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Numerical Solution of Two-Point Boundary Value Problem in Linear Differential Equation

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Abstract

A numerical method for solving linear differential equation with two-point boundary value condition is presented. The method is based upon function approximations.The properties of Legendre polynomials are presented. The opperational matrices of integration and product are then utilized to reduce our problem to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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Keywords: Differential equation; Legendre polynomials; Function approximation

1 Introduction

There are classes of sets of orthogonal functions wich are widely used. The first includes sets of piece-wise constant basis functions(e.g.,walsh,block-pulse,etc.). The second consists of sets of orthogonal polynomials(e.g.,Laguerre,Legendre, Chebyshev,etc.). The third is widely used sets of sin-cosine functions in Fourier series. While orthogonal polynomials and sine-cosine functions together from a class of continuous basis functions, piece-wise constant basis functions(PCBFs) have inherent discontinuities or jumps. The paper is organized as follows: in section 2 we describe the basis formulation of the Legendre polynomials required for our subsequent development. Section 3 is devoted to the formulation of the two-point boundary value problem. In section 4 the proposed method is used to approximate the two-point boundary value problem. In section 5 the proposed method is applied to the numerical examples.

2 Preliminaries

2.1 Legendre polynomials

The well-known Legendre polynomials are orthogonal in the interval $[-1, 1]$ with respect to the weight function $w(t) = 1$ and can be determined with the aid of following recurrence formula:

$$p_0(t) = 1,$$
$$p_1(t) = t,$$
$$p_{m+1}(t) = (\tfrac{2m+1}{m+1})tp_m(t) - (\tfrac{m}{m+1})p_{m-1}(t) \qquad m = 2, 3, \ldots$$

This functions are orthogonal in subinterval $[a, b]$ with the aid of the following linear transformation:

$$\tau = (\frac{2}{b-a})t - (\frac{b+a}{b-a})$$

that are named shifted Legendre polynomials. So in interval $[0, t_f]$ we have:

$$L_0(t) = 1,$$
$$L_1(t) = (\frac{2}{t_f})t - 1 \tag{1}$$
$$L_m(t) = p_m(\tau) = p_m((\tfrac{2}{t_f})t - 1) \qquad m = 2, 3, \ldots$$

2.2 Function approximation

A function $f(t)$, defined over the interval 0 to t_f may be expanded as:

$$f(t) = \sum_{i=0}^{\infty} c_i L_i(t) \tag{2}$$

where

$$c_i = \frac{\int_0^{t_f} f(t)L_i(t)\,dt}{\int_0^{t_f} (L_i(t))^2\,dt} \tag{3}$$

If $f(t)$ in Eq.(2) is truncated up to the M^{th} terms, then Eq.(2) can be written as:

$$f(t) = \sum_{i=0}^{M-1} c_i L_i(t) = C^T B(t) \tag{4}$$

With

$$C = [c_0, c_1, \ldots, c_{M-1}]^T \tag{5}$$

$$B(t) = [L_0(t), L_1(t), \ldots, L_{M-1}(t)]^T \tag{6}$$

2.3 Operational matrix

The integral of the vector $B(t)$ defined in Eq.(5) can be expressed by

$$\int B(t)\,dt \simeq PB(t)$$

where P is $M \times M$ operational matrix for integration and is given in [1] as:

$$P = \frac{t_f}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \ldots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \frac{-1}{2m-3} & 0 & \frac{1}{2m-3} \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{-1}{2m-1} & 0 \end{bmatrix} \tag{7}$$

2.4 The product operational matrix

The following property of the product of two Legendre function vectors will also be used. Let

$$B(t)B^T(t)C = \tilde{C}B(t) \tag{8}$$

where $\tilde{c}$ is a $M \times M$ product operational matrix. To illustrate the calculation procedure we choose $M = 4$.
Thus we have

$$\tilde{c} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ \frac{1}{3}c_2 & \frac{2}{5}c_3 + c_1 & \frac{3}{7}c_4 + \frac{2}{3}c_2 & \frac{3}{5}c_3 \\ \frac{1}{5}c_3 & \frac{9}{35}c_4 + \frac{2}{5}c_2 & \frac{2}{7}c_3 + c_1 & \frac{4}{15}c_4 + \frac{3}{5}c_2 \\ \frac{1}{7}c_4 & \frac{9}{35}c_3 & \frac{4}{21}c_4 + \frac{3}{7}c_2 & \frac{4}{15}c_3 + c_1 \end{bmatrix} \tag{9}$$

The integral of the vector $B(t)$ over the interval $[t_f, t]$ can be approximated by

$$\int_{t_f}^{t} B(\tau)\, dt \simeq SB(t) \tag{10}$$

3 Problem statement

The problem is to solve the following differential equation

$$f_n(t)Y^{(n)}(t) + f_{n-1}(t)Y^{(n-1)}(t) + ... + f_1(t)Y'(t) + f(t)Y(t) = h(t) \tag{11}$$

$$Y^{(j)}(a) = Y_{j,a} \quad , \qquad j = 0, 1, \ldots, n-1$$

Where $f_i : \ i = 1, 2, \ldots, n, f(t)$ and $h(t)$ are known functions and $a\epsilon\{t_0, t_f\}$.

4 Direct method

In this paper we use the proposed method with $n = 2$. Thus the problem is to solve the following differential equation:

$$f_2(t)Y''(t) + f_1(t)Y'(t) + f_0(t)Y(t) = f_3(t)Y(0) = Y_0Y'(t_f) = Y_1 \tag{12}$$

By using Eq.(4),we have

$$Y''(t) = C^T B(t) \tag{13}$$

By integrating Eq.(13) from t_f to t we get

$$Y'(t) - Y(t_f) = C^T SB(t)Y'(t) = (C^T S + e_1^T)B(t) \tag{14}$$

Where S is given in Eq.(10) and e_1 is a vector of order $M \times 1$ given by

$$e_1 = (Y_1, 0, \ldots, 0)^T$$

similarly by integrating Eq.(14) from 0 to t we obtain:

$$Y(t) - Y(0) = (C^T S + e_1^T)PB(t)Y(t) = [(C^T S + E_1^T)P + e_2^T]B(t) \tag{15}$$

Where

$$e_2 = (Y_0, 0, \ldots, 0)^T$$

we also expand $f_i(t), i = 0, 1, 2, 3$ by Legendre polynomials as:

$$f_i(t) = F_i^t B(t) \quad i = 0, 1, 2, 3 \tag{16}$$

By substituting Eqs.(14),(15),(16) in Eq.(12) we get

$$C^T\tilde{F}_2B(t)+[C^TS+E_1^T]\tilde{F}_1B(t)+[(C^TS+e_1^T)P+e_2^T]\tilde{F}_0B(t)=F_3^TB(t) \quad (17)$$

Where F can be calculated similarly to matrix $\tilde{C}$ in Eq.(8).
Eq.(17) can be written as:

$$C^T\tilde{F}_2+[C^TS+e_1^T]\tilde{F}_1+[(C^TS+e_1^t)P+e_2^t]\tilde{F}_0=F_3^T$$

That is a system of algebraic equations. Consequently the unknown coefficients of the vector C in Eq.(15) can be calculated.

5 Illustrative examples

5.1 Example 1

Consider the following differential equation.

$t^2Y''-2tY'+2Y=-t^3sin(t)$
$Y(-1)=sin(1)$
$Y'(1)=sin(1)+cos(1)$

We applied the proposed method with $M=8$. The result is shown in Fig.1

5.2 Example 2

Consider the foolowing differential equation.

$2(t+1)Y''+Y'-Y=\sqrt{t+1}$
$Y(-1)=0$
$Y'(1)=\frac{1}{2\sqrt{2}}$

the result with $M=8$ is shown in Fig.2

6 Conclusion

The Legendre polynomials and the associated operational matrices of integration P , product $\tilde{C}$ are applied to solve the two-point boundary value problems in differential equations. The method is based upon reducing the system into a set of algebraic equations. The matrices $P,\tilde{C}$ have many zeros; hence the method is computationally very attractive. The convergence of this method make this approach very contributed to the good agreement between approximate and exact values.

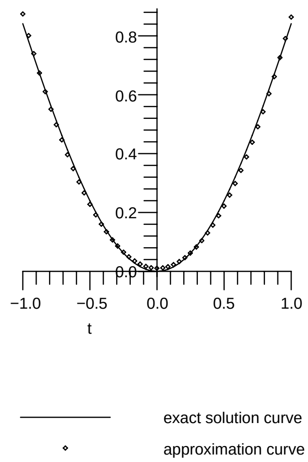

Figure 1:

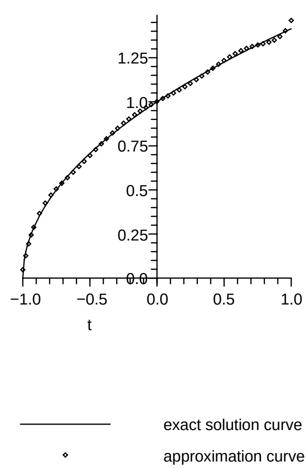

Figure 2:

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