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Exact and Numerical Solution of Kawahara Equation by the Variational Iteration Method

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Abstract

In this Letter, exact and numerical solutions are obtained for the Kawahara equation by the known variational iteration method (VIM). This method is based on Lagrange multipliers for identification of optimal value of parameters in a functional. Using this method creates a sequence which tends to the exact solution of the problem.

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1 Introduction

In this Letter, we consider the numerical solutions to a problem involving a nonlinear partial differential equation of the form

$$u_t + uu_x + u_{3x} - u_{5x} = 0, \tag{1}$$

which is called Kawahara equation [9]. We solve Eq. (1) subject to the initial condition

$$u(x, 0) = f(x), \qquad x \in \mathbb{R}. \tag{2}$$

The Kawahara equation occurs in the theory of magneto-acoustic waves in a plasma [7] and in the theory of shallow water waves with surface tension [5].

In order to solve this equation numerically, we use variational iteration method and the numerical results are compared with the exact solutions.

2 Variational iteration method

To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation

$$Lu + Nu = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to the VIM [1, 2, 3, 4, 8, 10] we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{Lu_n + N\tilde{u}_n - g(\tau)\}d\tau, \tag{4}$$

where $\lambda(\tau)$ is a general Lagrange multiplier [2, 3, 4, 6] which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation and $\tilde{u}_n$ is considered as a restricted variation [1, 2, 3, 4], i.e. $\delta\tilde{u}_n = 0$. In the next section, variational iteration method has been successfully used to study Kawahara equation.

3 VIM for Kawahara equation

Using the VIM we have

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{u_{n\tau} + \tilde{u}_n\tilde{u}_{n_x} + \tilde{u}_{n3x} - \tilde{u}_{n5x}\}d\tau,_{\, n\geq 0} \tag{5}$$

where $\tilde{u}_n$ is considered as restricted variations, i.e. $\delta\tilde{u}_n = 0$.
To find optimal value of $\lambda(\tau)$, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\tau)\{u_{n\tau} + \tilde{u}_n\tilde{u}_{n_x} + \tilde{u}_{n3x} - \tilde{u}_{n5x}\}d\tau, \tag{6}$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\tau)\{u_{n\tau}\}d\tau, \tag{7}$$

which results

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\lambda(\tau)u_n(x,\tau)|_{\tau=t} - \int_0^t \lambda'(\tau)u_n(x,\tau)d\tau = 0. \tag{8}$$

Therefore, the stationary conditions are obtained in the following form

$$1 + \lambda(\tau) = 0|_{\tau=t}\ , \tag{9}$$

$$\lambda'(\tau) = 0|_{\tau=t} \ , \tag{10}$$

which yields

$$\lambda(\tau) = -1, \tag{11}$$

substituting this value of the Lagrange multiplier into functional (5) give the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_{n\tau} + u_n u_{n_x} + u_{n_{3x}} - u_{n_{5x}}\}d\tau,_{\, n\geq 0}. \tag{12}$$

The iteration formula(12) will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations.

4 Implementation of the method

In this section, we first solve Eq. (1) subject to the initial condition:

$$u(x,0) = -\frac{72}{169} + \frac{105}{169}\text{sech}^4(kx),$$

where $k = \frac{1}{2\sqrt{13}}$. We start with initial approximation $u_0(x,t) = -\frac{72}{169} + \frac{105}{169}\text{sech}^4(kx)$ and using the iteration formula(12) and MATLAB software, we can obtain:

$$\begin{aligned}
u_1(x,t) &= u_0(x,t) - \int_0^t \{u_{0\tau} + u_0 u_{0_x} + u_{0_{3x}} - u_{0_{5x}}\}d\tau \\
&= -\frac{3}{28561}(4056\text{cosh}^9(kx) - 5915\text{cosh}^5(kx) \\
&+ 10080\sinh(kx)\text{cosh}^4(kx)kt - 14700\sinh(kx)kt \\
&- 378560\sinh(kx)\text{cosh}^4(kx)k^3t + 709800\sinh(kx)\text{cosh}^2(kx)k^3t \\
&+ 6056960\sinh(kx)\text{cosh}^4(kx)k^5t - 36909600\sinh(kx)\text{cosh}^2(kx)k^5t \\
&+ 39748800\sinh(kx)k^5t)/\text{cosh}^9(kx),
\end{aligned}$$

$$\begin{aligned}
u_2(x,t) &= u_1(x,t) - \int_0^t \{u_{1\tau} + u_1 u_{1_x} + u_{1_{3x}} - u_{1_{5x}}\}d\tau \\
&= \frac{3}{815730721}(168938315\text{cosh}^{15}(kx) - 115843416\text{cosh}^{19}(kx) \\
&+ 146747057766400\sinh(kx)\text{cosh}^{10}(kx)k^{11}t^3 \\
&+ 406425600\sinh(kx)\text{cosh}^{10}(kx)k^3t^3 \\
&+ 166944960000\sinh(kx)\text{cosh}^2(kx)k^5t^3 \\
&+ 88572331294720\text{cosh}^{15}(kx)k^{10}t^2 - 18425272320\text{cosh}^{15}(kx)k^4t^2
\end{aligned}$$

$$
\begin{aligned}
&- 1430956800\cosh^{11}(kx)k^2t^2 + 1565109000\cosh^7(kx)k^2t^2\\
&+ 640790026240\cosh^{15}(kx)k^6t^2 - 11071541411840\cosh^{15}(kx)k^8t^2\\
&+ 33020103261600\cosh^9(kx)k^6t^2 - 1695534750\cosh^5(kx)k^2t^2\\
&+ 73893019200\cosh^{11}(kx)k^4t^2 + 245306880\cosh^{15}(kx)k^2t^2\\
&+ 1609826400\cosh^9(kx)k^2t^2 + 74852668800\cosh^{13}(kx)k^4t^2\\
&- 5018954612672000\cosh^{13}(kx)k^{10}t^2\\
&- 135506979299200\cosh^{11}(kx)k^8t^2\\
&- 1421970391296000\,\sinh(kx)k^{11}t^3 - 1944810000\,\sinh(kx)k^3t^3\\
&+ 272679955475200\cosh^{13}(kx)k^8t^2\\
&- 403052832000\cosh^9(kx)k^4t^2\\
&- 7844167304000\cosh^7(kx)k^6t^2 + 277098822000\cosh^7(kx)k^4t^2\\
&+ 370052\cosh^5(kx)k^6t^2 - 112391282203200\cosh^7(kx)k^8t^2\\
&- 6658208866400\cosh^{13}(kx)k^6t^2 + 10517532480000\,\sinh(kx)k^7t^3\\
&- 306633600\cosh^{13}(kx)k^2t^2\\
&+ 2023043079657600000\cosh^7(kx)k^{10}t^2\\
&- 15914605559973120\cosh^9(kx)k^{10}t^2\\
&+ 4910250240255360\cosh^{11}(kx)k^{10}t^2\\
&+ 220695608689920\cosh^9(kx)k^8t^2\\
&- 8766520011849600\cosh^5(kx)k^{10}t^2\\
&+ 7935102630000\cosh^{11}(kx)k^6t^2\\
&- 18030055680000\,\sinh(kx)\cosh^2(kx)k^7t^3\\
&- 451419171840000\,\sinh(kx)\cosh^2(kx)k^9t^3\\
&- 2419033530368000\,\sinh(kx)\cosh^8(kx)k^{11}t^3\\
&+ 13745713782528000\,\sinh(kx)\cosh^6(kx)k^{11}t^3\\
&- 33446398806912000 sinh(kx)\cosh^4(kx)k^{11}t^3\\
&+ 36113533747200000\,\sinh(kx)\cosh^2(kx)k^{11}t^3\\
&- 1135265476800\,\sinh(kx)\cosh^{10}(kx)k^5t\\
&+ 1054175085600\,\sinh(kx)\cosh^{12}(kx)k^5t\\
&+ 2074464000\,\sinh(kx)\cosh^4(kx)k^3t^3\\
&- 508032000\,\sinh(kx)\cosh^8(kx)k^3t^3\\
&- 30527078400\,\sinh(kx)\cosh^{10}(kx)k^5t^3\\
&+ 1061663948800\,\sinh(kx)\cosh^{10}(kx)k^7t^3\\
&+ 1728720000\,\sinh(kx)\cosh^2(kx)k^3t^3\\
&+ 109706688000\,\sinh(kx)\cosh^8(kx)k^5t^3\\
&- 223984488000\,\sinh(kx)\cosh^4(kx)k^5t^3
\end{aligned}
$$

$$\begin{aligned}
&- \; 293544888000\,\sinh(kx)\cosh^4(kx)k^7t^3 \\
&+ \; 205646511616000\,\sinh(kx)\cosh^8(kx)k^9t^3 \\
&- \; 714209618304000\,\sinh(kx)\cosh^6(kx)k^9t^3 \\
&+ \; 972432123672000\,\sinh(kx)\cosh^4(kx)k^9t^3 \\
&- \; 7734586496000\,\sinh(kx)\cosh^8(kx)k^7t^3 \\
&- \; 19079424000\,\sinh(kx)\cosh^6(kx)k^5t^3 \\
&+ \; 14451471216000\,\sinh(kx)\cosh^6(kx)k^7t^3 \\
&+ \; 10812052160\,\sinh(kx)\cosh^{14}(kx)k^3t \\
&- \; 172992834560\,\sinh(kx)\cosh^{14}(kx)k^5t \\
&+ \; 419846700\,\sinh(kx)\cosh^{10}(kx)kt \\
&- \; 287894880\,\sinh(kx)\cosh^{14}(kx)kt \\
&- \; 1778112000\,\sinh(kx)\cosh^6(kx)k^3t^3 \\
&- \; 18343382220800\,\sinh(kx)\cosh^{10}(kx)k^9t^3 \\
&- \; 20272597800\,\sinh(kx)\cosh^{12}(kx)k^3t)/\cosh^{19}(kx),
\end{aligned}$$

$\vdots$

Thus the exact value of $u(x,t)$ in a closed form is

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = -\frac{72}{169} + \frac{105}{169}\text{sech}^4[k(x+ct)],$$

where $c = \frac{36}{169}$ and $k = \frac{1}{2\sqrt{13}}$. This results can be verified through substitution. Using the iteration formula (12) and MATLAB software, we may get $u_3(x,t)$ and other high order approximations. The numerical results of this example is given in Table 1. In the numerical calculation, $u_2(x,t)$ is approximation of $u(x,t)$ which is calculated by the recurrence formula (12) for the values of $t = 0.1(0.1)0.5$ and $x = 0.1(0.1)0.5$. By the variational iteration method, one can get a better result by calculating more terms of the sequence $\{u_n\}$.

In the second example, we solve Kawahara equation (1) subject to the initial condition:

$$u(x,0) = -\frac{72}{169} + \frac{420\text{sech}^2[kx]}{169(1+\text{sech}^2[kx])}$$

where $k = \frac{1}{2\sqrt{13}}$. We start with initial approximation $u(x,0) = -\frac{72}{169} + \frac{420\text{sech}^2[kx]}{169(1+\text{sech}^2[kx])}$ and using the iteration formula(12) and MATLAB software,

$t_j\backslash x_i$	0.1	0.2	0.3	0.4	0.5
$\lvert u-u_2\rvert_{ij}$					
0.1	5.6269e-009	4.5919e-008	1.5795e-007	3.8142e-007	7.5870e-007
0.2	1.1101e-008	8.9681e-008	3.0562e-007	7.3139e-007	1.4421e-006
0.3	1.6477e-008	1.3267e-007	4.5068e-007	1.0751e-006	2.1133e-006
0.4	2.1711e-008	1.7453e-007	5.9189e-007	1.4097e-006	2.7665e-006
0.5	2.6760e-008	2.1490e-007	7.2808e-007	1.7324e-005	3.3963e-006

Table 1: *The numerical results for $u_2$in comparison with the exact solution of u.*

we can obtain:

$$\begin{aligned} u_1(x,t) &= u_0(x,t) - \int_0^t \{u_{0\tau} + u_0 u_{0_x} + u_{0_{3x}} - u_{0_{5x}}\} d\tau \\ &= -\frac{12}{28561}(-4901 - 38870\cosh^6(kx) - 23491\cosh^2(kx) \\ &+ 169\cosh^{10}(kx) - 43940\cosh^4(kx) + 1014\cosh^{12}(kx) \\ &+ 236600\sinh(kx)\cosh^7(kx)k^3t + 189280\sinh(kx)\cosh^9(kx)k^5t \\ &- 47320\sinh(kx)\cosh^3(kx)k^3t + 425880\sinh(kx)\cosh^5(kx)k^3t \\ &+ 5040\sinh(kx)\cosh^9(kx)kt - 68040\sinh(kx)\cosh^3(kx)kt \\ &- 57960\sinh(kx)\cosh^5(kx)kt - 9240\sinh(kx)\cosh^7(kx)kt \\ &- 24360\sin(kx)\cosh(kx)kt + 36625680\sinh(kx)\cosh^5(kx)k^5t \\ &- 7760480\sinh(kx)\cosh^7(kx)k^5t - 189280\sinh(kx)\cosh(kx)k^3t \\ &- 47320\sinh(kx)\cosh^9(kx)k^3t - 14365\cosh^8(kx) \\ &+ 7287280\sinh(kx)\cosh(kx)k^5t \\ &- 38991680\sinh(kx)\cosh^3(kx)k^5t)/(\cosh^2(kx)+1)^6. \end{aligned}$$

$\vdots$

Thus the exact value of $u(x,t)$ in a closed form is

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = -\frac{72}{169} + \frac{420\text{sech}^2[(x+ct)]}{169(1+\text{sech}^2[k(x+ct)])},$$

where $c = \frac{36}{169}$ and $k = \frac{1}{2\sqrt{13}}$. This results can be verified through substitution. Using the iteration formula (12) and by MATLAB software, we may get $u_2(x,t)$ and other high order approximations. The numerical results of this example is given in Table 2. In the numerical calculation, $u_1(x,t)$ is approximation of $u(x,t)$ which is calculated by the recurrence formula (12) for the values of $t = 0.1(0.1)0.5$ and $x = 0.1(0.1)0.5$. By the variational iteration method, one can get a better result by calculating more terms of the sequence $\{u_n\}$.

$t_j \backslash x_i$	0.1	0.2	0.3	0.4	0.5
$\lvert u - u_1 \rvert_{ij}$					
0.1	2.4345e-004	4.9774e-004	7.6287e-004	1.0388e-003	1.3256e-003
0.2	4.8119e-004	9.7321e-004	1.4761e-003	1.9898e-003	2.5143e-003
0.3	7.1835e-004	1.4475e-003	2.1875e-003	2.9383e-003	3.6999e-003
0.4	9.5465e-004	1.9201e-003	2.8964e-003	3.8834e-003	4.8813e-003
0.5	1.1898e-003	2.3904e-003	3.6017e-003	4.8239e-003	6.0568e-003

Table 2: *The numerical results for* u_1 *in comparison with the exact solution of* u.

5 Conclusion

In this Letter, variational iteration method was employed successfully for solving the Kawahara equation. The solutions obtained by means of the variational iteration method is an infinite power series for appropriate initial condition, which can be, in turn, expressed in a closed form. The numerical solutions are compared with the exact solutions. The results show that the VIM is a powerful mathematical tool for finding the exact and numerical solutions of nonlinear equations. In our work we use the MATLAB to calculate the series obtained from the variational iteration method.

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