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A Descent Algorithm for Solving Variational Inequality Problems

Zahira Kebaili and A. Krim Keraghel

Department of Mathematics, Sétif University
19000 Sétif, Algeria
z_kebaili@yahoo.fr
a_keragheldz@yahoo.fr

Abstract

Variational inequality problem can be formulated as a differentiable optimization problem [3]. We propose a descent method to solve the equivalent optimization problem inspired by the approach of Fukushima, then we give theoretical results and numerical experimentations.

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1 Introduction

We consider the problem of finding a point $\overline{x} \in C$ such that:

$$\langle F(\overline{x}), x - \overline{x} \rangle \geq 0, \ for \ \ all \ x \in C \tag{1}$$

Where C is noempty closed convex subset of R^n, and F is continuously differentiable mapping from R^n into itself, and $\langle .,. \rangle$ denotes the inner product in R^n. This problem is called variational inequality problem. The relation between this problem and the optimization goes up in the Seventies there were a difficulties in the obtained problem of optimization (non-differentiability of the objective function). In the Nineties Fukushima and others established the equivalence between the variational inequality and a differentiable optimization problem with sufficient conditions of global optimality. However, the algorithms remain insufficient for a suitable treatment: restrictive assumptions and slow convergence witch sometimes difficult to establish.

In our study, we introduce some algorithmic modifications leading to interesting implementation.

The paper is organized as follows. In section 2, we introduce some definitions and basic result concerning the connection between problem (1) and the optimization. Section 3 is devoted to the description of Fukushima's method and its related algorithm is presented. In section 4, we applied the algorithm for some variational inequality problems [4]. Finally, we give in section 5 a conclusion and future remarks.

2. Preliminaries

First R^n the space of $n-$dimensional vectors, let $\|.\|_G$ denote the norm in R^n defined by $\|x\|_G = \langle x, Gx\rangle^{\frac{1}{2}}$, where G is a symmetric positive defined matrix. Given the projection of the point x onto the closed convex set C, denotes by $\text{proj}_{C,G}$ is defined as the (unique) solution of the problem:

$$y \in C \quad \min \quad \|y - x\|_G$$

For a vector valued mapping F from $\mathbb{R}^n$into itself, we have:
$F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^T$ with $x \in \mathbb{R}^n$.

Definition1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a multivalued vector mapping.

• F is monotone on $\mathbb{R}^n$, if for all $x, y \in \mathbb{R}^n$,

$$(x - y)^T (F(x) - F(y)) \geqslant\geq 0.$$

• F is strictly monotone, if for all $x, y \in \mathbb{R}^n$, $x \neq y$,

$$(x - y)^T (F(x) - F(y)) > 0$$

• F is strongly monotone if: there exists $c > 0$,such that, for all $x, y \in \mathbb{R}^n$:

$$(x - y)^T (F(x) - F(y)) > c \|x - y\|^2 .$$

2. **Resolution of the problem (1) by the optimization's techniques**

2.1 **Traditional approach**

The problem when F is the gradient of a differentiable convex function $h : R^n \to R$,i.e. $(F(x) = \nabla h(x))$, then (1) is noting other than the necessary and sufficient condition $(\langle \nabla h(\overline{x}), x - \overline{x} \rangle \geq 0 for \quad x \in C)$ for the following convex optimization problem **:**

$$\min h(x) \qquad subject \quad to \quad x \in C \tag{2}$$

Reciprocally, if F is differentiable and the Jacobienne matrix $\nabla F(x)$ is symmetric $\forall x \in C$, then we can associate to (1) an equivalent differentiable optimization problem of type (2). The difficulty arises in the case of asymmetric problems (very frequent in practice).

Auslender [1] was of the first having proposed an encouraging answer but with restricted hypoteses particularly for the numerical aspect with the help of the gap function: $g(x) = y \in C \max \langle F(x), x - y \rangle$
Thus (1) is equivalent to:

$$\min g(x) \qquad subject \quad to \quad x \in C \tag{3}$$

Obviously g is not différentiable. Auslender shows that it is it if C is strongly convex i.e ($\forall\ x_1, x_2 \in C\ (x_1 \neq x_2)$, $\forall \lambda \in \left]0,1\right[$,$\exists\ r > 0$ such as:
$B((1-\lambda)x_1 - \lambda x_2, r) \subseteq C$), difficult property to have in practice.

Thus, we can say that the contribution of Auslender is interesting in theory but certainly not in practice, the assumption of strong convexity of C is too restrictive, what encouraged the researchers to develop other functions of which the differentiability depends on F and not on C. Let us note, however, that the problem of associate to asymmetric variational inequality problem (1) a practical differentiable optimization problem remained open question hang several years.

Also, even work of Auslender does not have numerical impact, they constitute with our direction a starting reference for all the later development , in the occurrence the method of Fukushima which represents mainly and even the results obtained by Auslender especially in theory. This method will be the object of following section.

3. Method of Fukusima

The principal idea consists in building a differentiable optimization problem equivalent to 1, what constitutes the interest of this method.

3.1 Description of the method

Let $F : R^n \longrightarrow R^n$ a differentiable mapping, $C \subseteq R^n$ is noempty closed convex set , and $x \in R^n$ be temporarily fixed and consider the following problem:

$$\left\{ y \in C \max \left[\langle F(x), x - y \rangle - \frac{1}{2} \|y - x\|_G^2 \right] \right. \tag{4}$$

Proposition 3.1 The problem (4) is equivalent to the following problem:

$$\left\{ -y \in C \min \frac{1}{2} \left\| y - \left(x - G^{-1} F(x) \right) \right\|_G^2 \right. \tag{5}$$

Proof. see [4]

Let us note that the solution y_x of the problem (4) is nothing other than the ortogonal projection of the element $(x - G^{-1}F(x))$ onto the set C with respect to $\|.\|_G$. From where the existence and the uniquness of y_x (theorem traditional of projection). However we can prove the existence and the uniqueness of the solution for (5) by noticing that the objective is strongly convex in y and C is noempty closed, The result is a traditional consequence of Wieirstrass's theorem. Thus for each $x \in R^n$ there exists a single y_x what makes it possible to define the mapping:

$H : R^n \longrightarrow R^n$

$x \longmapsto y_x = H(x) = \Pr oj_{G,C} \left(x - G^{-1} F(x) \right)$

Proposition 3.2 $\bar{x}$ is a solution for the variational inequality problem (1) $\Leftrightarrow \bar{x} = H(\bar{x})$.

Now we state the optimization problem equivalent to the variational inequality problem (1) and the optimization problem of Fukushima [3].

3.2 equivalent optimization problem

Let the following optimization problem

$$\min f(x) \qquad subject \quad to \quad x \in C \tag{6}$$

Where $f : R^n \longrightarrow R$ is defined by:
$f(x) = \text{-}\langle F(x), H(x) - x\rangle - \frac{1}{2}\|H(x) - x\|_G^2$

It is about the objective function of (4) in which y is replaced there by its value.What correspond exactly with the function of Auslender increased by a quadratic term. The following theorem established the equivalence between the problem (1) and the (6) .

Theorem 3.1 [3]
Let f the function defined in (6) . Then $f(x) \geq 0$ for all $x \in C$, and $f(x) = 0$ if and only if x solves the optimization problem (6)
Proof. Again see [3] .

Remark. The function of Fukushima has the same properties as the operator F
in particular we have:
1. F continuous $\Longrightarrow f$ continuous.
2. $F \in C^1(R^n) \Longrightarrow \in C^1(R^n)$
in particular we have:

$$\nabla f(x) = F(x) + (\nabla F(x) - G)(H(x) - x) \tag{7}$$

What in favour of the aspect practises contrary to the Auslender's result, the function f is not necessarily convex, which involves difficulties if the problem (6) has local minimum or only stationary points.These difficulties can be overcome with the help of this result [3] .

Theorem 3.2 [3]
Assume that the mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable the Jacobian $\nabla F(x)$ is positive definite for all $x \in C$. If x is a stationary point of the problem (6) , i.e.,

$$\langle \nabla f(x), x - y\rangle\ forallx \in C$$

then x is a global optimal solution of (6), hence $\bar{x}$ is a solution of the problem (1) .
Proof. See [3] .

3.3 **Principle of the method**
It means to solve (1) instead of (6) by a descent method, where the direction in x is given by:

$$d = H(x) - x$$

Proposition 3.3 Assume that the mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable and the Jacobian $\nabla F(x)$ is positive definite for all $x \in C$ for each $x \in C$, then the vector $d = H(x) - x$ satisfies the descent conditions:

$$\langle \nabla f(x), d \rangle < 0$$

Proposition 3.4
F is strongly monotone $\Longrightarrow d = H(x) - x$ is a descent direction.

Now, we describe the:

3.3 **Basic Algorithm**

Step0: Data an initialization : Let $\varepsilon > 0$, $G-$ $n \times n$ symetric positive definite matrix, $x^0 \in C$, and $k := 0$;
Step1: compute the direction from: $d_k = H\left(x^k\right) - x^k$ such that
$H\left(x^k\right) = \Pr oj_{C,G}\left(x^k - G^{-1}F\left(x^k\right)\right)$;
While $\|d_k\| > \varepsilon$ do:

Step2: compute the steplength: $t_k = \arg\min_{t \in [0,1]} f(x^k + td_k)$.

Step3: compute the following iteration: $x^{k+1} = x^k + t_k d_k$, $k = k + 1$ and go to **Step1**.

For the calculation of steplength t_k Fukushima the consider the following Armijo-type $f(x^k + td_k) \leqslant f(x^k) - \lambda\beta^k \left\|d^k\right\|^2$, such that $\lambda > 0$ and $0 < \beta < 1$, then t_k the first β^k the last inequality.

Theorem 3.3 of convergence [3]**.** Suppose that is C compact and the mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable and strongly monotone, suppose also that $F(x)$ and $\nabla F(x)$ are Lipsschitz continuous on C, and let the sequence generated by the iteration $x^{k+1} = x^k + t_k d_k$, $k = 0, 1, 2...$,then for any starting point $x^0 \in c$, $\left\{x^k\right\}$lies in C and converges to the unique solution of the problem (1).

3.4 **Difficulties of the algorithm** [4]

The method of Fukushima includes two operations particularly difficult to realize for unspecified convex, the initialization to start the algorithm and the calculation of the projection which intervenes in the compute of the direction.

4. **Implementation and experimental results**

In this section, we present some numerical experiments applied to different classes problems given in [4]. Our algorithms were programmed in Pascal 7, the tolerance is $\varepsilon = 10^{-4}$. we display the following quantities iter means iterations, size is the dimension of the problem, and Pr denotes the problem.
The choice of the matrix G used in the projection and the steplength have a great influence on the performance of the algorithm. Here, G is chosen as a diagonal matrix (the easier case). For the steplength, we have adopted two strategies: the constant step of type fixed point ($lessexpensive$) ($Conststep$) and the variable step" rule of Armijo" ($Varstep$).

1- Nonlinear system of equations
In this case $C = R^n$,then the descent direction in x^k is given by $d^k = -G^{-1}F\left(x^k\right)$.

Table 1

Problem[4]	Size	Iter k		Time	
		$Const\ \ step$	Var step	$Const\ \ step$	Var step
1	4	5	22	0.27	0.39
2	4	35	37	0.24	0.33
3	20	41	29	0.38	0.38

2- Complementarity problem including the minimization of a differentiable function on R_+^n
If $C = R_+^n$, then the problem (1) is defined by:

$$\left\{ \begin{array}{c} finding\ \overline{x} \in R_+^n\ \ such\ \ that : \} \\ F\left(\overline{x}\right) \geqslant 0;\ \ \overline{x}^t F\left(\overline{x}\right) = 0 \end{array} \right.$$

is called the complementarity problem. Thus the descent direction in x^k is given by:

$$d^k = \left\{ \begin{array}{ll} -x^k & six^k - G^{-1}F\left(x^k\right) \leqslant 0 \\ -G^{-1}F\left(x^k\right) & six^k - G^{-1}F\left(x^k\right) \geqslant 0 \end{array} \right.$$

Table 2

Probl	Size	Iter k		Time	
		$Const\ \ step$	Var step	$Const\ \ step$	Var step
1	3	15	27	0.22	0.24
2	4	34	36	0.22	0.33
3	6	6	38	0.22	0.32
4	9	8	8	0.38	0.54
5	20	19	47	0.33	0.44

3- Variational inequality with hypercube

When ce take $C = \prod_{i=1}^{n} [a_i, b_i]$ $(hypercube)$then we calculate the descent direction in x^k as following:

$$d^k = \begin{cases} a_i - x^k & si x^k - G^{-1}F\left(x^k\right) \leq a_i \\ -G^{-1}F\left(x^k\right) & si\ a_i < x^k - G^{-1}F\left(x^k\right) < b_i \\ b_i - x^k & si \quad x^k - G^{-1}F\left(x^k\right) \geqslant b_i \end{cases}$$

Table 3

Problem[4]	Size	Iter k		Time	
		$Const\ \ step$	Pas var	$Const\ \ step$	Pas var
1	4	9	29	0.22	0.25
2	4	141	47	0.33	0.39
3	9	19	40	0.38	0.39
4	20	39	40	0.39	0.43

5. **GENERAL COMMENTS AND CONCLUSION**

The techniques of optimization constitute a suitable tool for the resolution of the variational inequality problem and have the necessary ingredients to progress more in the numerical aspect.

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