{i+I+1}}{\alpha_{i+I+2}}$ which implies

$$\| F^i(\frac{\alpha_{k+I+1}}{\alpha_{k+I+2}}F\xi) \| \leq \alpha_{i+I+1}$$

and consequently

$$\frac{\alpha_{k+I+1}}{\alpha_{k+I+2}}F\xi \in \mathcal{S}_k$$

We deduce, then by iteration that

$$(\frac{\alpha_{k+I+1}}{\alpha_{k+I+2}})^j F^j\xi \in \mathcal{S}_k, \quad \forall j \geq 0$$

or equivalently

$$\| (\frac{\alpha_{k+I+1}}{\alpha_{k+I+2}})^j F^{i+j}\xi \| \leq \alpha_{i+I+1}, \quad \forall 0 \leq i \leq k \ \textit{ and } \ \forall j \geq 0.$$

Particulary for $i = k$, we have

$$\| F^{k+j}\xi \| \leq \frac{(\alpha_{k+I+2})^j}{(\alpha_{k+I+1})^{j-1}}, \quad \forall j \geq 1.$$

We show by iteration that

$$\frac{(\alpha_{k+I+2})^j}{(\alpha_{k+I+1})^{j-1}} \leq \alpha_{k+I+j+1}, \quad \forall j \geq 1$$

which implies

$$\| F^{k+j}\xi \| \leq \alpha_{k+j+I+1}, \quad \forall j \geq 1.$$

Or equivalently

$$\| F^{i}\xi \| \leq \alpha_{i+I+1}, \quad \forall i \geq k.$$

And then we deduce that $\xi \in \mathcal{S}$ so $\mathcal{S}_k \subset \mathcal{S}$ hence $\mathcal{S} = \mathcal{S}_k$. ■

Remark 4.1 *As a natural consequence of the previous proposition, we shall give in section 5 an algorithm which allows to determine the smallest integer* k^* *such that* $\mathcal{S} = \mathcal{S}_{k^*}$.

Before taking interest on the determination of k^*, it is desirable to have simple condition which assure the finite accessibility of $\mathcal{S}$. Our main result in this direction is the following theorem.

Theorem 4.1 *If the following condition hold*

$$\lim_{i \to +\infty} \frac{\| F^i \|}{\alpha_{i+I+1}} = 0.$$

Then $\mathcal{S}$ *is finitely accessible.*

Proof.
The hypothesis of the theorem implies that it exists k_0 such that

$$\frac{\| F^{k_0+1} \|}{\alpha_{k_0+I+2}} < \frac{1}{\alpha_{I+1}}.$$

Let $\xi \in \mathcal{S}_{k_0}$, then

$$\begin{array}{rcl} \| F^{k_0+1}\xi \| & \leq & \| F^{k_0+1} \| \| \xi \| \\ & \leq & \frac{\alpha_{k_0+I+2}}{\alpha_{I+1}} \| \xi \| \\ & \leq & \alpha_{k_0+I+2} \end{array}$$

which gives $\xi \in \mathcal{S}_{k_0+1}$ and that completes the demonstration. ■

Remark 4.2 *In the theorem 3.1 and proposition 3.2, we need the value of* $\gamma = \sup_{i\geq 0} \frac{\| F^i \|}{\alpha_{i+i+1}}$ *to check the sufficient condition. In practice, it is not always easy to calculate this supremum . So we establish in the following proposition, sufficient condition wich ensure the feasibility of this calculus.*

Proposition 4.2 *Suppose that* $\lim_{i\to+\infty} \frac{\| F^i \|}{\alpha_{i+I+1}} = 0$*, and let* $\mu = \max_{0\leq i\leq k^*} \frac{\| F^i \|}{\alpha_{i+I+1}}$*, where* k^* *is the smallest integer such that* $\mathcal{S} = \mathcal{S}_{k^*}$*. Then if* $diam\mathcal{P} < \frac{1}{2\mu\|T\|}$*, we have*

$$0 \in int\mathcal{D}_{z_0}(\alpha, \mathcal{P}), \quad for \;\; every \quad z_0 \in conv(\bigcup_{k=1}^{s} B(Tv_k, \beta))$$

where $\beta = \frac{1}{2\mu} - ||T||diam\mathcal{P}$*.*

Proof.
Let $\gamma = \sup_{i\geq 0} \frac{\| F^i \|}{\alpha_{i+I+1}}$. It's evident that $\gamma \geq \mu$.
Suppose that $\gamma > \mu$ which implies that there exists $i_0 > k^*$ such that $\| F^{i_0} \| > \mu\alpha_{i_0+I+1}$
what implies that there exists $\xi_0 \in B(0,1)$ such that

$$\| F^{i_0}(\frac{1}{\mu}\xi_0) \| > \alpha_{i_0+I+1}$$

what yields $\frac{1}{\mu}\xi_0 \notin \mathcal{S}$.

On the other hand, for $0 \leq i \leq k^*$ we have

$$\| F^i(\frac{1}{\mu}\xi_0) \| \leq \frac{1}{\mu}\alpha_{i+I+1}\mu = \alpha_{i+I+1},$$

consequently $\frac{1}{\mu}\xi_0 \in \mathcal{S}_{k^*}$, which is contradiction. So $\gamma = \mu$.
Then application of proposition 3.2 with $\gamma = \mu$ completes the proof. ■

5 Algorithmic approach

From the previous results we can deduce an algorithm for determination of k^*, the smallest integer such that $\mathcal{S} = \mathcal{S}_{k^*}$, and consequently the $(\alpha, \mathcal{P})$-admissible set $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$.

Let $\mathbb{R}^p$ be endowed with the infinite norm

$$\|\xi\| = \max |\xi_i|, \quad for\ all \quad \xi = (\xi_1, ..., \xi_p) \in \mathbb{R}^p.$$

The set $\mathcal{S}_k$ is then described as follows

$$\mathcal{S}_k = \{\xi \in \mathbb{R}^p : f_j(\frac{1}{\alpha_{i+I+1}} F^i \xi) \leq 0 \quad pour \quad j = 1, 2, ..., 2p \quad et \quad i = 0, 1, ..., k\}$$

where the functions $f_j : \mathbb{R}^p \to \mathbb{R}$, are defined for every $y = (y_1, ..., y_p) \in \mathbb{R}^p$ by

$$f_{2l-1}(y) = y_l - 1, \quad \forall l \in \{1, 2, ..., p\}$$
$$f_{2l}(y) = -y_l - 1, \quad \forall l \in \{1, 2, ..., p\}.$$

It follows from remark (4.1(i)),that :

$$\mathcal{S}_{k+1} = \mathcal{S}_k \Leftrightarrow \mathcal{S}_k \subset \mathcal{S}_{k+1}$$

or equivalently

$$\forall \xi \in \mathcal{S}_k, \quad \forall j \in \{1, 2, ..., 2p\} \quad , f_j(\frac{1}{\alpha_{k+I+2}} F^{k+1} \xi) \leq 0,$$

or yet

$$\sup_{\xi \in \mathcal{S}_k} f_j(\frac{1}{\alpha_{k+I+2}} F^{k+1} \xi) \leq 0 \quad for \quad j \in \{1, 2, ..., 2p\}.$$

what is equivalent to

$$\sup f_j(\frac{1}{\alpha_{k+I+2}} F^{k+1} \xi) \leq 0 \quad for \quad j \in \{1, 2, ..., 2p\}.$$

with the constraints

$$\begin{cases} f_j(\frac{1}{\alpha_{l+I+1}} F^l \xi) \leq 0, \\ j = 1, 2, \ldots, 2p, \\ l = 0, 1, \ldots, k. \end{cases}$$

Finally, we deduce an algorithm that, when it converges, calculate k^*.

Algorithm

step 1 : initialise $k := 0$;
step 2 : for $i = 1, \ldots, 2p$, do :

$$\text{Maximize } J_i(x) = f_i\left(\frac{1}{\alpha_{k+I+2}} F^{k+1} x\right)$$
$$\begin{cases} f_j\left(\frac{1}{\alpha_{l+I+1}} F^l x\right) \leq 0, \\ j = 1, \ldots, 2p, \\ l = 0, \ldots, k. \end{cases}$$

Let J_i^* be the maximum value of $J_i(x)$.
If $J_i^* \leq 0$, for $i = 1, \ldots, 2p$ then
set $k^* := k$ and stop.
Else continue.

step 3 : Replace k by $k + 1$ and return to step 2.

Remark 5.1 *The optimization problem cited in step* 2 *is a mathematical programming problem and can be solved by standard methods, in particular the method of simplex.*

To illustrate this work we give in the following section a numerical example.

6 Numerical Example

Consider the following perturbed system:

$$\begin{cases} x_{i+1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x_i + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_i + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} d_i, \quad \forall i \geq 0 \\ x_0 \end{cases} \tag{7}$$

With the corresponding perturbed output signal:

$$y_i = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} x_i + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} d_i$$

We suppose that the age of disturbances $(d_i)_{0 \leq i \leq I}$ is $I = 1$ We consider the identity observer:

$$z_{i+1} = \begin{pmatrix} 0.5 & 0 \\ 1 & 0.5 \end{pmatrix} z_i + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_i + \begin{pmatrix} 0.25 & -0.25 \\ 0 & 0.5 \end{pmatrix} y_i, \quad \forall i \geq 0 \tag{8}$$

It is obvious that

1. P = TB
2. TA - FT = DC
3. The eigenvalues of F are 0.5 and 0.5, then F is stable.

Let $\alpha_i = \frac{1}{i+1}$
The algorithm established in section 5 gives $k^* = 3$
We have by proposition 4.2, $\mu = \max\limits_{0 \leq i \leq 3} \frac{\|F^i\|}{\alpha_i + I + 1} = 6.25$.

For $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the polyhedron $\mathcal{P}$ with vertices

$$v_1 = \begin{pmatrix} 0.02 \\ 0.02 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} -0.02 \\ 0.02 \end{pmatrix} \quad , \quad v_3 = \begin{pmatrix} -0.02 \\ -0.02 \end{pmatrix} \quad \textit{and} \quad v_4 = \begin{pmatrix} 0.02 \\ -0.02 \end{pmatrix}$$

A simple calculation gives $diam\mathcal{P} \simeq 0.056$ then $diam\mathcal{P} < \frac{1}{2\mu\|T\|} = 0.08$, thus proposition 4.2 insures that the set $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$ of the $(\alpha, \mathcal{P})$-admissible disturbances corresponding to the polyhedron $\mathcal{P}$ and to the observer initial state $z_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is nonempty and is entirely characterized by proposition 3.1, and we have the following representation

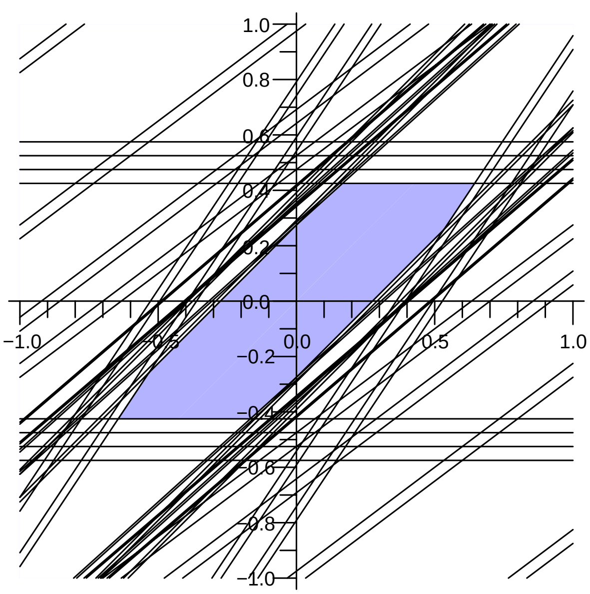

Figure 1: Graphic representation of the set $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$

Remark 6.1 *The procedure suggested requires a great amount of computational work if the state-space dimension or the age of disturbances or the num-*

ber of the vertices of $\mathcal{P}$ are large, because the set $\mathcal{D}_{z_0}(\alpha, \mathcal{P})$ is then obtained by the resolution of a large set of linear inequalities.

7 Conclusion

The problem of improving some performances of an observer of a discrete linear system in presence of disturbances has been considered.
It was proved that the characterization of the set of the disturbances that realize the desired performance is achieved by the selection of an adequate class of initial state observer.
It has been shown that with the hypothesis that the unknown initial state belongs to polyhedral set, the solution involves simple linear programming algorithms.

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Received: December 4, 2006