

Numerical Solution of Nonlinear Volterra-Hammerstein Integral Equations Using the Hybrid of Block-pulse and Rationalized Haar Functions

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Abstract

A numerical method for finding the solution of nonlinear Volterra-Hammerstein integral equations is proposed. The properties of the hybrid functions which consists of block-pulse functions plus rationalized Haar functions are presented. The hybrid functions together with the operational matrices of integration and product are then utilized to reduce the solution of nonlinear Volterra-Hammerstein integral equations to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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1 Introduction

Orthogonal functions have also received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. The available sets of orthogonal functions can be divided into three classes. The first includes set of piecewise constant basis functions (PCBFs)(e.g., Walsh, block-pulse, Haar, etc.). The second consists of set of orthogonal polynomials (e.g., Laguerre, Legendre, Chebyshev, etc.). The third is the widely used set of sine-cosine functions in Fourier series. In these methods, a truncated orthogonal series is used for numerical integration of nonlinear Volterra-Hammerstein integral

equations, with the goal of obtaining efficient computational solutions. Typical examples are the Walsh functions [3], block-pulse functions [6], Laguerre polynomials [5], Legendre polynomials [2], Chebyshev polynomials [4], Fourier series [13], rationalized Haar series [11] and Hartley series [15]. The utilization of these series has the common objective of representing models efficiently, and calculating intermediate parameters rapidly for the given problem.

The orthogonal set of Haar functions is a group of square waves with magnitude of $+2^{\frac{i}{2}}$, $-2^{\frac{i}{2}}$, and 0, $i = 0, 1, 2, \dots$ [10]. The use of Haar functions comes from the rapid convergence feature of Haar series in expansion of functions compared with that of Walsh series [1]. Lynch and Reis [7] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral power of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipeline architecture [7]. The RH functions are composed of only three amplitudes $+1$, -1 and 0. Ohkita and Kobayashi [8,9] applied the RH functions to solve linear ordinary differential equations [8] and linear first and second order partial differential equations [8]. In [8], the solution $y(t)$, $0 \leq t \leq T$ for a linear second order ordinary differential equation is obtained at the points $t_k = \frac{kT}{N}$, $k = 0, 1, \dots, N - 1$, where N is an integral value given by a power of two. Further, Razzaghi and Ordokhani [12], used RH functions to solve nonlinear Volterra-Hammerstein integral equations. In [12], the number of RH functions in approximate solution is given by $k = 2^{\alpha+1}$, $\alpha = 0, 1, 2, \dots$, hence for getting a high accuracy a large number of basis functions should be used. However, this is not economical because of the requirement of large computer memory.

In the present work we introduce an alternative computational method for the solution of nonlinear Volterra-Hammerstein integral equations. This method consists of reducing the solution of problem to a set of algebraic equations by expanding hybrid functions with unknown coefficients. These hybrid functions, which consist of block-pulse functions and RH functions are introduced. The operational matrices of integration and product are calculated and then utilized to evaluate the unknown coefficients.

The paper is organized as follows : Section 2 is devoted to the basic formulation of the hybrid functions of block-pulse and RH functions required for our subsequent development, the operational matrices of integration and product are also derived. In Section 3 we apply the proposed numerical method to the numerical solution of nonlinear Volterra-Hammerstein integral equations, and in Section 4 we report our numerical finding and demonstrate the accuracy of the proposed method.

2 Properties of Hybrid Functions

2.1 Hybrid of block-pulse and rationalized Haar functions

Hybrid functions $\phi_{nr}(t), n = 1, 2, \dots, N, r = 0, 1, \dots, k - 1, k = 2^{\alpha+1}, \alpha = 0, 1, 2, \dots$ have three arguments, n and r are the order for block-pulse and rationalized Haar functions respectively and t is the normalized time. They are defined on the interval $[0, 1)$ as

$$\phi_{nr}(t) = \begin{cases} \phi_r(Nt + 1 - n), & \frac{n-1}{N} \leq t < \frac{n}{N} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Here, $\phi_r(t) = RH(r, t)$ are the rationalized Haar functions of order r which are orthogonal in the interval $[0, 1)$ and satisfy [8,9,12]:

$$RH(r, t) = \begin{cases} 1, & J_1 \leq t < J_{1/2} \\ -1, & J_{1/2} \leq t < J_0 \\ 0, & \text{otherwise} \end{cases}$$

where

$$J_u = \frac{j - u}{2^i}, \quad u = 0, \frac{1}{2}, 1.$$

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0, 1, 2, 3, \dots, \quad j = 1, 2, 3, \dots, 2^i.$$

$RH(0, t)$ is defined for $i = j = 0$ and is given by

$$RH(0, t) = 1, \quad 0 \leq t < 1.$$

Since $\phi_{nr}(t)$ is the combination of rationalized Haar functions and block-pulse functions which are both complete and orthogonal, thus the set of hybrid functions are complete orthogonal set. The orthogonality property is given by

$$\int_0^1 \phi_{nr}(t)\phi_{n'r'}(t)dt = \begin{cases} \frac{2^{-i}}{N}, & n = n', r = r' \\ 0, & \text{otherwise} \end{cases}$$

where

$$r = 2^i + j + 1, \quad r' = 2^{i'} + j' + 1, \quad i' = 0, 1, 2, 3, \dots, \quad j' = 1, 2, 3, \dots, 2^{i'}.$$

2.2 Function approximation

A function $f(t)$ defined over $[0, 1)$ may be expanded in hybrid functions as

$$f(t) = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} a_{nr} \phi_{nr}(t), \quad (2)$$

where a_{nr} are given by

$$a_{nr} = 2^i N \int_0^1 f(t) \phi_{nr}(t) dt.$$

The series in Eq. (2) contains an infinite number of terms. If we let $i = 0, 1, 2, \dots, \alpha$ then the infinite series in Eq. (2) is truncated up to its first Nk terms as

$$f(t) \simeq \sum_{r=0}^{k-1} \sum_{n=1}^N a_{nr} \phi_{nr}(t) = A^T B(t),$$

where

$$k = 2^{\alpha+1}, \quad \alpha = 0, 1, 2, \dots,$$

$$A = [a_{10}, a_{11}, \dots, a_{1k-1}, a_{20}, a_{21}, \dots, a_{2k-1}, \dots, a_{N0}, a_{N1}, \dots, a_{Nk-1}]^T, \quad (3)$$

$$B(t) = [\phi_{10}(t), \phi_{11}(t), \dots, \phi_{1k-1}(t), \phi_{20}(t), \phi_{21}(t), \dots, \phi_{2k-1}(t), \dots, \phi_{N0}(t), \phi_{N1}(t), \dots, \phi_{Nk-1}(t)]^T. \quad (4)$$

Also, the integration of the cross-product of two vector $B(t)$ in Eq. (4) is

$$W = \int_0^1 B(t) B^T(t) dt = \frac{1}{N} \text{diag.}(D, D, \dots, D), \quad (5)$$

where W is the $Nk \times Nk$ matrix and

$$D = \frac{1}{N} \text{diag.}(1, 1, \underbrace{\frac{1}{2}, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^2}}_{2^2}, \dots, \underbrace{\frac{1}{2^\alpha}, \frac{1}{2^\alpha}, \dots, \frac{1}{2^\alpha}}_{2^\alpha}).$$

Now, let $\kappa(t, s)$ be a function of two independent variable defined for $t \in [0, 1)$ and $s \in [0, 1)$. Then κ can be expanded into hybrid functions as

$$\kappa(t, s) \simeq B^T(t) K B(s), \quad (6)$$

where K is the $Nk \times Nk$ matrix and

$$K = (k_{nr}), \quad k_{nr} = 2^{2i} N^2 \int_0^1 \int_0^1 \kappa(t, s) \phi_{nr}(t, s) ds dt, \quad (7)$$

$$n = 1, 2, \dots, N, \quad r = 0, 1, 2, \dots, k - 1.$$

2.3 Operational matrix of integration

The integration of the vector $B(t)$ defined in Eq. (4) is given by

$$\int_0^t B(t')dt' \simeq PB(t), \tag{8}$$

where P is the $Nk \times Nk$ operational matrix for integration and is given by

$$P = \frac{1}{N} \begin{bmatrix} \hat{P} & H & H & \dots & H & H \\ O & \hat{P} & H & \dots & H & H \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ O & O & O & \dots & \hat{P} & H \\ O & O & O & \dots & O & \hat{P} \end{bmatrix}.$$

Also H is the $k \times k$ matrix represented by

$$H = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and \hat{P} is the $k \times k$ operational matrix for rationalized Haar functions and is given in [11,12] as

$$\hat{P} = \hat{P}_{k \times k} = \frac{1}{2k} \begin{bmatrix} 2k\hat{P}_{\frac{k}{2} \times \frac{k}{2}} & -\hat{\Phi}_{\frac{k}{2} \times \frac{k}{2}} \\ \hat{\Phi}_{\frac{k}{2} \times \frac{k}{2}}^{-1} & 0 \end{bmatrix},$$

where $\hat{P}_{1 \times 1} = [\frac{1}{2}]$, $\hat{\Phi}_{1 \times 1} = [1]$ and $\hat{\Phi}_{k \times k}$ is given by

$$\hat{\Phi}_{k \times k} = [\Phi(\frac{1}{2k}), \Phi(\frac{3}{2k}), \dots, \Phi(\frac{2k-1}{2k})],$$

with

$$\Phi(t) = [\phi_0(t), \phi_1(t), \dots, \phi_{k-1}(t)]^T.$$

2.4 The product operational matrix

Let

$$\Omega(t) = B(t)B^T(t), \tag{9}$$

where $\Omega(t)$ is $Nk \times Nk$ matrix. Using Eq. (1) we get

$$\phi_{nr}(t)\phi_{n'r'}(t) = \begin{cases} 0, & n \neq n' \\ \phi_r(t)\phi_{r'}(t), & n = n'. \end{cases}$$

From [12], we have

$$\Omega_{(Nk) \times (Nk)}(t) = \text{diag.}(\Psi_{k \times k}^{(1)}, \Psi_{k \times k}^{(2)}, \dots, \Psi_{k \times k}^{(N)}), \tag{10}$$

where

$$\Psi_{k \times k}^{(n)} = \begin{bmatrix} \phi_{n0}(t) \\ \phi_{n1}(t) \\ \vdots \\ \phi_{n(k-1)}(t) \end{bmatrix} \left[\phi_{n0}(t), \phi_{n1}(t), \dots, \phi_{n(k-1)}(t) \right], \quad n = 1, 2, 3, \dots, N.$$

For $N = 2$ and $k = 4$ we have

$$\Omega_{8 \times 8} = \begin{bmatrix} \phi_{10} & \phi_{11} & \phi_{12} & \phi_{13} & 0 & 0 & 0 & 0 \\ \phi_{11} & \phi_{10} & \phi_{12} & -\phi_{13} & 0 & 0 & 0 & 0 \\ \phi_{12} & \phi_{12} & \frac{\phi_{10} + \phi_{11}}{2} & 0 & 0 & 0 & 0 & 0 \\ \phi_{13} & -\phi_{13} & 0 & \frac{\phi_{10} - \phi_{11}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{20} & \phi_{21} & \phi_{22} & \phi_{23} \\ 0 & 0 & 0 & 0 & \phi_{21} & \phi_{20} & \phi_{22} & -\phi_{23} \\ 0 & 0 & 0 & 0 & \phi_{22} & \phi_{22} & \frac{\phi_{20} + \phi_{21}}{2} & 0 \\ 0 & 0 & 0 & 0 & \phi_{23} & -\phi_{23} & 0 & \frac{\phi_{20} - \phi_{21}}{2} \end{bmatrix},$$

where ϕ_{ij} , $i = 1, 2$, $j = 0, 1, 2, 3$ are functions of t .

Furthermore, by multiplying the matrix $\Omega(t)$ in Eq. (9) by the vector A in Eq. (3) we obtain

$$\Omega(t)A = \tilde{A}B(t), \tag{11}$$

where \tilde{A} is $Nk \times Nk$ matrix and is given by

$$\tilde{A} = \text{diag.}(\tilde{A}_{k \times k}^{(1)}, \tilde{A}_{k \times k}^{(2)}, \dots, \tilde{A}_{k \times k}^{(N)}),$$

with

$$\tilde{A}_{k \times k}^{(n)} = \begin{bmatrix} \tilde{A}_{\frac{k}{2} \times \frac{k}{2}}^{(n)} & \tilde{H}_{\frac{k}{2} \times \frac{k}{2}}^{(n)} \\ \tilde{A}_{\frac{k}{2} \times \frac{k}{2}}^{(n)} & \tilde{D}_{\frac{k}{2} \times \frac{k}{2}}^{(n)} \end{bmatrix}, \quad n = 1, 2, 3, \dots, N$$

where

$$\begin{aligned} \tilde{A}_{1 \times 1}^{(n)} &= [a_{n0}], \quad n = 1, 2, \dots, N, \\ \tilde{H}_{\frac{k}{2} \times \frac{k}{2}}^{(n)} &= \hat{\Phi}_{\frac{k}{2} \times \frac{k}{2}} \text{diag.} [a_{n(k/2)}, a_{n(k/2+1)}, \dots, a_{n(k-1)}], \end{aligned}$$

$$\hat{H}_{\frac{k}{2} \times \frac{k}{2}}^{(n)} = \text{diag} \cdot [a_{n(k/2)}, a_{n(k/2+1)}, \dots, a_{n(k-1)}] \hat{\Phi}_{\frac{k}{2} \times \frac{k}{2}}^{-1},$$

$$\tilde{D}_{\frac{k}{2} \times \frac{k}{2}}^{(n)} = \text{diag} \cdot [[a_{n0}, a_{n1}, \dots, a_{n(k/2-1)}] \hat{\Phi}_{\frac{k}{2} \times \frac{k}{2}}].$$

For $N = 2$ and $k = 4$ we have

$$\tilde{A}_{8 \times 8} = \begin{bmatrix} a_{10} & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 \\ a_{11} & a_{10} & a_{12} & -a_{13} & 0 & 0 & 0 & 0 \\ \frac{a_{12}}{2} & \frac{a_{12}}{2} & a_{10} + a_{11} & 0 & 0 & 0 & 0 & 0 \\ \frac{a_{13}}{2} & -\frac{a_{13}}{2} & 0 & a_{10} - a_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{20} & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & 0 & a_{21} & a_{20} & a_{22} & -a_{23} \\ 0 & 0 & 0 & 0 & \frac{a_{22}}{2} & \frac{a_{22}}{2} & a_{20} + a_{21} & 0 \\ 0 & 0 & 0 & 0 & \frac{a_{23}}{2} & -\frac{a_{23}}{2} & 0 & a_{20} - a_{21} \end{bmatrix}.$$

3 Solution of Nonlinear Volterra-Hammerstein Integral Equations

Consider the following nonlinear Volterra-Hammerstein integral equations

$$y(t) = f(t) + \int_0^t \kappa(t, s)g(s, y(s))ds, \quad 0 \leq t \leq 1 \tag{12}$$

where f, g and κ are given continuous functions, which $g(s, y(s))$ nonlinear in $y(s)$. We assume that Eq. (12) has a unique solution y to be determined. To solve for $y(t)$, we approximate the solution not to Eq. (12), but rather to an equivalent equation:

$$z(t) = g(t, y(t)), \quad 0 \leq t \leq 1. \tag{13}$$

From Eq. (12) we get

$$z(t) = g(t, f(t) + \int_0^t \kappa(t, s)z(s)ds). \tag{14}$$

Suppose $z(t)$ can be expressed approximately as

$$z(t) = A^T B(t). \tag{15}$$

Using Eqs. (6),(11) and (15) we get

$$\int_0^t \kappa(t, s)g(s, y(s))ds = \int_0^t B^T(t)K \tilde{A}B(s)ds.$$

From Eqs. (8) and (14) we get

$$z(t) = g(t, f(t) + B^T(t)K \tilde{A}PB(t)). \tag{16}$$

In order to construct the approximations for $z(t)$ we collocate Eq. (16) in Nk points. For a suitable collocation points we choose Newton-Cotes nodes as

$$t_p = \frac{2p-1}{2Nk}, \quad p = 1, 2, \dots, Nk. \quad (17)$$

Equation (16) can be expressed as

$$z(t_p) = g(t_p, f(t_p) + B^T(t_p)K\tilde{A}PB(t_p)), \quad p = 1, 2, \dots, Nk. \quad (18)$$

Equation (18) can be solved for the unknowns $a_{nr}, n = 1, 2, \dots, N, r = 0, 1, \dots, k-1$. The required approximations to the solution $y(t)$ in Eq. (12) are obtained as

$$y(t) = f(t) + \int_0^t \kappa(t, s)z(s)ds, \quad 0 \leq t \leq 1.$$

Using Eqs. (6), (8) and (11) we get

$$y(t) = f(t) + B^T(t)K\tilde{A}PB(t). \quad (19)$$

4 Illustrative Examples

We applied the method presented in this article to 2 examples given by Razzaghi and Ordokhani [12] with $N = 2$ and $k = 8$. These examples were solved by Razzaghi and Ordokhani [12] using rationalized Haar method with $k = 16$ and their result are provide for comparison.

4.1 Example 1

Consider the nonlinear Volterra-Hammerstein integral equation

$$y(t) = f(t) + \int_0^t \kappa(t, s)y^2(s)ds, \quad 0 \leq t \leq 1 \quad (20)$$

where $\kappa(t, s) = ts + 1$, and

$$f(t) = -\frac{1}{4}t^5 - \frac{2}{3}t^4 - \frac{5}{6}t^3 - t^2 + 1$$

The exact solution [12] is $y(t) = t + 1$. By using Eq. (18) the solution in Eq. (20) is calculated. Table (1) represents the approximate solution using the method in [12] with $k = 16$ together with the results obtained using the present method with $N = 2$ and $k = 8$ and exact solution.

t	Method in [12] with $k = 16$	Present method with $N = 2$ and $k = 8$	Exact
0.0	1.005	1.0001	1
0.2	1.204	1.2003	1.2
0.4	1.400	1.4001	1.4
0.6	1.599	1.5998	1.6
0.8	1.801	1.8002	1.8
1	1.999	1.9999	2

Table 1. Approximate and exact solutions for Example 1.

4.2 Example 2

Consider the equations

$$y(t) = 1 + \sin^2(t) - \int_0^t 3\sin(t-s)y^2(s)ds. \quad 0 \leq t \leq 1, \quad (21)$$

We solve Eq. (21) using the method in section 3. The computational result for $N = 2, k = 8$ using the present method together with the rationalized Haar method [12] for $k = 16$ and the exact solution $y(t) = \cos t$.

t	Method in [12] with $k = 16$	Present method with $N = 2$ and $k = 8$	Exact
0.0	1.0000	1.0000	1
0.2	0.9800	0.9801	0.9801
0.4	0.9210	0.9211	0.9211
0.6	0.8255	0.8253	0.8253
0.8	0.6969	0.6967	0.6967
1	0.5405	0.5403	0.5403

Table 2. Approximate and exact solutions for Example 2.

5 Conclusion

The hybrid of block-pulse and rationalized Haar functions and the associated operational matrices of integration P and product \tilde{A} are applied to solve the nonlinear Volterra-Hammerstein integral equations. The method is based upon

reducing the system into a set of algebraic equations. The matrices P and \tilde{A} have many zeros; hence is much faster than rationalized Haar functions and reduces the CPU time and the computer memory, at the same time keeping the accuracy of the solution. The numerical examples support this claim.

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References

- [1] K. G. Beauchamp, *Walsh functions and their applications*, 1975.
- [2] R. Y. Chang and M. L. Wang, Shifted Legendre direct method for variational problems series, *J. Optimazation Theory Appl.*, **39** (1983), 299-307.
- [3] C. F. Chen and C. H. Hasio, A Walsh series direct method for solving variational problems, *Journal of the Franklin Institute*, **300** (1975), 265-280.
- [4] I. R. Horng and J. H. Chou, Shifted Chebyshev direct method for solving variational problems, *International Journal of Systems Science*, **16** (1985), 855-861.
- [5] C. Hwang and Y. P. Shih, Laguerre series direct method for variational problems, *Journal of Optimization Theory and Applications*, **39** (1983), 143-149.
- [6] C. Hwang and Y.P. Shih, Optimal control of delay systems via block pulse functions, *Journal of Optimization Theory and Applications*, **45** (1985), 101-112.
- [7] R. T. Lynch and J. J. Reis, Haar transform image coding, *Proc. National Telecommun. Conf. ,Dallas,TX* (1976), 44.3-1-44.3.
- [8] M. Ohkita and Y. Kobayashi, An application of rationalized Haar functions to solution of linear differential equations, *IEEE Trans, on Circuit and systems*, **9** (1986), 853-862.
- [9] M. Ohkita and Y. Kobayashi, An application of rationalized Haar functions to solution of linear partial differential equations, *Math. Comput. in Simulations*, **30** (1988), 419-428.
- [10] M. Razzaghi and J. Nazarzadeh, Walsh functions, *Wiley Encyclopedia of Electrical and Electronics Engineering*, **23** (1999), 429-440.

- [11] M. Razzaghi and Y. Ordokhani, Solution for a classical problems in the calculus of variations via rationalized Haar functions, *Kybernetika*, **37(5)** (2001), 575-583.
- [12] M. Razzaghi and Y. Ordokhani, Solution of nonlinear Volterra-Hammerstein integral equations via rationalized Haar functions, *Mathematical Problems in Engineering*, **7** (2001), 205-219.
- [13] M. Razzaghi and M. Razzaghi, Fourier series direct method for variational problems, *International Journal of Control*, **48** (1988), 887-895.
- [14] J. J. Reis, R. T. Lynch and J. Butman, Adaptive Haar transform video bandwidth reduction system for RPV's, *Proc. Ann. Meeting Soc. Photo Optic Inst. Eng.(SPIE), San Diego, CA* (1976), 24-35.
- [15] J. J. Ricot and E. Acha, Analysis of linear time-varying systems via Hartley series, *International Journal of Systems Science*, **29** (1998), 541-549.

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