# Stability of limit cycle in a predator-prey model with modified Leslie-Gower and Holling-type $II$ schemes with time delay

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## Abstract

In this paper we study the stability of the periodic solutions of a model set forth by M. A. Aziz Alaoui et al. [1, 11] with time delay, which describes the competition between the predator and prey. This model incorporates a modified version of Leslie-Gower functional response as well as that of the Holling-type $II$. In this paper we consider the model with one delay and a unique non trivial equilibrium $E^*$. its dynamics are studied in terms of the local stability and of the description of the Hopf bifurcation at $E^*$, that is proven to exists as the delay (taken as a parameter of bifurcation) crosses some critical values. The main result of this paper is to establish an explicit algorithm for determining the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions, using the methods presented by T. Faria et al. [5, 6].

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**Keywords:** Delay differential equations, stability/unstability, Hopf bifurcation, periodic solutions, direction of Hopf bifurcation, stability of periodic solutions.

# 1   Introduction and mathematical models

Time delays of one type or another have been incorporated into biological models by many researchers, we refer to the monographs of Cushing [4], Gopalsamy [8], Kuang [9] and MacDonald [10] for general delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate.

In this paper, we shall consider a two-dimensional system with discrete delay proposed recently by Aziz Alaoui et al. [1, 11], see also [1, 12, 13] which modelling a predator-prey competition. This model incorporates a modified version of Leslie-Gower functional response as well as that of the Holling-type $II$.

The first model proposed in this optic is given by an ordinary differential equations [1] as follows

$$\begin{cases} \frac{dx}{dt} = (a_1 - bx - \frac{c_1 y}{x+k_1})x, \\[2mm] \frac{dy}{dt} = (a_2 - \frac{c_2 y}{x+k_2})y \end{cases} \tag{1}$$

with initial conditions $x(0) > 0$ and $y(0) > 0$.

This two species food chain model describes a prey population $x$ which serves as food for a predator $y$. The model parameters $a_1$, $a_2$, $b$, $c_1$, $c_2$, $k_1$ and $k_2$ are assuming only positive values. These parameters are defined as follows: $a_1$ is the growth rate of prey $x$, $b$ measures the strength of competition among individuals of species $x$, $c_1$ is the maximum value of the per capita reduction rate of $x$ due to $y$, $k_1$ (respectively, $k_2$) measures the extent to which environment provides protection to prey $x$ (respectively, to the predator $y$), $a_2$ describes the growth rate of $y$, and $c_2$ has a similar meaning to $c_1$.

In this paper we consider the delayed model of (1) see [11]

$$\begin{cases} \frac{dx(t)}{dt} = (a_1 - bx(t) - \frac{c_1 y(t)}{x(t)+k_1})x(t), \\[2mm] \frac{dy(t)}{dt} = (a_2 - \frac{c_2 y(t-\tau)}{x(t-\tau)+k_2})y(t) \end{cases} \tag{2}$$

for all $t > 0$. Here, the discrete delay $\tau > 0$ has been incorporated in the negative feedback of the predator's density.

The notion of global stability is studied by many authors in the predator-prey

systems with delay [2, 12, 14].

In [1], the authors study the boundeness and global stability of system (1).

In [11], the authors study the global stability and persistence of the delayed system (2) by using liapunov functional.

In [15], the authors study the occurrence of Hopf bifurcation at the third trivial equilibrium and at the non trivial positive equilibrium when the delay crosses some critical values.

Our goal in this paper is to consider the system (2) the non trivial equilibrium $E^*$. We study the stability of limit cycle around the non trivial equilibrium $E^*$ which is the most biologically meaningful one. We establish an explicit algorithm for determining the direction of the Hopf bifurcation.

This paper is organized as follows. In the next section, we recall some results on the existence and the change of stability of equilibrium points $E^*$ and the occurrence of the Hopf bifurcation. The main result is given in sections 3, we show the stability or instability of the bifurcating periodic solutions and the direction of Hopf bifurcation via normal form theory. In the end, we give an application.

## 2   Stability and Hopf bifurcation

Consider again the system (2), then we have the following result on the existence of equilibrium points:

**Proposition 2.1.** *[11] i) The system have three equilibrium points $E_0 = (0,0)$, $E_1 = (\frac{a_1}{b}, 0)$ and $E_2 = (0, \frac{a_2 k_2}{c_2})$,*
*ii) if the following condition holds*

$$\frac{a_2 k_2}{c_2} < \frac{a_1 k_1}{c_1},$$

*then the system (2) has a unique non trivial positive equilibrium $E^* = (x^*, y^*)$, where*

$$x^* = \frac{1}{2c_2 b}(-(c_1 a_2 - a_1 c_2 + c_2 b k_1) + \sqrt{\Delta}),$$

*and*

$$y^* = \frac{a_2(x^* + k_2)}{c_2},$$

*and*

$$\Delta = (c_1 a_2 - a_1 c_2 + c_2 b k_1)^2 - 4c_2 b(c_1 a_2 k_2 - c_2 a_1 k_1) > 0.$$

Normalizing the delay $\tau$ by the time scaling $t \longrightarrow \frac{t}{\tau}$, (2) is transformed

into

$$\begin{cases} \frac{dx(t)}{dt} = \tau((a_1 - bx(t) - \frac{c_1 y(t)}{x(t)+k_1})x(t)), \\ \\ \frac{dy(t)}{dt} = \tau((a_2 - \frac{c_2 y(t-1)}{x(t-1)+k_2})y(t)) \end{cases} \tag{3}$$

By the translation $z(t) = (u(t), v(t)) = (x(t), y(t)) - E^* \in \mathbb{R}^2$ and by linearizing system (3) around the equilibrium $E^*$, (3) is written as an FDE in $C := C([-1, 0], \mathbb{R}^2)$ as

$$\frac{dz}{dt}(t) = L(\tau)z_t + f(z_t, \tau) \tag{4}$$

where $z_t(\theta) = z(t + \theta), \forall \theta \in [-1, 0]$ and $L_0(\tau) : C \longrightarrow \mathbb{R}^2$, $f_0 : C \times \mathbb{R}^+ \longrightarrow \mathbb{R}^2$ are given by

$$L(\tau)(\varphi) = \tau \begin{pmatrix} A_{11}\varphi_1(0) + A_{12}\varphi_2(0) \\ \\ A_{21}\varphi_1(-1) + A_{22}\varphi_2(-1) \end{pmatrix}$$

where

$$\begin{aligned} A_{11} &= -bx^* + \frac{c_1 y^*}{(x^* + k_1)^2}x^*, \\ A_{12} &= -\frac{c_1 x^*}{x^* + k_1}, \\ A_{21} &= \frac{a_2{}^2}{c_2}, \\ A_{22} &= -a_2, \end{aligned}$$

and

$$f(\varphi, \tau) = \tau \begin{pmatrix} (a_1 - b(\varphi_1(0) + x^*) - \frac{c_1(\varphi_2(0)+y^*)}{\varphi_1(0)+x^*+k_1})(\varphi_1(0) + x^*) \\ -A_{11}\varphi_1(0) - A_{12}\varphi_2(0) \\ \\ (a_2 - \frac{c_2(\varphi_2(-1)+y^*)}{\varphi_1(-1)+x^*+k_2})(\varphi_2(0) + y^*) - A_{21}\varphi_1(-1) - A_{22}\varphi_2(-1) \end{pmatrix}$$

where $\varphi = (\varphi_1, \varphi_2) \in C$.
The characteristic equation of the linear equation

$$\frac{dz(t)}{dt} = L_0(\tau)z_t \tag{5}$$

is given by

$$\Delta_1(\lambda, \tau) = \lambda^2 + \tau p\lambda + \tau^2 r + (s\tau\lambda + q\tau^2)e^{-\lambda} = 0, \tag{6}$$

where $p$, $s$, $r$, and $q$ have the following expressions:

$$
\begin{aligned}
p &= -A_{11} \\
r &= 0 \\
s &= -A_{22} \\
q &= \det(J).
\end{aligned}
$$

where

$$
J = \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right).
$$

This type of characteristic equation is studied by many authors see [3].
The following theorem gives the result of change of stability of the non trivial steady state $E^*$.

**Theorem 2.2.** *[15] Assume $\frac{a_2 k_2}{c_2} < \frac{a_1 k_1}{c_1}$ and $a_1 < bk_1$. Then, there exists a critical value $\tau_0$ of the time delay, such that the non trivial steady state $E^*$ is asymptotically stable for $\tau \in [0, \tau_0[$ and unstable for $\tau > \tau_0$, where*

$$
\tau_0 = \frac{1}{\zeta_+} \arccos \left\{ \frac{q\zeta_+^2 - ps\zeta_+^2}{s^2\zeta_+^2 + q^2} \right\}, \tag{7}
$$

*and*

$$
\zeta_+^2 = \frac{1}{2}(s^2 - p^2) + \frac{1}{2}[(s^2 - p^2)^2 + 4q^2]^{\frac{1}{2}}. \tag{8}
$$

The next theorem gives a result on the existence of limit cycle of system (4) at the non trivial steady state $E^*$.

**Theorem 2.3.** *[15] Assume $\frac{a_2 k_2}{c_2} < \frac{a_1 k_1}{c_1}$ and $a_1 < bk_1$. Then, there exists $\varepsilon_0 > 0$ such that, for each $0 \leq \varepsilon < \varepsilon_0$, equation (4) has a family of periodic solutions $p_l(\varepsilon)$ with period $T_l = T_l(\varepsilon)$, for the parameter values $\tau = \tau(\varepsilon)$ such that $p_l(0) = E^*$, $T_l(0) = \frac{2\pi}{\zeta_+}$ and $\tau(0) = \tau_0$, where $\tau_0$ and $\zeta_+$ are given respectively in equations (7) and (8) and $\omega_0 = \tau_0 \zeta_+$ is the purely imaginary root of equation (6).*

# 3 Direction of Hopf bifurcation

Consider (4) in the phase space $C$, let $\Lambda = \{-i\omega_0, i\omega_0\}$. Introducing the new parameter $\alpha = \tau - \tau_0$, (4) is rewritten as

$$
\frac{dz}{dt}(t) = L(\tau_0)z_t + F(z_t, \alpha) \tag{9}
$$

where $F(\varphi, \alpha) = L(\alpha)(\varphi) + f(\varphi, \tau_0 + \alpha)$. Using the formal adjoint theory for FDEs in [7], we decompose $C$ by $\Lambda$ as $C = P \oplus Q$, where P is the center space for

$$\frac{dz}{dt}(t) = L(\tau_0)z_t.$$

Considering complex coordinates, $P = span\{\phi_1, \phi_2\}$, with $\phi_1(\theta) = e^{i\omega_0\theta}v$, $\phi_2(\theta) = \overline{\phi_1(\theta)}$, $-1 \leq \theta \leq 0$, where the bar means complex conjugation, and $v$ is a vector in $\mathbb{C}^2$ that satisfies

$$L(\tau_0)(\phi_1) = i\omega_0 v. \tag{10}$$

Then

$$v = (v_1, v_2) = (1, \frac{i\omega_0 - \tau_0 A_{11}}{\tau_0 A_{12}})$$

For $\Phi = [\phi_1, \phi_2]$, note that $\dot{\Phi} = B\Phi$, where $B$ is the $2 \times 2$ diagonal matrix

$$B = \begin{pmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{pmatrix}.$$

Choose a basis $\Psi$ for the adjoint space $P^*$, such that $(\Psi, \Phi) = (\psi_i, \phi_j)_{i,j=1}^2$, where $(.,.)$ is the bilinear form on $C^* \times C$ associated with the adjoint equation. Thus, $\Psi(s) = col(\psi_1(s), \psi_2(s)) = col(u^T e^{-i\omega_0 s}, \overline{u}^T e^{i\omega_0 s})$, $s \in [0, 1]$, for $u \in \mathbb{C}^2$ such that

$$(\psi_1, \phi_1) = 1, \qquad (\psi_1, \phi_2) = 0. \tag{11}$$

A further computation leads to

$$u = (u_1, u_2) = (\frac{A_{12}}{A_{12} + A_{21(i\omega_0 + \tau_0 A_{11})e^{-i\omega_0}}}, 0).$$

We take the enlarged phase space

$$BC = \{\varphi : [-1, 0] \longrightarrow \mathbb{C}^2/\varphi \text{ continuous on } [-1, 0), \exists \lim_{\theta \to 0^-} \varphi(\theta)\},$$

we can see that the projection of $C$ upon $P$, associated with the decomposition $C = P \oplus Q$, is now replaced by $\pi : BC \longrightarrow P$, xhich leads to the decomposition

$$BC = P \oplus Ker\pi.$$

Using the decomposition

$$z_t = \Phi X(t) + Y_t,$$

where $X(t) \in \mathbb{C}^2$, $Y_t \in Q^1$, we decompose (9) as

$$\begin{cases} \frac{dX}{dt} = BX + \Psi(0)F(\Phi X + Y, \alpha) \\ \\ \frac{dY}{dt} = A_{Q^1}Y + (I - \pi)X_0 F(\Phi X + Y, \alpha), \end{cases} \tag{12}$$

where here and throughout this section we refer to [6] for results and explanations of several notations involced. We write the taylor formula

$$\Psi(0)F(\Phi X + Y, \alpha) = \frac{1}{2}f_2^1(X, Y, \alpha) + \frac{1}{3!}f_3^1(X, Y, \alpha)$$

$$(I - \pi)X_0 F(\Phi X + Y, \alpha) = \frac{1}{2}f_2^2(X, Y, \alpha) + \frac{1}{3!}f_3^2(X, Y, \alpha),$$

where $f_j^1(X, Y, \alpha)$, $f_j^1(X, Y, \alpha)$ are homogeneous polynomials in $(X, Y, \alpha)$ of degree $j$, $j = 1, 3$, with coefficients in $\mathbb{C}^2$, $Ker\pi$, respectively.

The normal form method gives for (9) a normal form on the center manifold of the origin at $\alpha = 0$, written as

$$\frac{dX}{dt} = BX + \frac{1}{2}g_2^1(X, 0, \alpha) + \frac{1}{3!}g_3^1(X, 0, \alpha) + h.o.t., \qquad (13)$$

where $g_2^1$, $g_3^1$ are the second and third order terms in $(X, \alpha)$, respectively, and h.o.t. stands for higher order terms.

The normal form procedure will show that these terms have the form

$$\frac{1}{2}g_2^1(X, 0, \alpha) = \begin{pmatrix} A_1 X_1 \alpha \\ B_1 X_2 \alpha \end{pmatrix},$$

and

$$\frac{1}{3!}g_3^1(X, 0, \alpha) \begin{pmatrix} A_2 X_1^2 X_2 \\ B_2 X_1 X_2^2 \end{pmatrix} + O(|X|\alpha^2).$$

Moreover, it will be turn out that $B_1 = \overline{A_1}$, $B_2 = \overline{A_2}$, because the coefficients in (9) are real.

We continue this section with the computation of $g_2^1$, $g_3^1$, omitting some details. Always following [5], we first recall the operators, $M_j^1$,

$$M_j^1(p)(X, \alpha) = D_X p(X, \alpha)BX - Bp(X, \alpha), \qquad j \geq 2.$$

In particular,

$$M_j^1(\alpha^l X^q e_k) = i\omega_0(q_1 - q_2 + (-1)^k)\alpha^l X^q e_k, \qquad l + q_1 + q_2 = j, k = 1, 2,$$

for $j = 1, 2$, $q = (q_1, q_2) \in \mathbb{N}_0^2$, $l \in \mathbb{N}_0$, and $e_1, e_2$ the canonical basis for $\mathbb{C}^2$. Hence,

$$Ker(M_2^1) = span\left\{ \begin{pmatrix} X_1\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X_2\alpha \end{pmatrix} \right\}$$

$$Ker(M_3^1) = span\left\{ \begin{pmatrix} X_1^2 X_2 \\ 0 \end{pmatrix}, \begin{pmatrix} X_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X_1 X_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ X_2\alpha^2 \end{pmatrix} \right\}$$

$$(14)$$

From equation (9), it is

$$f_2^1(X, Y, \alpha) = 2\Psi(0)[L(\alpha)(\Phi X + Y) + f(\Phi X + y, \alpha)] \qquad (15)$$

and we have,

$$f_2^1(X, 0, \alpha) = 2\left\{ \frac{\alpha}{\tau_0} i\omega_0 \begin{pmatrix} u^T v X_1 - u^T \overline{V} X_2 \\ \overline{u}^T v X_1 - \overline{u}^T \overline{v} X_2 \end{pmatrix} + \right.$$

$$\left. \tau_0 \begin{pmatrix} (u_1 a_{11} + u_2 b_{11})X_1^2 + (u_1 a_{12} + u_2 b_{12})X_1 X_2 + (u_1 a_{13} + u_2 b_{13})X_2^2 \\ (\overline{u}_1 a_{11} + \overline{u}_2 b_{11})X_1^2 + (\overline{u}_1 a_{12} + \overline{u}_2 b_{12})X_1 X_2 + (\overline{u}_1 a_{13} + \overline{u}_2 b_{13})X_2^2 \end{pmatrix} \right\}$$

where

$$a_{11} = \frac{1}{2}f_{20}^1 = -b + \frac{c_1 y^*}{(x^* + k_1)^2} - \frac{c_1 x^* y^*}{(x^* + k_1)^3},$$

$$a_{12} = f_{11}^1 = -\frac{c_1}{x^* + k_1} + \frac{c_1 x^*}{(x^* + k_1)^2},$$

$$a_{13} = \frac{1}{2}f_{20}^1 = 0$$

and

$$b_{11} = \frac{1}{2}f_{20}^2 = -\frac{c_2 y^{*2}}{(x^* + k_2)^3},$$

$$b_{12} = f_{11}^2 = \frac{2c_2 y^*}{(x^* + k_2)^2},$$

$$b_{13} = \frac{1}{2}f_{20}^2 = -\frac{1}{2}\frac{c_2}{x^* + k_2}$$

where

$$f_{ij}^1 = \frac{\partial^{i+j} f^1}{\partial^i x \partial^j y}\Big|_{(x^*, y^*)}$$

$$f_{ij}^2 = \frac{\partial^{i+j} f^2}{\partial^i x \partial^j y}\Big|_{(x^*, y^*)}$$

and $f^1$ and $f^2$ are the components of the function $f$ defined in equation (4). Therefore, the second order terms in $(X, \alpha)$ of the normal form on the center manifold are given by

$$\begin{aligned} g_2^1(X, 0, \alpha) &= Proj_{Ker(M_2^1)} f_2^1(X, 0, \alpha) \\ &= \begin{pmatrix} A_1 X_1 \alpha \\ B_1 X_2 \alpha \end{pmatrix} \end{aligned}$$

where

$$A_1 = \overline{B_1} = \frac{i\omega_0 u^T v}{\tau_0}.$$

To eliminate these nonresonant terms in the quadratic terms $f_2^1(X, 0, \alpha)$, we have to make a series of transformations of variables, which can change the coefficients of the cubic terms of $f_3^1(X, 0, \alpha)$. By some computations, the expression of $f_3^1(X, 0, \alpha)$ is given as follows

$$f_3^1(X, 0, \alpha) = \tau_0 \begin{pmatrix} (u_1 a_{21} + u_2 b_{21})X_1^3 + (u_1 a_{22} + u_2 b_{22})X_1^2 X_2 + \\ (u_1 a_{23} + u_2 b_{23})X_1 X_2^2 + (u_1 a_{24} + u_2 b_{24})X_2^3 \\ \\ (\overline{u_1} a_{21} + \overline{u_2} b_{21})X_1^3 + (\overline{u_1} a_{22} + \overline{u_2} b_{22})X_1^2 X_2 + \\ (\overline{u_1} a_{23} + \overline{u_2} b_{23})X_1 X_2^2 + (\overline{u_1} a_{24} + \overline{u_2} b_{24})X_2^3 \end{pmatrix} \tag{16}$$

where

$$a_{21} = \frac{1}{3!} f_{30}^1 v_1^3 + \frac{1}{2} f_{21}^1 \overline{v_1}^2 v_2 + \frac{1}{2} f_{12}^1 v_1 v_2^2 + \frac{1}{3!} f_{03}^1 v_2^3,$$

$$a_{22} = \frac{1}{3!} f_{30}^1 3 v_1 |v_1|^2 + \frac{1}{2} f_{21}^1 (2|v_1|^2 v_2 + \overline{v_1}^2 \overline{v_2}) + \frac{1}{2} f_{12}^1 (2v_1 |v_2|^2 + \overline{v_1} v_2^2) + \frac{1}{3!} f_{03}^1 3 \overline{v_2} |v_2|^2,$$

$$a_{23} = \frac{1}{3!} f_{30}^1 3 \overline{v_1} |v_1|^2 + \frac{1}{2} f_{21}^1 (\overline{v_1}^2 v_2 + 2|v_1|^2 \overline{v_2}) + \frac{1}{2} f_{12}^1 (\overline{v_1} v_2^2 + 2\overline{v_1} |v_2|^2) + \frac{1}{3!} f_{03}^1 3 \overline{v_2} |v_2|^2,$$

$$a_{24} = \frac{1}{3!} f_{30}^1 \overline{v_1}^3 + \frac{1}{2} f_{21}^1 \overline{v_1}^2 \overline{v_2} + \frac{1}{2} f_{12}^1 \overline{v_1} \overline{v_2}^2 + \frac{1}{3!} f_{03}^1 \overline{v_2}^3,$$

and

$$b_{21} = \frac{1}{3!} f_{30}^2 v_1^3 e^{-3i\omega_0} + \frac{1}{2} f_{21}^2 v_1^2 v_2 e^{-2i\omega_0}(1 + e^{-i\omega_0}) + \\ \frac{1}{2} f_{12}^2 v_1 v_2^2 e^{-i\omega_0}(1 + e^{-2i\omega_0}) + \frac{1}{3!} f_{03}^2 v_2^3(1 + e^{-3i\omega_0}),$$

$$b_{22} = \frac{1}{3!} f_{30}^2 3 v_1 |v_1|^2 e^{-i\omega_0} + \frac{1}{2} f_{21}^2 (2|v_1|^2 v_2 + v_1^2 \overline{v_2} e^{-i\omega_0})(1 + e^{-i\omega_0}) + \\ \frac{1}{2} f_{12}^2 (2v_1 e^{-i\omega_0}(|v_1|_2 + |v_2|^2) + \overline{v_1} v_2^2 (e^{i\omega_0} + e^{-i\omega_0}) + \frac{1}{3!} f_{03}^2 3 v_2 |v_2|^2(1 + e^{-i\omega_0}),$$

$$b_{23} = \frac{1}{3!} f_{30}^2 3 \overline{v_1} |v_1|^2 e^{i\omega_0} + \frac{1}{2} f_{21}^2 (2|v_1|^2 v_2 + \overline{v_1}^2 v_2 e^{i\omega_0})(1 + e^{i\omega_0}) + \\ \frac{1}{2} f_{12}^2 (\overline{v_1} e^{i\omega_0}(3|v_1|_2 + 2|v_2|^2) + v_1 \overline{v_2}^2 e^{-i\omega_0}) + \frac{1}{3!} f_{03}^2 3 \overline{v_2} |v_2|^2(1 + e^{i\omega_0}),$$

$$b_{24} = \frac{1}{3!} f_{30}^2 \overline{v_1}^3 e^{3i\omega_0} + \frac{1}{2} f_{21}^2 \overline{v_1}^2 \overline{v_2} e^{2i\omega_0}(1 + e^{i\omega_0}) + \\ \frac{1}{2} f_{12}^2 (\overline{v_1 v_2}^2 e^{i\omega_0} + \overline{v_1}^3 e^{3i\omega_0}) + \frac{1}{3!} f_{03}^2 \overline{v_2}^3(1 + e^{3i\omega_0}).$$

Notice that

$$Ker(M_3^1) = span\left\{\begin{pmatrix} X_1^2 X_2 \\ 0 \end{pmatrix}, \begin{pmatrix} X_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X_1 X_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ X_2\alpha^2 \end{pmatrix}\right\}.$$

However, the terms $O(|X|\alpha^2)$ are irrelevant to determine the generic Hopf bifurcation. Hence, we only need to compute the coefficients of $X_1^2 X_2$. After some computations we find that the coefficient of $X_1^2 X_2$ is

$$A_2 = \frac{i\tau_0^2}{2\omega_0}((u_1 a_{11} + u_2 b_{11})(u_1 a_{12} + u_2 b_{12}) - 2|u_1 a_{12} + u_2 b_{12}|^2 -$$
$$\frac{1}{3}|u_1 a_{13} + u_2 b_{13}|^2) + \tau_0(u_1 a_{22} + u_2 b_{22}).$$

Thus

$$\frac{1}{3!}g_3(X, 0, \alpha) = \begin{pmatrix} A_2 X_1^2 X_2 \\ \overline{A_2} X_1 X_2^2 \end{pmatrix} + O(|X|\alpha^2).$$

Then one prove that the normal form (13) has the form

$$\frac{dX}{dt} = \begin{pmatrix} A_1 X_1\alpha \\ \overline{A_1} X_2\alpha \end{pmatrix} + \begin{pmatrix} A_2 X_1^2 X_2 \\ \overline{A_2} X_1 X_2^2 \end{pmatrix} + O(|X|\alpha^2). \tag{17}$$

The normal form relative to $P$ can be written in real coordinates $(x, y)$ through the change of variables $X_1 = x - iy$, $X_2 = x + iy$. Followed by the use of polar coordinates $(r, \theta)$, $x = r\cos(\theta)$, $y = r\sin(\theta)$, this normal form becomes

$$\begin{cases} \frac{dr}{dt} = K_1\alpha r + K_2 r^3 + O(\alpha^2 + |(r, \alpha)|) \\ \\ \frac{d\theta}{dt} = -\omega_0 + O(|(r, \alpha)|). \end{cases} \tag{18}$$

where $K_1 = Re(A_1)$ and $K_2 = Re(A_2)$.
We are now in the position of the computation of the expressions of $K_1$ and $K_2$.
From the expression of $A_1$, we have

$$K_1 = \frac{1}{\tau_0}\frac{\omega_0 A_{12} A_{21}(\omega_0\cos(\omega_0) - A_{11}\sin(\omega_0))}{M^2 + N^2}$$

where

$$M = A_{12} + A_{21}(A_{11}\cos(\omega_0) + \omega_0\sin(\omega_0))$$
$$N = A_{21}(\omega_0\cos(\omega_0) - A_{11}\sin(\omega_0)),$$

and from the expression of $A_2$, we have

$$K_2 = (\frac{A_{12}}{M^2+N^2})\{M(f_{20}^1 - \frac{2A_{11}}{A_{12}}f_{11}^1 + f_{02}^1(\frac{\omega_0^2+A_{11}^2}{A_{12}^2}))\{M(\frac{\omega_0}{A_{12}}f_{11}^1 - \frac{\omega_0 A_{11}}{A_{12}^2}f_{02}^1) - N(\frac{1}{2}f_{20}^1 -$$
$$\frac{A_{11}}{A_{12}}f_{11}^1 + f_{02}^1(\frac{-\omega_0^2+A_{11}^2}{2A_{12}^2}))\} + N(f_{20}^1 - \frac{2A_{11}}{A_{12}}f_{11}^1 + f_{02}^1(\frac{\omega_0^2+A_{11}^2}{A_{12}^2}))\{M(\frac{1}{2}f_{20}^1 - \frac{A_{11}}{A_{12}}f_{11}^1 +$$
$$f_{02}^1(\frac{-\omega_0^2+A_{11}^2}{2A_{12}^2})) + N(\frac{\omega_0}{A_{12}}f_{11}^1 - \frac{\omega_0 A_{11}}{A_{12}^2}f_{02}^1)\}\}.$$

Then we have the following theorem

**Theorem 3.1.** *If $K_2 \neq 0$, then system (9) exhibits a generic Hopf bifurcation. The periodic orbits of system (9) bifurcating from the origin and $\alpha = 0$ satisfy*

$$r(t, \alpha) = \sqrt{\frac{-K_1 \alpha}{K_2}} + O(\alpha),$$

$$\theta(t, \alpha) = -\omega_0 t + O(|\alpha|^{\frac{1}{2}}),$$

*so that*
*1) if $K_1 K_2 < 0$ ($K_1 K_2 > 0$ respectively), there exists a unique nontrivial periodic orbit in the neighborhood of $r = 0$ for $\alpha > 0$ ($\alpha < 0$ respectively) and no nontrivial periodic orbits for $\alpha < 0$ ($\alpha > 0$ respectively);*
*2) the nontrivial periodic solutions in the center manifold are stable $K_2 < 0$ and unstable if $K_2 > 0$.*

# 4    Application

Although we have explicit formulas to compute the quantities $K_1$ and $K_2$, it is complicated to find the sign of $K_i$ $i = 1, 2$ for an ungiven values of the parameters, as one can see from the calculus above.
As an application, hare we complete the calculus only for $a_1 = 20$, $b = 10$, $c_1 = 5$, $k_1 = 8$, $a_2 = 15$, $c_2 = 12$ and $k_2 = 11$. Simplifying the above formulas and using Matlab 6.5, we obtain $x^* = 1.1705$, $y^* = 11.4632$, $K_1 = -0.1937$ and $K_2 = 3.5923$ and $K_1 K_2 = -0.6958$. For this rather particular situation, the Hopf bifurcation analysis is completed, since theorems 2.3 and 3.1 imply the following statement:

**Proposition 4.1.** *consider $a_1 = 20$, $b = 10$, $c_1 = 5$, $k_1 = 8$, $a_2 = 15$, $c_2 = 12$ and $k_2 = 11$, and let $\tau_0$ be defined as above. Then, for equation (9), at $\tau = \tau_0 = 0.1932$ there exists a generic supercritical Hopf bifurcation on a locally unstable two-dimensional center manifold of the positive equilibrium $E^* = (1.1705, 11.4632)$; moreover, the associated non trivial periodic solutions are unstable.*

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