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Blow-up of the Solution for a kind of Six Order Hyperbolic and Parabolic Evolution Systems

Chen Ning

Department of Mathematics and Physics
Southwest University of Science and Technology
Mianyang 621010 Sichuan, P.R. China

Abstract: In this paper, we give some results on the blow-up behaviors of the solution to the mixed problem for some higher nonlinear hyperbolic evolution equation in finite time .By introducing the ``blow-up factor K (u, u_t)'' we get some new results, which generalize the conclusions of [5] , [6] and [7] .

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1.Introduction and Lemma

A nonlinear pseudo-hyperbolic equation is important in biological mechanics and other fields. In paper [4], Zhang Jian discussed the generalized neural pseudo –hyperbolic equation $u_{tt} - \Delta u_t = F(x, t, u, \nabla u, \nabla u_t)$.

In this paper, we discuss a mixed problem of six-order system. The results obtained generalize some results of [5], [6] and [7] . Let Ω be a bounded domain of R^n having sufficiently smooth boundary $\partial\Omega$, γ denotes the derivative for outward normal direction, u (t, x) is a real function, and $\Delta = \sum_{i=1-}^{n} \frac{\partial^2}{\partial x_i^2}$, $\Delta^2 -= \Delta(\Delta), \ldots$

where x =(x $_{1,}$ x $_{2,}$ …,x $_n$) $\in \Omega$,(t, x) $\in R^+ \times \Omega$ .Let D u=(u $_t$,D $_x$ u)=(u $_t$,u $_{x_1}$,u $_{x_2}$,…u $_{x_n}$),

(D u) $_{x_i}$ =(u $_{tx_i}$,u $_{x_1 x_i}$,…u $_{x_n x_i}$),(i=1,2,…n); and D $_x$ D u=((D u) $_{x_1}$,…(D u) $_{x_n}$) .The following lemma (see lemma 2.1 in [5]) will be needed for our discussion below.

Lemma 1 .Let J $\in C'(0,\infty)$, J (0)>0, C>0, α >0, such that J'(t) $\geq c|J(t)|^{1+\alpha}$,then there exists a T $_0 < \infty$,

such that $\underset{t \to T_0^-}{Lim}$ J(t)= ∞ .

Throughout this paper, we always suppose that F, $\eta, \xi, R, S, u_{0,} u_{1,}$ a_0 and b $_1$ are appropriately smooth.

2. ``Blow-up'' factor K (u, u $_t$)

Let us consider a class of Six -order hyperbolic evolution system:

$$\begin{cases} u_{tt} - \Delta^3 \eta - S(x)\Delta^2 \eta - S(x)\Delta\eta = F(x, u, Du, D_x Du), x \in \Omega, t > 0, \\ u(0, x) = u_0(x), , u(0, x) = u_1(x); \\ [a_1(t, x)\frac{\partial \Delta^2 \eta}{\partial \gamma} + b_1(t, x)u]_{\partial\Omega} = 0, [a_2(t, x)\frac{\partial \Delta\eta}{\partial \gamma} + b_2(t, x)u]_{\partial\Omega} = 0, (1) \end{cases}$$

where $\eta = \eta(u, u_t)$ is an appropriately smooth function of 2-variables..By introducing the blow-up of the above equation ,we obtain some results. We assume that the initial value problem of (1) is compatible with the boundary problem of one , and that

(i). $\int_\Omega K(u_0$ (x), u $_1$ (x))d x>0;

(ii). $K_{u_t}[\frac{\partial\Delta^2\eta}{\partial\gamma}]_{\partial\Omega}=0, K_{u_t}S(x)[\frac{\partial\Delta\eta}{\partial\gamma}]_{\partial\Omega}=0,$ K_{u_t} S (x)[$\frac{\partial\eta}{\partial\gamma}$]$_{\partial\Omega}$ = 0 ;

(iii). K_{u_t}.F $\geq c\,|K(u,u_t)|^{1+\alpha}$ -K_u .u_t + G ,

where G= $\sum_{i=1}^{n}[K_{u_t}]_{x_i}[\Delta^2\eta]_{x_i}+\sum_{i=1}^{n}[K_{u_t}S]_{x_i}[\Delta\eta]_{x_i}$ + $\sum_{i=1}^{n}\left[K_{u_t}S\right]_{x_i}\ [\eta]_{x_i}$,

c>0, $\alpha>0$, .

Theorem 1. Assume that there is a real value function K (u, u_t) satisfying (i),(ii),(iii), and that $u\in C^2(0,T;H^6(\Omega)),$ is a solution of the mixed problems (1) where 0<T$\leq\infty$.Then there exists a T_0 $<\infty$ such that

$$\lim_{t\to T_0^-}\int_\Omega K(u,u_t)dx=\infty .$$

K(u ,u_t) is called the ``Blow-up factor ʻʼ of (1) .

Proof: We take $J(t)=\int_\Omega K(u,u_t)dx$,then

$$J'(t)=\int_\Omega[K_u u_t+K_{ut}u_{tt}]dx$$

$$=\int_\Omega[K_u u_t+K_{u_t}(\Delta^3\eta+S(x)\Delta^2\eta+S(x)\Delta\eta)]\ dx$$

.

By using Green identity , we have

$$\int_\Omega[K_{u_t}[\Delta^3\eta]dx=\int_{\partial\Omega}K_{u_t}[\frac{\partial\Delta^2\eta}{\partial\gamma}]ds-\sum_{i=1}^{n}[K_{u_t}]_{x_i}[\Delta^2\eta]_{x_i}\ dx$$

$$\int_\Omega K_{u_t}\ S\ (x).\ [\ \Delta^2\ \eta\ ]\ d\ x\ =\int_{\partial\Omega}K_{u_t}\ S(x)[\ \frac{\partial\Delta\eta}{\partial\gamma}\ ]\ ds-\sum_{i=1}^{n}\int_\Omega[K_{u_t}\ S]_{x_i}.[\Delta\eta]_{x_i}\ d\,x\ .$$

By the same way, we have

$\int_{\Omega} K_{u_t} \; S \; (x). \; [\; \Delta\eta \;] \; d \; x = \int_{\partial\Omega} K_{u_t} \; .S(x)[\; \frac{\partial\eta}{\partial\gamma} \;]d \; s \; -$

$\int_{\Omega} \; \sum_{i=1}^{n} \left[K_{u_t} S\right]_{x_i} \left[\eta\right]_{x_i} d\,x$.

By Holder inequality , we get J'(t) $\geq$ c $\int_{\Omega} \; \left|K(u,u_t)\right|^{1+\alpha} d\,x \geq c\,.\,c_1 \left|J(t)\right|^{1+\alpha}$,

where $c_1 = (\int_{\Omega} dx)^{-1}$.>0. We choose $T_0 = \frac{1}{c.c.\alpha[J(0)]^{\alpha}} < \infty$, by Lemma 1, we

have $\quad \underset{t\to T_0^-}{Lim} \int_{\Omega} K(u,u_t)dx = \infty$. This ends the proof.□

3. ``Blow-up factor'' K (x , u_t ,v_t)

Now we use the method of ``blow-up factor'' to discuss the following mixed problem, some more general results are obtained. We consider

$$\begin{cases} \\ u_{tt} - L^3\eta - S(x)L^2\eta - T(x)L\eta - R(x)L\xi = f(u,v,u_t), x\in\Omega, t>0, \\ v_{tt} - L^3\xi - S(x)L^2\xi - T(x)L\xi - R(x)L\eta = g(u,v,u_t) \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), v(0,x) = v_0(x), v_t(0,x) = v_1(x), \\ [\frac{\partial L^2\eta}{\partial\gamma}]\big|_{x\in\partial\Omega} = 0, [\frac{\partial L^2\xi}{\partial\gamma}]\big|_{x\in\partial\Omega} = 0, [\frac{\partial L\,\eta}{\partial\gamma}]\big|_{x\in\partial\Omega} = 0, [\frac{\partial L\,\xi}{\partial\gamma}]\big|_{x\in\partial\Omega} = 0, \\ [L^2\eta] = 0, [L^2\xi] = 0, [L\eta] = 0, [L\xi] = 0, x\in\partial\Omega, \end{cases}$$

(2)

where L= $\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left\{a_{ij}(x) \frac{\partial}{\partial x_j} \right\}$, $|\Omega| = \int_{\Omega} dx$ .Throughout this paper we suppose also that f, g S, R are smooth , and $\eta = \eta(u,u_t)$, $\xi = \xi$ (v, v_t) are real functions of two variables ,we get the following theorems;

Theorem 2. Assume that there is a real value function K (x , u_t ,v_t) of 3 - variables satisfying :

$\int_{\Omega} K(x,u_1(x),v_1(x))\,\mathrm{d}x > 0$, and $K_{u_t} f + K_{v_t} g \geq C\left|K(x,u_t,v_t)\right|^{1+\alpha} + G_L$,

where C>0, α >0, $G_L = \sum_{i=1}^{n} [(K_{u_t})_{x_i} (L^2 \eta)_{x_i} + (K_{v_t})_{x_i} (L^2 \xi)_{x_i} + (K_{u_t} S)_{x_i} (L \eta)_{x_i} + (K_{v_t} S)_{x_i} (L \xi)_{x_i} + (K_{u_t} T)_{x_i} (\eta)_{x_i} + (K_{v_t} T)_{x_i} (\xi)_{x_i} + (K_{u_t} R)_{x_i} (\xi)_{x_i} + (K_{v_t} R)_{x_i} (\eta)_{x_i}]$. Then there exists a $T_0 < \infty$ such that

$$\lim_{t \to T_0^-} \int_{\Omega} K(x,u_t,v_t)\,\mathrm{d}x = \infty,$$

where (u (t, x), v (t x)) is the classical solution of the systems (2) .

Proof. Let $J(t) = \int_{\Omega} K(x,u_t,v_t)\,\mathrm{d}x$, then

$$J'(t) = \int_{\Omega} [K_{u_t} \cdot u_{tt} + K_{v_t} \cdot v_{tt}]\,\mathrm{d}x$$

$$= \int_{\Omega} [K_{u_t}(L^3\eta + S(x)L^2\eta + T(x)L\eta + R(x)L\xi) + K_{v_t}(L^3\xi + S(x)L^2\xi + T(x)L\xi + R(x)L\eta)]dx$$

Similar to the proof of Theorem 1, we use Green identity to obtain $J'(t) \geq C\int_{\Omega}\left|K(x,u_t,v_t)\right|^{1+\alpha}dx$,that is $J'(t) \geq C\left|J(t)\right|^{1+\alpha}$. By Lemma 1, and $J(0) > 0$,we get $\lim_{t \to T_0^-}\int_{\Omega} K(x,u_t,v_t)dx = \infty$ .This completes the proof of the theorem.□

4 . ``Blow-up factor'' $K(u_{t,}v)$

We consider again the coupled systems for higher order hyperbolic and parabolic equations:

$$\begin{cases} u_{tt} - \Delta^3\eta - S(x)\Delta^2\eta - T(x)\Delta\eta = f(u,v,u_t), x \in \Omega, t > 0, \\ v_t - \Delta^2\xi - R(x)\Delta\xi = g(u,v,u_t) \end{cases} \qquad (3)$$

where the initial condition: $u\big|_{t=0}=u_0(x), u_t\big|_{t=0}=u_1(x); \quad v\big|_{t=0}=v_0(x)$, (3)';

And the boundary value condition:

$$[a_1(t,x)\frac{\partial\Delta^2\eta}{\partial\gamma}+b_1(t,x)u]_{\partial\Omega}=0, \qquad [a_2(t,x)\frac{\partial\Delta\eta}{\partial\gamma}+b_2(t,x)u]_{\partial\Omega}=0,$$

$$[a_3(t,x)\frac{\partial\eta}{\partial\gamma}+b_3(t,x)]_{\partial\Omega}=0; \qquad [a_4(t,x)\frac{\partial\Delta\xi}{\partial\gamma}+b_4(t,x)u]_{\partial\Omega}=0.$$

$$[a_5(t,x)\frac{\partial\xi}{\partial\gamma}+b_5(t,x)u]_{\partial\Omega}=0. \text{(3)''}$$

there functions f, g ,S,T,R,…are suite smooth ,and satisfying following conditions

(i) $\int_\Omega K(u_1(x),v_0(x))dx>0,$

(ii) $K_{u_t}[\frac{\partial\Delta^2\eta}{\partial\gamma}]_{\partial\Omega}=0,$

$$K_{u_t}[\frac{\partial\Delta\eta}{\partial\gamma}]_{\partial\Omega}=0,\ K_{u_t}[\frac{\partial\eta}{\partial\gamma}]_{\partial\Omega}=0,\ K_v[\frac{\partial\Delta\xi}{\partial\gamma}]_{\partial\Omega}=0,\ K_v[\frac{\partial\xi}{\partial\gamma}]_{\partial\Omega}=0,$$

(iii) $K_{u_t}.f+K_v.g\ge c\left|K(u_t,v)\right|^{1+\alpha}+G,(\alpha>0),$

where

$$G=\sum_{i=1}^{n}\{[K_{u_t}]_{x_i}[\Delta^2\eta]_{x_i} \quad +[K_{u_t}S]_{x_i}[\Delta\eta]_{x_i} \quad +[K_{u_t}T]_{x_i}[\eta]_{x_i} \quad +[K_v]_{x_i}[\Delta\xi]_{x_i}$$

$$+[K_vR]_{x_i}[\xi]_{x_i}\}.$$

Theorem 3. Assume that there is a real value function K (u_t,v) satisfying (i),(ii),(iii), and that $u\in C^2(0,T;H^6(\Omega)), \mathrm{v}\in C^2(0,T;H^4(\Omega)),$ are a solution of the mixed problems (1) where 0<T$\le\infty$.Then there exists a T $_0$

$<\infty$ such that

$$\lim_{t\to T_0^-}\int_\Omega K(u_t,v)dx=\infty,$$

where $K(u_t,v)$ is called the ``Blow-up factor ‘’ of (3) .

Proof. Let J (t) = $\int_\Omega K(u_t,v)dx,$ then

$$J'(t)=\int_\Omega (K_{u_t}.u_{tt}+K_v.v_t)dx=\int_\Omega [K_{u_t}(\Delta^3\eta+S(x)\Delta^2\eta+T(x)\Delta\eta)]dx+\int_\Omega [K_v(\Delta^2\xi+R(x)\Delta\xi)]dx$$

$$+\int_\Omega [K_{u_t}.f+K_v.g]dx$$

By Green identity we get:

$$\int_\Omega K_{u_t}[\Delta^3\eta]dx=\int_{\partial\Omega} K_{u_t}[\frac{\partial\Delta^2\eta}{\partial\gamma}]ds-\sum_{i=1}^{n}\int_\Omega [K_{u_t}]_{x_i}[\Delta^2\eta]_{x_i}dx=$$

$$-\sum_{i=1}^{n}\int_\Omega [K_{u_t}]_{x_i}.[\Delta^2\eta]_{x_i}dx,$$

$$\int_\Omega K_{u_t}S(x).[\Delta^2\eta]dx=\int_{\partial\Omega} K_{u_t}S(x)[\frac{\partial\Delta\eta}{\partial\gamma}]ds-\sum_{i=1}^{n}\int_\Omega [K_{u_t}S]_{x_i}.[\Delta\eta]_{x_i}dx.$$

$$=-\sum_{i=1}^{n}\int_\Omega [K_{u_t}.S]_{x_i}.[\Delta\eta]_{x_i}dx,$$ and $$\int_\Omega K_{u_t}[\Delta\eta]dx=\int_{\partial\Omega} K_{u_t}[\frac{\partial\eta}{\partial\gamma}]ds-\sum_{i=1}^{n}\int_\Omega [K_{u_t}]_{x_i}[\eta]_{x_i}dx.$$

$$\int_\Omega K_v.[\Delta^2\xi]dx=\int_{\partial\Omega} K_v[\frac{\partial\Delta\xi}{\partial\gamma}]ds-\sum_{i=1}^{n}\int_\Omega [K_v]_{x_i}[\Delta\xi]_{x_i}dx,$$

$$\int_\Omega K_vR(x)[\Delta\xi]dx=\int_{\partial\Omega} K_vR[\frac{\partial\xi}{\partial\gamma}]ds-\int_\Omega \{\sum_{i=1}^{n}[K_vR]_{x_i}[\xi]_{x_i}\}dx$$

Thus, $\int_\Omega (K_{u_t}.f+K_v.g)dx\geq c\int_\Omega |K(u_t,v)|^{1+\alpha}dx,$

that is J'(t) $\geq c.c_1.\left|J(t)\right|^{1+\alpha}$. Where $c_1 = (\int_\Omega dx)^{-1} > 0$. By Lemma 1, we get $\underset{t\to T_0^-}{Lim}\int_\Omega K(u_t,v)dx = \infty$. This ends this proof .□

Remark .We consider again this problems with no homogeneous condition.

The system of coupled equations for hyperbolic and parabolic are appearing in thermal elastic mechanics . We consider the more generalized stronger type systems for higher order evolution systems:

$$\begin{cases} u_{tt} - \beta\Delta^2 u - \Delta u = f(u,v,u_t), x\in\Omega, t>0, \\ v_t - \beta\Delta^2 v - \Delta v = g(u,v,u_t), \end{cases} \tag{4}$$

the initial condition : $u\big|_{t=0} = u_0(x), u_t\big|_{t=0} = u_1(x), v\big|_{t=0} = v_0(x),$ (4)'the boundary condition : $[\frac{\partial\Delta u}{\partial\gamma}]_{\partial\Omega} = k_1(t,x), [\frac{\partial v}{\partial\gamma}]_{\partial\Omega} = k_2(t,x)$,

$[\frac{\partial u}{\partial\gamma}]_{\partial\Omega} = k_3(t,x), [\frac{\partial v}{\partial\gamma}]_{\partial\Omega} = k_4(t,x)$,(4)''.

where functions f,g,…etc are suite smooth ,they satisfy conditions:

(i) $\int_{\partial\Omega}[ak_1(t,x)+bk_2(t,x)]ds \geq 0, \int_{\partial\Omega}(ak_3(x,t)+bk_4(x,t))ds \geq 0,$

(ii) $af + bg \geq c_1\left|au_t + bv\right|^{1+\alpha}, c_1 > 0, \alpha > 0,$

(iii) $\int_\Omega [au_1(x) + bv_0(x)]dx > 0.$

which generalized some results along the direction in [9] ,and we get the following theorem.

Theorem 4.suppose that conditions (i), (ii), (iii) holds .If the system (4) exits this solution $u\in C^2(0,T;H^4), v\in C^1(0,T;H^4)$ .Then there exists a $T_0 < \infty$ such that

$$\underset{t\to T_0^-}{Lim}\int_\Omega (au_t + bv)dx = \infty,$$

that is $\limsup_{t\to T_0^-}\left|u_t(x,t)\right| = \infty$,or $\limsup_{t\to T_0^-}\left|v(x,t)\right| = \infty$,where a >0,b>0.In other words, the solution (u, v) of (4) blow-up in the finite time.

Proof. Let J (t)=$\int_{\Omega}(au_t+bv)dx,$ then by using condition (i) and Green identity , we obtain :

J'(t)= $\int_{\Omega}(au_{tt}+bv_t)dx$ = $\int_{\Omega}\beta(a\Delta^2u+b\Delta^2v)dx$ + $\int_{\Omega}(a\Delta u+b\ \ \Delta v)dx$ + $\int_{\Omega}(af+bg)dx$.

$$=\int_{\partial\Omega}\beta[ak_1(x,t)+bk(x,t)]ds+\delta\int_{\partial\Omega}[ak_3(x,t)+bk_4(x,t)]ds+\int_{\Omega}(af+bg)dx,$$

that is J'(t)$\geq c_1\int_{\Omega}|au_t+bv|^{1+\alpha}dx$. Thus, by Holder inequality, we get easy that J'(t)$\geq c|J(t)|^{1+\alpha}$, where C= $c_1(\int_{\Omega}dx)^{-1}$.>0. Similar to the proof of theorem 1, and J(0)>0. By Lemma 1 ,there exists a $T_0<\infty,$ such that

$$\lim_{t\to T_0^-} J(t)=\lim_{t\to T_0^-}\int_{\Omega}(au_t+bv)dx=\infty.$$

That is $\lim_{t\to T_0} Sup|u_t(x,t)|=\infty,$ or $\lim_{t\to T_0^-} Sup|v(x,t)|=\infty$. Thus the solution (u ,v) of system (4) blow-up in the finite time. This completes this proof. □

Corollary. Let $\beta=0$, we get the systems of complete equation with 2- order that is also the similar results.

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