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Solution of the Topology Optimization Problem Based Subdomains Method

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Abstract

In topology optimization problems, we are often forced to deal with large-scale numerical problems, so that the domain decomposition method occurs naturally. Consider a typical topology optimization problem, the minimum compliance problem of a linear isotropic elastic continuum structure, in which the constraints are the partial differential equations of linear elasticity. We subdivide the Partial differential equations into two subproblems posed on non-overlapping subdomains, each of which has boundary data that depends on the solution of the other subproblem. In this paper we present a new formulation of the minimum compliance problem based on the domain decomposition methods, and then we prove the equivalence of the two problems..

Mathematics Subject Classification: 74P, 65M55

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1 Introduction

The topology optimization has for objective to find an optimal shape without any a priori assumption about its topology, i.e., on the nature and the connectivity of elements which constitute it. Mathematically, the topology optimization problem takes the form:

$$\min_{\omega \subset \Omega} f(u(\omega), \omega) \tag{1}$$

s.t:

$$\begin{cases} g_i(\omega) \leq 0 & 1 \leq i \leq m \\ h_j(\omega) = 0 & 1 \leq j \leq n \end{cases}$$

f is the objective function, g_i and h_j are the functions defining the constraints, in practice they are implicit and nonlinear functions in ω. Their evaluation then requires the resolution of a state equation and the topology optimization problem (1) is reformulated as follows:

$$\min_{\omega \subset \Omega} f(u(\omega), \omega) \tag{2}$$

s.t:

$$\begin{cases} g_i(u(\omega), \omega) \leq 0 & 1 \leq i \leq m \\ h_j(u(\omega), \omega) = 0 & 1 \leq j \leq n \end{cases}$$

where u is the solution of the state equation $L(u(\omega), \omega) = 0$.

One can find various methods of topology optimization in the literature for solving the problem (2), methods based on the shape gradient, evolutionary methods [7, 12], and methods which employ a material distribution approach for a fixed reference domain, especially the homogenization methods [2, 20, 1], and the fictitious or power-law materials also called SIMP (Solid Isotropic Material with Penalization) method which has seen widespread academic use and has proven very popular and extremely tempting to solve practical applications [3, 4].

In spite of its effectiveness in structural design, topology optimization is not yet largely widespread in the industry, the principal reason is that the topology optimization problem is a large scale optimization problem; it is characterized by a very significant number of design variables, which amplifies the difficulty of its resolution. It is common to introduce 1000 to 10000 design variables to solve a real problem, thus the computation time is typically very high since the problem requires repeated solution of finite element analysis of the equilibrium equations.

However, during two last decades, the parallel computers knew a great evolution, in particular in computing power and storage capacity [16]. Domain decomposition methods (called also subdomains methods) are a valuable approach when solving partial differential equation (PDE) problems on parallel computers [10, 15].

Any domain decomposition method is based on the assumption that the given computational domain is partitioned into subdomains which may or may not overlap. Next, the original problem can be reformulated upon each subdomain, yielding a family of subproblems of reduced size, that are coupled one to another through the values of the unknown solution at sub-domain interfaces.

Reviewing the literature, it seems that the application of parallel computing in topology optimization is rare, and devoted only to the discrete case [21, 13, 5], there is no mathematical formulation of the topology optimization problem in the continuum case. Thus, The main objective of the present work is to propose a new mathematical formulation of the minimum compliance problem of an isotropic linear elastic structure based on domain decomposition methods when the design domain is partitioned into two non-overlapping subdomains, the domain decomposition method for the problem of linear elasticity is then based on a constrained minimization problem for which the objective functional measures the jumps in the solution across the interface between subdomains, the constraints are the partial differential equations.

The remainder of this paper is organized as follows. Section 2 describes the formulation of the topology optimization problem which ensures at least the existence of the solution in a simple case of linear elasticity. In section 3, the equivalent formulation is given when the design domain is partitioned into two non-overlapping sub-domains, then we propose an algorithm of resolution of the finding optimality system.

2 Preliminary Notes

The topology optimization problem is a nonlinear optimization problem, often non convex, the objective function depends on a state variable describing the operational mode and the design variables determine the shape and topology, the state variable must satisfy a boundary value problem, here we deals with a typical problem of topology optimization which consists in minimizing the compliance of an isotropic linear elastic structure (see figure 1).

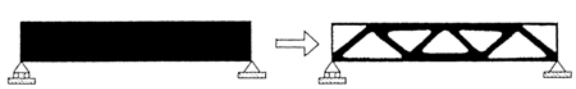

Figure 1: Topology optimization of the MBB-beam

Consider an elastic body in the configuration region $\Omega \subset \mathbb{R}^d$,$(d = 2, 3)$ with boundary Γ. The problem of linear elasticity is given as follows:

$$\text{Find } u : \Omega \to \mathbb{R}^d \text{ such that:}$$
$$\begin{cases} (-div\sigma(u))_i = f_i & \text{in} \quad \Omega, i = 1, ..., d, \\ u = \varphi_D & \text{on} \quad \Gamma_D \\ \displaystyle\sum_{j=1}^{d} \sigma_{ij}(u)\, n_j = (\varphi_N)_i & \text{on} \quad \Gamma_N \quad i = 1, ..., d. \end{cases} \tag{3}$$

where n denotes the unit outward normal vector on Γ. f is the vector of volume forces acting on the body, φ_D is the given displacement on the portion of the domain boundary Γ_D, while φ_N are the tractions applied on the complementary part Γ_N, and $\sigma = (\sigma_{ij})_{1\le i,j\le d}$ the stress tensor.

Take, for simplicity, $\Gamma_N = \emptyset$ and $\varphi_D = 0$, thus the system of equations of linear elasticity (3) becomes the following Dirichlet boundary value problem:

$$\text{Find } u : \Omega \to \mathbb{R}^d \text{ such that:}$$
$$\begin{cases} -2\mu \sum\limits_{j=1}^{d} \frac{\partial}{\partial x_j}\varepsilon_{ij}(u) - \lambda \frac{\partial}{\partial x_i} div(u) = f_i & \text{in } \Omega \quad 1 \le i \le d \\ u_i = 0 & \text{on } \Gamma \quad 1 \le i \le d \end{cases} \tag{4}$$

where $\mu > 0$, $\lambda \ge 0$ are the Lamé's constants and $\varepsilon = (\varepsilon_{ij})_{1\le i,j\le d}$ is the strain tensor given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The variational formulation of (4) reads:

$$\text{Find } u \in H_0^1(\Omega)^d \text{ such that :} \quad a(u, v) = l(v) \quad \forall v \in H_0^1(\Omega)^d \tag{5}$$

where

$$a(u, v) = 2\mu \sum_{i,j=1}^{d} \int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(v)d\Omega + \lambda \int_\Omega div(u)div(v)d\Omega,$$

and

$$l(v) = \int_\Omega f v d\Omega.$$

The Korn's inequality states that there exists a constant $C_\Omega > 0$ such that [8]:

$$\sum_{i,j=1}^{d} \int_\Omega (\varepsilon_{ij}(v))^2 d\Omega \ge C_\Omega \|v\|^2_{H_0^1(\Omega)^d} \qquad \forall v \in H_0^1(\Omega)^d \tag{6}$$

Hence, the form a is coercive, furthermore it is easily seen that a (resp. l) is bilinear and continuous in $H_0^1(\Omega)^d$ (respectively linear and continuous in

$H_0^1(\Omega)^d$), when the problem (5) admits a unique solution $u \in H_0^1(\Omega)^d$ by a straightforward application of the Lax-Milgram theorem [6]. In orthonormal base, we have [17]:

$$a(u,v) = \int_\Omega E_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)d\Omega$$

The medium supposed non homogeneous thus we uses $E_{ijkl}(x)$ instead of E_{ijkl} for each $x \in \Omega$. Consequently the minimum compliance (maximum global stiffness) problem takes the following form in the SIMP approach [4]:

$$\begin{cases} \min\limits_\rho l(u) & \\ a_\rho(u,v) = l(v) & \forall v \in U \subseteq H_0^1(\Omega)^d \\ E_{ijkl}(x) = \rho^p(x)E^0_{ijkl} & \end{cases} \tag{7}$$

with the following constraints on ρ:

$$\int_\Omega \rho(x)d\Omega \leq V, \quad 0 < \rho_{\min} < \rho(x) \leq 1, \quad \forall x \in \Omega$$

U is the set of admissible displacements, V is a limit on the amount of material at our disposal, E^0_{ijkl} represents the material properties of a given isotropic material, ρ which is interpreted as a density of material is the design variable and p is the penalty factor which penalizes intermediate densities in order to end up with (nearly) 'solid and void' distributions. Normally, one writes $E_{ijkl} \in L^\infty(\Omega)$ to indicate the relevant functional space for our problem, unfortunately, in this case, the problem (7) lacks existence of solutions in its general continuum setting. To ensure existence of solutions, the power-law approach must be combined with a perimeter constraint, a gradient constraint or with filtering techniques [18]. Here we use a gradient constraint by which we mean the norm of the function ρ in the Sobolev space $H^1(\Omega)$, see [4]:

$$\|\rho\|_{H^1(\Omega)} = \left[\int_\Omega \left(\rho^2 + \|\nabla\rho\|^2\right) d\Omega\right]^{\frac{1}{2}} \leq M \quad \text{where} \quad 1 < p < \frac{d}{d-2} \quad \left(\Omega \subset \mathbb{R}^d\right)$$

where

$$\|\nabla\rho\|^2 = \sum_i \left(\frac{\partial\rho}{\partial x_i}\right)^2.$$

Bendsøe has proved existence of solutions when including this bound in the minimum compliance problem [4]. Thus, we will choose the new formulation due to Bendsøe:

$$\begin{cases} \min\limits_{u,\rho\in H^1(\Omega)} l(u) \\ a_\rho(u,v) = l(v) \quad \forall v \in U \end{cases} \tag{8}$$

with the following constraints on ρ:

$$\int_{\Omega} \rho(x) d\Omega \leq V, \quad 0 < \rho_{min} < \rho(x) \leq 1, \quad \forall x \in \Omega$$

and the gradient constraint

$$\|\rho\|_{H^1(\Omega)} = \left[\int_{\Omega} (\rho^2 + (\nabla\rho)^2) d\Omega \right]^{1/2} \leq M$$

where:

$$a_{\rho}(u, v) = \int_{\Omega} \rho^p(x) E^0_{ijkl} \epsilon_{ij}(u) \epsilon_{kl}(v) d\Omega.$$

3 Main Results

Let Ω be a bounded domain in $\mathbb{R}^d$ where $d = 2, 3$ with Lipschitz boundary Γ. Further, we suppose that Ω is partitioned into two non-overlapping subdomains Ω_1 and Ω_2 with interface Γ_0 i.e. $\Gamma_0 = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Let $\Gamma_i = \overline{\Omega}_i \bigcap \Gamma \quad i = 1, 2$ (see figure 2).

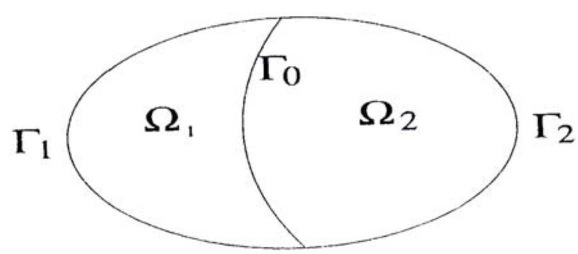

Figure 2: Decomposition of Ω

The problem of linear elasticity (5) can be written as:
Find $u_i \in H^1_{\Gamma_i}(\Omega_i)^d, \ i = 1, 2;$ such that:

$$a_{\rho_1}(u_1, v_1) = (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \tag{9}$$

$$a_{\rho_2}(u_2, v_2) = (f, v_2)_{\Omega_2} - (g, v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d \tag{10}$$

$$u_1 = u_2 \quad \text{on } \Gamma_0. \tag{11}$$

where

$$g_l = \sum_{j=1}^{d} \sigma_{lj}(u_1) n^1_j = -\sum_{j=1}^{d} \sigma_{lj}(u_2) n^2_j$$

and

$$g = (g_l)_{1 \leq l \leq d} \quad \text{with} \quad (g, v)_{\Gamma_0} = \int_{\Gamma_0} g v d\Gamma_0$$

The existence and uniqueness of the solution to both problems (9) and (10) is a straightforward consequence of the Lax-Milgram Theorem. In fact, for the problem (9), the bilinear form a_{ρ_1} is continuous in $H^1_{\Gamma_1}(\Omega_1)^d$, the coerciveness follows from the Korn's inequality (6) (see also [14]), in addition, let us define:

$$l_1(v_1) = \int_{\Omega_1} f v_1 d\Omega_1 + \int_{\Gamma_0} g v_1 d\Gamma_0 \qquad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d.$$

It is clear that l_1 is a continuous linear form in $H^1_{\Gamma_1}(\Omega_1)^d$ therefore the problem (9) has a unique solution $u_1 \in H^1_{\Gamma_1}(\Omega_1)^d$, the same argument for the problem (10). In addition, as a_{ρ_1} is coercive then

$$\exists K_1 > 0 \,/\, \|u_1\|^2_{H^1_{\Gamma_1}(\Omega_1)^d} \leq K_1 a_{\rho_1}(u_1, u_1)$$

i.e.:

$$\begin{aligned} \|u_1\|^2_{H^1_{\Gamma_1}(\Omega_1)^d} &\leq K_1 \left(\int_{\Omega_1} f u_1 d\Omega_1 + \int_{\Gamma_0} g u_1 d\Gamma_0 \right) \\ &\leq K_1 \left(\|f\|_{L^2(\Omega_1)^d} \|u_1\|_{H^1(\Omega_1)^d} + \|g\|_{L^2(\Gamma_0)^d} \|u_1 \mid \Gamma_0\|_{L^2(\Gamma_0)^d} \right) \end{aligned}$$

where $u_1 \mid \Gamma_0$ denotes the trace of u_1 on Γ_0, and according to the trace inequality

$$\exists K_2 > 0 \,/\, \|u_1 \mid \Gamma_0\|_{L^2(\Gamma_0)^d} \leq K_2 \|u_1\|_{H^1_{\Gamma_1}(\Omega_1)^d}$$

Consequently,

$$\exists C_1 > 0 \,/\, \|u_1\|_{H^1_{\Gamma_1}(\Omega_1)^d} \leq C_1 \left(\|f\|_{L^2(\Omega_1)^d} + \|g\|_{L^2(\Gamma_0)^d} \right),$$

the same argument for u_2, hence

$$\exists C > 0 \,/\, \|u_i\|_{H^1_{\Gamma_i}(\Omega_i)^d} \leq C \left(\|f\|_{L^2(\Omega_i)^d} + \|g\|_{L^2(\Gamma_0)^d} \right) \qquad i = 1, 2. \tag{12}$$

For an arbitrary choice for the control g, the solutions u_1 and u_2 of the problem (9) and the problem (10), respectively, do not agree with the solution u of (5) in the respective sub-domains, i.e., $u_1 \neq u \mid \Omega_1$ and $u_2 \neq u \mid \Omega_2$. The discrepancy is due to the fact that for an arbitrary choice of g, we have that $u_1 \neq u_2$ along Γ_0, even in a weak sense. In addition, there exists clearly a choice of g, namely such that the solutions of the problems (9) and (10) coincide with the solution of (5) on the corresponding subdomains. Thus, we consider a functional that measures the jumps of solutions across the interface between subdomains with a penalty term to regularize the problem, put:

$$\mathcal{J}_\delta(u_1, u_2, g) = \frac{1}{2} \int_{\Gamma_0} (u_1 - u_2)^2 d\Gamma_0 + \frac{\delta}{2} \int_{\Gamma_0} g^2 d\Gamma_0.$$

Let $(\rho_1, \rho_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$ then we consider the following optimization problem:

$$\min_g \mathcal{J}_\delta(u_1, u_2, g) \tag{13}$$

$$a_{\rho_1}(u_1, v_1) = (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \tag{14}$$

$$a_{\rho_2}(u_2, v_2) = (f, v_2)_{\Omega_2} - (g, v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma 2}(\Omega_2)^d \tag{15}$$

Let the admissibility set be defined by:

$$\mathcal{U}_{ad} = \left\{ \begin{array}{c} (u_1, u_2, g) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma 2}(\Omega_2)^d \times L^2(\Gamma_0)^d / \\ (9) \text{ and } (10) \text{ are satisfied and } \mathcal{J}_\delta(u_1, u_2, g) < \infty \end{array} \right\} \tag{16}$$

we have the following result:

Theorem 3.1 *The problem (13)-(15) has a unique optimal solution.*

Proof. Let $\rho \in H^1(\Omega)$, it was seen that the problem (5) admits a unique solution $u \in H^1_0(\Omega)^d$ while $u_i = u_{/\Omega_i}$, $\rho_i = \rho_{/\Omega_i}$ and $g_l = \sum_{j=1}^{d} \sigma_{lj}(u) n_j$, we have $(u_1, u_2, g) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma 2}(\Omega_2)^d \times L^2(\Gamma_0)^d$ satisfying (9) and (10), hence $(u_1, u_2, g) \in \mathcal{U}_{ad}$ i.e. $\mathcal{U}_{ad} \neq \emptyset$. Let then $\left\{ \left(u_1^{(n)}, u_2^{(n)}, g^{(n)} \right) \right\}$ be a minimizing sequence in $\mathcal{U}_{ad}$. Then, from (16), we have that the sequence $\left\{ g^{(n)} \right\}$ is uniformly bounded in $L^2(\Gamma_0)^d$. And, by (12), $(u_1^{(n)})_n$ and $(u_2^{(n)})_n$ are uniformly bounded. Consequently, there exists a subsequence $\left\{ \left(u_1^{(n_i)}, u_2^{(n_i)}, g^{(n_i)} \right) \right\}$ such that:

$$u_1^{(n_i)} \to \hat{u}_1 \ \text{ in } \ H^1_{\Gamma_1}(\Omega_1)^d$$

$$u_2^{(n_i)} \to \hat{u}_2 \ \text{ in } \ H^1_{\Gamma_2}(\Omega_2)^d$$

$$g^{(n_i)} \to \hat{g} \ \text{ in } \ L^2(\Gamma_0)^d$$

By the process of passing to the limit, we have that $(\hat{u}_1, \hat{u}_2, \hat{g})$ satisfies (9) and (10) therefore $(\hat{u}_1, \hat{u}_2, \hat{g}) \in \mathcal{U}_{ad}$. Also, the fact that the functional $\mathcal{J}_\delta(.,.,.)$ is lower semi-continuous implies that

$$\inf_{(u_1,u_2,g)\in\mathcal{U}_{ad}} \mathcal{J}_\delta(u_1, u_2, g) = \lim_{n_i\to\infty} \inf \mathcal{J}_\delta(u_1^{(n_i)}, u_2^{(n_i)}, g^{(n_i)}) \geq \mathcal{J}_\delta(\hat{u}_1, \hat{u}_2, \hat{g})$$

We conclude $\mathcal{J}_\delta(\hat{u}_1, \hat{u}_2, \hat{g}) = \inf \mathcal{J}_\delta(u_1, u_2, g)$ then $(\hat{u}_1, \hat{u}_2, \hat{g})$ is an optimal solution. Uniqueness follows from the convexity of the functional $\mathcal{J}_\delta$, $\mathcal{U}_{ad}$ and the linearity of the constraints [9]. ■

Theorem 3.2 *For each $\delta > 0$, let $(u_1^\delta, u_2^\delta, g^\delta)$ denotes the optimal solution of the problem (13)-(15). If $\hat{u}$ is the solution of (5), putting $\hat{u}_i = \hat{u}_{/\Omega_i \cup \Gamma_0}$ then $\left\| u_i^\delta - \hat{u}_i \right\|_{H^1_{\Gamma_i}(\Omega_i)^d} \to 0$ as $\delta \to 0$, for $i = 1, 2$.*

Proof. Let $\hat{g}_l = \sum_{j=1}^{d} \sigma_{lj}(\hat{u}_1)n_j \quad \text{on } \Gamma_0 \quad 1 \leq l \leq d.$

Let $(u_1^\delta, u_2^\delta, g^\delta)_\delta$ denotes a sequence of optimal solutions, then

$$\mathcal{J}_\delta(u_1^\delta, u_2^\delta, g^\delta) \leq \mathcal{J}_\delta(\hat{u}_1, \hat{u}_2, \hat{g}) \quad \forall\delta > 0$$

i.e.

$$\frac{1}{2}\int_{\Gamma 0}(u_1^\delta - u_2^\delta)^2 d\Gamma_0 + \frac{\delta}{2}\int_{\Gamma 0}(g^\delta)^2 d\Gamma_0 \leq \frac{\delta}{2}\int_{\Gamma 0}(\hat{g})^2 d\Gamma_0 \quad \forall\delta > 0$$

Then, $\left\|g^\delta\right\|_{L^2(\Gamma_0)^d}$ is uniformly bounded in $L^2(\Gamma_0)^d$ and

$$\left\|u_1^\delta - u_2^\delta\right\|_{L^2(\Gamma_0)^d} \to 0 \quad \text{as} \quad \delta \to 0.$$

By (12), $\left\|u_1^\delta\right\|_{H^1_{\Gamma_1}(\Omega_1)^d}$ and $\left\|u_2^\delta\right\|_{H^1_{\Gamma_2}(\Omega_2)^d}$ are also uniformly bounded. Hence, as $\delta \to 0$, there exists a subsequence which converges to some $(u_1^*, u_2^*, g^*) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma 2}(\Omega_2)^d \times L^2(\Gamma_0)^d$ and the fact that $\left\|u_1^\delta - u_2^\delta\right\|_{L^2(\Gamma_0)^d} \to 0$ yields $u_1^* = u_2^*$ on Γ_0. By passing to the limit u_1^* and u_2^* satisfy (9) and (10) respectively. Let

$$u^* = \begin{cases} u_1^* & \text{in} \quad \Omega_1 \cup \Gamma_0 \\ u_2^* & \text{in} \quad \Omega_2 \cup \Gamma_0 \end{cases}$$

Then u^* satisfies (5) and by the uniqueness of the solution of (5), we conclude that $\hat{u} = u^*$. ■

Remark 3.3 *In the problem (13)-(15) for each* (ρ_1, ρ_2) *and each* $\delta > 0$ *there exists a unique optimal solution* $(u_1^\delta, u_2^\delta, g^\delta)$ *without* $u_1^\delta = u_2^\delta$ *on* Γ_0*, but according to Theorem 3.2, if* $\delta \to 0$*, the sequence of optimal solutions* $(u_1^\delta, u_2^\delta, g^\delta)_\delta$ *converges to the unique optimal solution* (u_1^*, u_2^*, g^*) *for which* $u_i^* = u \mid \Omega_i$ $(i = 1, 2)$ *where* u *is the unique solution of the problem*
$a_\rho(u, v) = (f, v)_\Omega$ *with*

$$\rho = \begin{cases} \rho_1 & \mathit{in} \quad \Omega_1 \\ \rho_2 & \mathit{in} \quad \Omega_2 \end{cases}$$

which yields the following corollary.

Corollary 3.4 *For each* $(\rho_1, \rho_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$*, an admissible solution* (u_1, u_2, g) *is the optimal solution of (13)-(15) corresponding (see remark 3.3) if and only if* $u_1 = u_2$ *on* Γ_0*.*

Proof. For $(\rho_1, \rho_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$ if (u_1, u_2, g) is the optimal solution, it follows from the above mentioned remark that $u_1 = u_2$ on Γ_0.

On the other hand, if $u_1 = u_2$ on Γ_0, let $(\hat{u}_1, \hat{u}_2, \hat{g})$ be the optimal solution corresponding to some $(\rho_1, \rho_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$, hence $\hat{u}_1 = \hat{u}_2$ on Γ_0. Setting

$$u = \begin{cases} u_1 & \text{in} \quad \Omega_1 \cup \Gamma_0 \\ u_2 & \text{in} \quad \Omega_2 \cup \Gamma_0 \end{cases} \quad \text{and} \quad \hat{u} = \begin{cases} \hat{u}_1 & \text{in} \quad \Omega_1 \cup \Gamma_0 \\ \hat{u}_2 & \text{in} \quad \Omega_2 \cup \Gamma_0 \end{cases}$$

As $(\hat{u}_1, \hat{u}_2, \hat{g})$ is the optimal solution then for

$$\rho = \begin{cases} \rho_1 & \text{in} \quad \Omega_1 \cup \Gamma_0 \\ \rho_2 & \text{in} \quad \Omega_2 \cup \Gamma_0 \end{cases}$$

We have: $a_\rho(\hat{u}, v) = (f, v)_\Omega$ for all $v \in H_0^1(\Omega)^d$, whereas $a_\rho(u, v) = (f, v)_\Omega$ for all $v \in H_0^1(\Omega)^d$ for the same ρ. By the uniqueness of the solution of (5), we have $u = \hat{u}$ therefore (u_1, u_2, g) is the optimal solution. ■

Consequently, we have the fundamental Theorem of this paper.

Theorem 3.5 *The problem (8) can be equivalently reformulated as:*

$$\begin{cases} \min\limits_{u_1, u_2, \rho_1, \rho_2} l_1(u_1) + l_2(u_2) \\ \min\limits_{g} \mathcal{J}_\delta(u_1, u_2, g) \\ a_{\rho_1}(u_1, v_1) = (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \\ a_{\rho_2}(u_2, v_2) = (f, v_2)_{\Omega_2} - (g, v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d \end{cases} \tag{17}$$

with the following constraints on ρ_i:

$$\sum_{i=1}^{2} \int_{\Omega_i} \rho_i(x) d\Omega_i \leq V \quad 0 < \rho_i \leq 1 \ \ \forall x \in \Omega_i, \ \ i = 1, 2.$$

and the gradient constraint $\|\rho_i\|_{H^1(\Omega_i)} \leq M_i$ with $i = 1, 2$, where
$l_i(u_i) = \displaystyle\int_{\Omega_i} f u_i d\Omega_i + (-1)^{i+1} \int_{\Gamma_0} g u_i d\Gamma_0.$

Let us start by defining the admissibility set to each problem;

- For the problem (8):

$$\mathcal{U}^* = \left\{ u \in H_0^1(\Omega)^d / \exists \rho \in \mathcal{G}^*, a_\rho(u, v) = (f, v)_\Omega \ \ \forall v \in H_0^1(\Omega)^d \right\}$$

  where

$$\mathcal{G}^* = \left\{ \rho \in H^1(\Omega) / \text{the constraints of the problem (8) on} \, \rho \right\}.$$

- For the problem (17), the admissibility set is defined by:

$$\mathcal{U}_* = \left\{ \begin{array}{c} (u_1, u_2, g) \in \mathcal{U}_{ad} \,/\, (u_1, u_2, g) \,\text{is the optimal solution} \\ \text{of}\ (13) - (15) \ \text{corresponding to some}\, (\rho_1, \rho_2) \in \mathcal{G}_* \end{array} \right\}$$

where

$$\mathcal{G}_* = \left\{ \begin{array}{c} (\rho_1, \rho_2) \in H^1(\Omega_1) \times H^1(\Omega_2)/\text{the constraints} \\ \text{of the problem (17) on}\, \rho_i \ i = 1, 2. \end{array} \right\}$$

In order to prove Theorem 3.5, we need to show the following Lemma:

Lemma 3.6 *$u \in \mathcal{U}^*$ if and only if $(u_1, u_2, g) \in \mathcal{U}_*$ where $u_i = u_{/\Omega_i}$ and $g = (g_l)_{1 \leq l \leq d}$ with $g_l = \sum_{j=1}^{d} \sigma_{lj}(u) n_j$.*

Proof. Let $u \in \mathcal{U}^*$ then $\exists \rho \in \mathcal{G}^*$, $a_\rho(u, v) = (f, v)_\Omega$ for all $v \in H^1_0(\Omega)^d$ and set $u_i = u_{/\Omega_i}$, $\rho_i = \rho_{/\Omega_i}$ and $g = (g_l)_{1 \leq l \leq d}$ where $g_l = \sum_{j=1}^{d} \sigma_{lj}(u) n_j$ while $n = (n_j)_{1 \leq j \leq d}$ denotes the unit outward normal vector on Γ, this yields immediately (9) and (10).
It is clear that $(u_1, u_2, g) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma_2}(\Omega_2)^d \times L^2(\Gamma_0)^d$, consequently, $(u_1, u_2, g) \in \mathcal{U}_{ad}$. Moreover, $\rho \in H^1(\Omega)$ implies that
$(\rho_1, \rho_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$, as $\|\rho\|_{H^1(\Omega)} \leq M$, then it exists $M_1, M_2 \in \mathbb{R}^+$, $\|\rho_i\|_{H^1(\Omega_i)} \leq M_i$ with $i = 1, 2$, in addition

$$\int_\Omega \rho(x) d\Omega \leq V \Rightarrow \int_{\Omega_1} \rho_1(x) d\Omega_1 + \int_{\Omega_2} \rho_2(x) d\Omega_2 \leq V$$

we have

$$0 < \rho(x) \leq 1, x \in \Omega \Rightarrow 0 < \rho_i(x) \leq 1, x \in \Omega_i$$

hence $(\rho_1, \rho_2) \in \mathcal{G}_*$. Put : $n^1 = (n^1_j)_j = (n_j)_j = n = (-n^2_j)_j = -n^2$; thus:

$$\sum_{j=1}^{d} \sigma_{lj}(u) n_j = \sum_{j=1}^{d} \sigma_{lj}(u_1) n^1_j = -\sum_{j=1}^{d} \sigma_{lj}(u_2) n^2_j = g_l \ \text{ on } \Gamma_0$$

thus, $(u_1, u_2, g) \in \mathcal{U}_*$ if and only if for $(\rho_1, \rho_2) \in \mathcal{G}_*$ defined in the beginning of this proof, (u_1, u_2, g) is the optimal solution of (13)-(15) corresponding, this is equivalent to $u_1 = u_2$ on Γ_0, which is true since $u_i = u_{|\Omega_i}$, with $i = 1, 2$.

On the other hand, let $(u_1, u_2, g) \in \mathcal{U}_*$. Setting

$$u = \left\{ \begin{array}{ll} u_1 & \text{in} \quad \Omega_1 \\ u_2 & \text{in} \quad \Omega_2 \end{array} \right. \qquad \text{and} \qquad \rho = \left\{ \begin{array}{ll} \rho_1 & \text{in} \quad \Omega_1 \\ \rho_2 & \text{in} \quad \Omega_2 \end{array} \right.$$

while

$$g_l = \sum_{j=1}^{d} \sigma_{lj}(u_1) n_j = \sum_{j=1}^{d} \sigma_{lj}(u_2) n_j$$

Given that $(u_1, u_2, g) \in \mathcal{U}_*$ then $\exists(\rho_1, \rho_2) \in \mathcal{G}_*$ such that (u_1, u_2, g) is the optimal solution of (13)-(15) corresponding, hence $u_1 = u_2$ on Γ_0, thus one can put:

$$u = \begin{cases} u_1 & \text{in} \quad \Omega_1 \cup \Gamma_0 \\ u_2 & \text{in} \quad \Omega_2 \cup \Gamma_0 \end{cases}$$

Then, taking $v \in H_0^1(\Omega)^d$ and $v_i = v_{/\Omega_i} \in H^1_{\Gamma_i}(\Omega_i)^d$, we have that

$$\begin{aligned} a_\rho(u, v) &= \int_\Omega \rho^p(x) E^0_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) d\Omega \\ &= \int_{\Omega_1} \rho_1^p(x) E^0_{ijkl} \varepsilon_{ij}(u_1) \varepsilon_{kl}(v_1) d\Omega_1 + \int_{\Omega_2} \rho_2^p(x) E^0_{ijkl} \varepsilon_{ij}(u_2) \varepsilon_{kl}(v_2) d\Omega_2 \\ &\quad + \int_{\Gamma_0} \rho^p(x) E^0_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) d\Gamma_0 \\ &= (f, v)_{\Omega_1} + (g, v)_{\Gamma_0} + (f, v)_{\Omega_2} - (g, v)_{\Gamma_0} + (f, v)_{\Gamma_0} \\ &= (f, v)_\Omega \end{aligned}$$

Finally, we have: $a_\rho(u, v) = (f, v)_\Omega$ for all $v \in H_0^1(\Omega)^d$.
Given that $(u_1, u_2, g) \in \mathcal{U}_*$, therefore $(u_1, u_2) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma_2}(\Omega_2)^d$ with $u_1 = u_2$ on Γ_0, thus $u \in H_0^1(\Omega)^d$, in addition $(\rho_1, \rho_2) \in \mathcal{G}_*$ then we check easily that $\rho \in \mathcal{G}^*$ such that $a_\rho(u, v) = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega)^d$, consequently $u \in \mathcal{U}^*$. ■

Proof of Theorem 3.5 Let $\hat{u} \in \mathcal{U}^*$ be an optimal solution of (8), $\hat{u}_i = \hat{u}_{/\Omega_i}$ and $\rho_i = \rho_{/\Omega_i}$ with $i = 1, 2$. Putting $\hat{g}_l = \sum_{j=1}^{d} \sigma_{lj}(\hat{u}) n_j$, according to Lemma 3.6, $(\hat{u}_1, \hat{u}_2, \hat{g}) \in \mathcal{U}_*$. To show that (8) implies (17), it remains to be shown that $(\hat{u}_1, \hat{u}_2, \hat{g})$ is a corresponding optimal solution (see Remark 3.3), when $u_1 = u_2$ on Γ_0, thus one can put:

$$u = \begin{cases} u_1 & \text{in} \quad \Omega_1 \cup \Gamma_0 \\ u_2 & \text{in} \quad \Omega_2 \cup \Gamma_0 \end{cases}$$

it follows from Lemma 3.6, that $u \in \mathcal{U}^*$, and as $\hat{u}$ is an optimal solution of (8), we obtain $l(\hat{u}) \leq l(u)$, we finally have $l_1(\hat{u}_1) + l_2(\hat{u}_2) \leq l_1(u_1) + l_2(u_2)$.

On the other hand ; let $(\hat{u}_1, \hat{u}_2, \hat{g}) \in \mathcal{U}_*$ be an optimal solution of (17), therefore one can put:

$$\hat{u} = \begin{cases} \hat{u}_1 & \text{in} \quad \Omega_1 \cup \Gamma_0 \\ \hat{u}_2 & \text{in} \quad \Omega_2 \cup \Gamma_0 \end{cases}$$

because $\hat{u}_1 = \hat{u}_2$ on Γ_0, thus, by Lemma 3.6, $\hat{u} \in \mathcal{U}_*$.

Let $v \in \mathcal{U}^*$, $v_i = v_{/\Omega_i}$, if we put $g_l = \sum_{j=1}^{d} \sigma_{lj}(v) n_j$, then a new application of the Lemma 3.6 enables us to have $(v_1, v_2, g) \in \mathcal{U}_*$, whereas $(\hat{u}_1, \hat{u}_2, \hat{g})$ is an optimal solution of (17) implies

$$l_1(\hat{u}_1) + l_2(\hat{u}_2) \leq l_1(v_1) + l_2(v_2)$$

that is $l(\hat{u}) \leq l(v)$ which is true for all $v \in \mathcal{U}^*$, hence, $\hat{u}$ is an optimal solution of (8).

The lagrangian of the minimization problem (13)-(15):
$\mathcal{L}(u_1, u_2, g, \lambda_1, \lambda_2) = \mathcal{J}_\delta(u_1, u_2, g) - a_{\rho_1}(u_1, \lambda_1) + (f, \lambda_1)_{\Omega_1} + (g, \lambda_1)_{\Gamma_0} - a_{\rho_2}(u_2, \lambda_2) + (f, \lambda_2)_{\Omega_2} - (g, \lambda_2)_{\Gamma_0}$
where
$(u_1, u_2, g, \lambda_1, \lambda_2) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma 2}(\Omega_2)^d \times L^2(\Gamma_0)^d \times H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma 2}(\Omega_2)^d$

The optimality system is derived by setting to zero $\dfrac{\partial \mathcal{L}}{\partial u_i}, \dfrac{\partial \mathcal{L}}{\partial \lambda_i}$ and $\dfrac{\partial \mathcal{L}}{\partial g}$ and given by the following equations:

$$\begin{array}{rcll} a_{\rho_1}(u_1, v_1) & = & (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0} & \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \\ a_{\rho_2}(u_2, v_2) & = & (f, v_2)_{\Omega_2} - (g, v_2)_{\Gamma_0} & \forall v_2 \in H^1_{\Gamma 2}(\Omega_2)^d \\ a_{\rho_1}(\xi, \lambda_1) & = & (u_1 - u_2, \xi)_{\Gamma_0} & \forall \xi \in H^1_{\Gamma_0}(\Omega_1)^d \\ a_{\rho_2}(\xi, \lambda_2) & = & -(u_1 - u_2, \xi)_{\Gamma_0} & \forall \xi \in H^1_{\Gamma_0}(\Omega_2)^d \\ (g, r)_{\Gamma_0} & = & -\dfrac{1}{\delta}(\lambda_1 - \lambda_2, r)_{\Gamma_0} & \forall r \in L^2(\Gamma_0)^d \end{array} \tag{18}$$

This optimality system may be viewed as a weak formulation of the problems respectively:
for $i = 1, ..., d$:

$$(-div\sigma(u_1)_i = f_i \quad \text{in} \quad \Omega_1; u_1 = 0 \;\text{ on }\; \Gamma_1; \sum_{j=1}^{d} \sigma_{ij}(u_1) n^1_j = g_i$$

$$(-div\sigma(u_2)_i = f_i \;\text{ in }\; \Omega_2; u_2 = 0 \;\text{ on }\; \Gamma_2; -\sum_{j=1}^{d} \sigma_{ij}(u_2) n^2_j = g_i$$

$$(div\sigma(\lambda_1)_i = 0 \quad \text{in} \quad \Omega_1; \lambda^i_1 = 0 \;\text{ on }\; \Gamma_1 \;\text{ and }\; \sum_{j=1}^{d} \sigma_{ij}(\lambda_1) n^1_j = u^i_1 - u^i_2$$

$$(div\sigma(\lambda_2)_i = 0 \quad \text{in} \quad \Omega_2; \lambda^i_2 = 0 \;\text{ on }\; \Gamma_2 \;\text{ and }\; \sum_{j=1}^{d} \sigma_{ij}(\lambda_2) n^2_j = -(u^i_1 - u^i_2)$$

and

$$g = (g_i)_{1 \le i \le d} \quad \text{where} \quad g_i = \frac{-1}{\delta}(\lambda_1^i - \lambda_2^i) \quad \text{on} \quad \Gamma_0 \tag{19}$$

We choose a gradient method to obtain a parallelizable algorithm.
Define: $\mathcal{M}_\delta(g) = \mathcal{J}_\delta(u_1(g), u_2(g), g)$ where for a given g

$$u_i(g) : g \in L^2(\Gamma_0)^d \to H^1_{\Gamma_i}(\Omega_i)^d \qquad i = 1, 2$$

are the solutions of (14) and (15) respectively, then the minimization problem (13)-(15) is equivalent to determine $g \in L^2(\Gamma_0)^d$ which minimize $\mathcal{M}_\delta(g)$, combining some previous results yields that the first derivative of $\mathcal{M}_\delta(g)$ is :

$$\frac{d\mathcal{M}_\delta(g)}{dg} = \delta g + (\lambda_1 - \lambda_2)/\Gamma_0$$

hence for $n = 1, 2, \ldots$ $\quad g^{(n+1)} = g^{(n)} - \dfrac{\alpha}{\delta}\dfrac{d\mathcal{M}_\delta(g)}{dg}$ where $\dfrac{\alpha}{\delta}$ is the step size, combining with the formule (19) we obtain an update formula for g:

$$g^{(n+1)} = (1 - \alpha)g^{(n)} - \frac{\alpha}{\delta}(\lambda_1^{(n)} - \lambda_2^{(n)})$$

and the algorithm is given as follows:

Step1: Choose $g^{(0)}$
For n=0,1,2,...

Step2: Choose $\rho_1^{(0)}$ and $\rho_2^{(0)}$

1. Solve the topology optimization problem on each subdomain to determine $\rho_1^{(opt)}$ and$\rho_2^{(opt)}$
   For m=0,1,2,...
   $\min\limits_{\rho_i^{(m)}} l_i(u_i)$
   $a_{\rho_1^{(m)}}(u_1, v_1) = (f, v_1)_{\Omega_1} + (g^{(n)}, v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d$
   $a_{\rho_2^{(m)}}(u_2, v_2) = (f, v_2)_{\Omega_2} - (g^{(n)}, v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma 2}(\Omega_2)^d$
   $\displaystyle\int_{\Omega_i} \rho_i^{(m)}(x) d\Omega_i \le V_i$
   $0 < \rho_{min} \le \rho_i^{(m)}(x) \le 1 \quad i = 1, 2 \quad \text{such that } V_1 + V_2 \le V$
   and the constraint on the gradient to ensure existence of the solution [4] :
   $\|\rho_i\|_{H^1(\Omega_i)} \le M_i \ \ i = 1, 2.$

2. determine $\lambda_1^{(n)}, \lambda_2^{(n)}$ from:

$$a_{\rho_1^{opt}}(\lambda_1^{(n)}, R) = (u_1^{(n)opt} - u_2^{(n)opt}, R)_{\Gamma_0} \quad \forall R \in H^1_{\Gamma_1}(\Omega_1)^d$$
$$a_{\rho_2^{opt}}(\lambda_2^{(n)}, R) = -(u_1^{(n)opt} - u_2^{(n)opt}, R)_{\Gamma_0} \quad \forall R \in H^1_{\Gamma_2}(\Omega_2)^d$$

3. update g:
$$g^{(n+1)} = (1-\alpha)g^{(n)} - \frac{\alpha}{\delta}(\lambda_1^{(n)} - \lambda_2^{(n)})$$
with the optimality condition given previously:
$$g_i = -\frac{1}{\delta}(\lambda_1^i - \lambda_2^i)$$
δ is fixed, for a suitable choice of the step size $\dfrac{\delta}{\alpha}$ we control the value of α

Remark 3.7 *We need a good "g" or an optimal g, g^{opt} which satisfy*

$$(g^{opt}, u_1)_{\Gamma_0} = (g^{opt}, u_2)_{\Gamma_0}$$

hence the decomposition of the compliance $l(u)$ on Ω in $l(u_1, u_2, g)$ on Ω_1 and Ω_2 is given by:

$$l(u_1, u_2, g) = l_1(u_1) + l_2(u_2) \tag{20}$$

where :

$$l_i(u_i) = \int_{\Omega_i} f u_i d\Omega_i + (-1)^{i+1} \int_{\Gamma_0} g u_i d\Gamma_0 \quad i = 1, 2$$

allow us to retrieve the global compliance $l(u)$ in a unique choice of g "$g = g^{opt}$" that is :

$$(g, u_1)_{\Gamma_0} = (g, u_2)_{\Gamma_0} \quad \textit{if and only if} \quad g = g^{opt}$$

or

$$l(u) = l(u_1, u_2, g) \Leftrightarrow g = g^{opt}$$

and the problem (17) takes the following form :

$$\begin{array}{c} \min\limits_{\rho_1,\rho_2} l(u_1, u_2, g^{opt}) \\ \min\limits_{g} \mathcal{J}_\delta(u_1, u_2, g) \\ a_{\rho_1}(u_1, v_1) = (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \\ a_{\rho_2}(u_2, v_2) = (f, v_2)_{\Omega_2} - (g, v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma 2}(\Omega_2)^d \end{array} \tag{21}$$

with the constraints on ρ_i :

$$\sum_{i=1}^{2} \int_{\Omega_i} \rho_i(x) d\Omega_i \le V \quad 0 < \rho_i \le 1 \ \ \forall x \in \Omega_i, \ \ i = 1, 2.$$

which is a bilevel formulation that we'll develop in

References

[1] G. Allaire, Z. Belhachimi and F. Jouve, *The homogenization method for topology and shape optimization, Single and multiple case,* European Journal of Finite Elements, **5** (1996), 649 - 672.

[2] M. P. Bendsøe and N. Kikuchi, *Generating optimal topology in structural design using a homogenization method,* Comput. Methods Appl. Mech. Eng., **71** (1988), 197- 224.

[3] M. P. Bendsøe, *Optimization of Structural Topology, Shape and Material,* Springer Verlag, Berlin Heidelberg, 1995.

[4] M. P. Bendsøe and O. Sigmund, *Topology Optimization, Theory, Methods, and Applications,* Springer Verlag, Berlin, 2003.

[5] T. Borvall and J. Petersson,*Large scale topology optimization in 3D using parallel computing,* Computer methods in applied mechanics and engineering, **190** (2001), 6201 - 6229.

[6] H. Brezis, *Analyse fonctionnelle, Théorie et applications,* Masson, Paris, 1983.

[7] C. D. Chapman, K. Saitou and J. Jakiela, *Genetic algorithms as an approach to configuration and topology design,* Journal of Mechanics Design **116** (1994), 1005 - 1012.

[8] G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics,* Springer-Verlag, Berlin, 1976.

[9] I. Ekeland and R. Témam, *Convex analysis and variational problems,* **28** of Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1999.

[10] A. El Hami and B. Radi,*Some decomposition methods in the analysis of repetitive structures,* Computers and Structures, **58(5)** (1996), 973 - 980.

[11] M. D. Gunzburger and J. Lee, *A domain decomposition method for optimization problems for partial differential equations,* International Journal Computers and mathematics with applications, Elsevier Science, **40** (2000), 177 - 192.

[12] C. Kane and M. Schoenauer,*Topological optimum design using genetic algorithms,* Control and Cybernetics, **25(5)** (1996), 1059 - 1088.

[13] A. Mahdavi, R. Balaji, M. Frecker and E. M. Mackensturn, *Parallel Optimality Criteria-based Topology Optimization for Minimum Compliance*

Design, Proc. of the Sixth World Congress of Structural and Multidisciplinary Optimization, Rio de Janeiro, 30 May-03 June 2005, Brezil.

[14] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations,* Clarendon Press, Oxford, 1999.

[15] B. Radi, J. -C. Gelin and A. Perriot, *Subdomain methods in structural mechanics,* International Journal for Numerical Methods in Engineering, **37** (1994), 3309 - 3322.

[16] B. Radi and J. -F. Estrade, *The ARPEGE model: Some strategies and performances,* Parallel Computing, **24** (1998) 1167 - 1175.

[17] J. Salençon, *Mécanique des milieux continus. Tome2 : Élasticité. Milieux curvilignes*, Ellipses, 1988.

[18] O. Sigmund and J. Petersson, *Numerical instabilities in topology optimization: a survey on procedures dealing with checkerboards, mesh-dependencies and local minima,* Struct.Optim., **16** (1998), 68 - 75.

[19] M. Sunar and R. Kahraman, *A comparative study of multiobjective optimization methods in Structural Design,* Turk J Engin Environ Sci, **25** (2001), 69 - 78.

[20] K. Suzuki and N. Kikuchi, *A homogenization method for shape and topology optimization,* Comp. Math. Appl. Mech. Eng., **93** (1991), 291 - 318.

[21] K. Vemaganti and W. E. Lawrence, *Parallel methods for topology optimization,* Elsevier science, 2004.

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