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Legendre Multi-Wavelets Direct Method for Linear Integro-Differential Equations

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Abstract

We use the continuous Legendre multi-wavelets on the interval $[0, 1)$ to solve the linear integro-differential equation. To do so, we reduced the problem into a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. Comparison has been done with two other methods and it shows that the accuracy of these results are higher than them.

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1 Introduction

In the recent years, there has been an increase usage among scientists and engineers to apply wavelet technique as well as a numerical solution to solve both linear and nonlinear problems. The main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations. As overview of this method can be found in [4, 5]. The approach is based on converting the underlying differential equations into an integral equations through integration, approximating various signals involved in the

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equation by truncated orthogonal series $\Psi(t)$ and using the operational matrix of integration P, to eliminate the integral operations. In this paper, we use linear Legendre multi-wavelets on the interval $[0, 1)$ to solve the linear integro-differential equation[4]. Numerical examples are provided to show the advantages of using linear Legendre multi-wavelets over the methods in[2, 3].

2 Linear Legendre multi-wavelets and its properties

2.1 Wavelets and linear Legendre multi-wavelets

Wavelet constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [1]

$$\psi_{a,b}(t) = |a|^{-1}\,\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0.$$

If we restrict the parameters a and b to the discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, where $a_0 > 1$, $b_0 > 0$ and n, and k are positive integers, we have the following of discrete wavelets:

$$\psi_{n,k}(t) = |a|^{\frac{k}{2}}\,\psi\left(a_0^k t - nb_0\right),$$

which from a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{n,k}(t)$ forms an orthonormal basis [1].

The linear Legendre multi-wavelets are described in [4]. Khellat [4]used this kind of wavelets to solve optimal control problem. For constructing the linear Legendre multi-wavelets, at first we describe scaling functions $\phi_0(x)$ and $\phi_1(x)$ as follows:

$$\phi_0(t) = 1, \quad \phi_1(t) = \sqrt{3}\,(2t-1), \quad 0 \leq t < 1.$$

Now let $\psi^0(t)$ and $\psi^1(t)$ be the corresponding mother wavelets, then by MRA and applying suitable conditions [4] on $\psi^0(t)$ and $\psi^1(t)$ the explicit formula for linear Legendre mother wavelets will obtain as

$$\psi^0(t) = \begin{cases} -\sqrt{3}\,(4t-1), & 0 \leq t < \frac{1}{2}, \\ \sqrt{3}\,(4t-3), & \frac{1}{2} \leq t < 1, \end{cases} \tag{1}$$

$$\psi^1(t) = \begin{cases} 6t-1, & 0 \leq t < \frac{1}{2}, \\ 6t-5, & \frac{1}{2} \leq t < 1, \end{cases} \tag{2}$$

and the family $\{\psi_{kn}^j\} = \left\{2^{\frac{k}{2}}\psi^j\left(2^k t - n\right)\right\}$, k is any nonnegative integer, $n = 0, 1, \cdots, 2^k - 1$ and $j = 0, 1$, forms an orthonormal basis for $L^2(\mathbb{R})$.

3 Linear Legendre multi-wavelets operational matrix of integration

Let us,

$$\Psi(t) = \Big[\phi_0(t), \phi_1(t), \psi^0_{00}(t), \psi^1_{00}(t), \cdots, \psi^0_{M0}(t), \psi^0_{M1}(t), \cdots, \psi^0_{M(2^M-1)}(t), \cdots \tag{3}$$

$$\psi^1_{M0}(t), \psi^1_{M1}(t), \cdots \psi^1_{M(2^M-1)}(t)\Big]^T.$$

Where M is a nonnegative integer. The integration of the vector $\Psi(t)$ defined in (3) can be obtained as

$$\int_0^t \Psi(t)\,dt \simeq P\Psi(t), \tag{4}$$

Where P is a $2^{M+2} \times 2^{M+2}$ matrix given by[4]

$$P = \begin{bmatrix} P_{2^{M+1}\times 2^{M+1}} & Q_{2^{M+1}\times 2^{M+1}} \\ -Q^T_{2^{M+1}\times 2^{M+1}} & R_{2^{M+1}\times 2^{M+1}} \end{bmatrix} \tag{5}$$

The submatrix $P_{2^{M+1}\times 2^{M+1}}$ in equation (5) is generated by

$$P_{2\times 2} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & 0 \end{bmatrix} \tag{6}$$

and the submatrix $R_{2^{M+1}\times 2^{M+1}}$ is generated by the formula

$$R_{2^{M+1}\times 2^{M+1}} = \frac{\sqrt{3}}{24} \times \frac{1}{2^M} \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \tag{7}$$

for $M = 0, 1, 2, \cdots$, where O and I are $2^M \times 2^M$ zero and identity matrix, respectively. To generate the submatrix $Q_{2^{M+1}\times 2^{M+1}}$ $(M = 1, 2, \cdots)$, suppose it has the block matrix form

$$Q_{2^{M+1}\times 2^{M+1}} = \begin{bmatrix} S & O \\ T & O \end{bmatrix} \tag{8}$$

where S and T are $2^M \times 2^M$ matrices and O is zero matrix. To characterize S, let $Q_{2^M\times 2^M}$ has the form

$$Q_{2^M\times 2^M} = [C_1\ C_2 \cdots C_{2^{M-1}}\ O\ O \cdots O], \tag{9}$$

where $C_i (1 \le i \le 2^{M-1})$ and O are $2^M \times 1$ column matrices. Then S can be obtained by

$$S = \frac{\sqrt{2}}{8} \left[C_1 \; C_1 \; C_2 \; C_2 \cdots C_{2^{M-1}} \; C_{2^{M-1}} \right]. \tag{10}$$

Hence, we need $Q_{2\times 2}$ which is the following matrix:

$$Q_{2\times 2} = \frac{1}{8} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{11}$$

To obtain matrix T, we begin by

$$T_{2\times 2} = \frac{\sqrt{2}}{23} \begin{bmatrix} -1 & 1 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \tag{12}$$

and for $M \ge 2$, consider

$K_1 = \frac{1}{2} \begin{bmatrix} I & O \\ O & O \end{bmatrix}$,

$K_2 = \frac{1}{2} \begin{bmatrix} O & I \\ O & O \end{bmatrix}$,

$K_3 = \frac{1}{2} \begin{bmatrix} O & O \\ I & O \end{bmatrix}$,

$K_4 = \frac{1}{2} \begin{bmatrix} O & O \\ O & I \end{bmatrix}$,

where I and O are identity and zero matrices of dimension $2^{M-2} \times 2^{M-2}$, respectively. Now if we put $H = T_{2^{M-1} \times 2^{M-1}}$, then T can be characterized by the following formula:

$$T = \begin{bmatrix} K_1 H & K_3 H \\ K_2 H & K_4 H \end{bmatrix} \tag{13}$$

Hence, the matrix P in equation (5) is obtained by equations (7) and (8).

4 Linear Legendre multi-wavelets direct method

In this section we have used the Linear Legendre multi-wavelets to approximate the functions with one variable or two variables then by substituting of these approximations in the linear integro-differential equation, the equation has transformed into a system of algebraic equations.

4.1 Function approximation

A function $f(t)$ defined over $[0,1)$ may be expanded as

$$f\left(t\right)=f_0\phi_0\left(t\right)+f_1\phi_1\left(t\right)+\sum_{k=0}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{\infty}f_{kn}^{j}\psi_{kn}^{j}\left(t\right), \tag{14}$$

where

$$f_0=\langle f(t),\phi_0(t)\rangle,\quad f_1=\langle f(t),\phi_1(t)\rangle,\quad f_{kn}^{j}=\langle f(t),\psi_{kn}^{j}(t)\rangle. \tag{15}$$

In equation (15), $\langle .,.\rangle$ denoting the inner product. If the infinite series of equation (14) is truncated, then it can be written as

$$f\left(t\right)\approx f_0\phi_0\left(t\right)+f_1\phi_1\left(t\right)+\sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1}f_{kn}^{j}\psi_{kn}^{j}\left(t\right)=F^T\Psi\left(t\right), \tag{16}$$

Where $\Psi\left(t\right)$ defined in (3) and F given by

$$F=\left[f_0,f_1,f_{00}^0,f_{00}^1,\cdots,f_{M0}^0,f_{M1}^0,\cdots,f_{M(2^M-1)}^0,\cdots,f_{M0}^1,f_{M1}^1,\cdots f_{M(2^M-1)}^1\right]^T, \tag{17}$$

4.2 Solving the linear integro-differential equations

Consider the following integro-differential equation:

$$\begin{cases} y'\left(t\right)=\int_0^1 k(t,s)y(s)ds+y(t)+x(t), \\ y(0)=y_0, \end{cases} \tag{18}$$

where $x\in L^2\left[0,1\right),\ k\in L^2\left(\left[0,1\right)\times\left[0,1\right)\right)$ and y is an unknown function.
If we approximate x,y' and K by (14)-(17) as follows:

$$x\left(t\right)=X^T\Psi\left(t\right),y'\left(t\right)=Y'^T\Psi\left(t\right),y\left(0\right)=Y_0^T\Psi\left(t\right),k\left(t,s\right)=\Psi^T\left(t\right)K\Psi\left(s\right).$$

Then

$$\begin{aligned} y\left(t\right)&=\int_0^t y'(s)ds+y(0)\simeq\int_0^t Y'^T\Psi\left(s\right)ds+Y_0^T\Psi\left(t\right)\\ &\simeq Y'^TP\Psi\left(t\right)+Y_0^T\Psi\left(t\right)=\left(Y'^TP+Y_0^T\right)\Psi\left(t\right). \end{aligned}$$

With substituting in (18) we have

$$\Psi^T\left(t\right)Y'=\int_0^1\Psi^T\left(t\right)K\Psi\left(s\right)\Psi^T\left(s\right)\left(P^TY'+Y_0\right)ds+\Psi^T\left(t\right)\left(P^TY'+Y_0\right)+\Psi^T\left(t\right)X$$

$$\Rightarrow \Psi^T(t)\, Y' = \Psi^T(t)\, K \left(P^T Y' + Y_0\right) + \Psi^T(t) \left(P^T Y' + Y_0\right) + \Psi^T(t)\, X$$

$$\Rightarrow \left(I - KP^T - P^T\right) Y' = KY_0 + Y_0 + X.$$

By solving this linear system we can find the vector Y', so

$$Y^T = Y'^T P + Y_0^T \Rightarrow y(t) \simeq Y^T \Psi(t)\,.$$

5 Numerical examples

In two examples below $|y - \tilde{y}|$ shows the absolute value of difference between the exact solution with the numerical solution. Table 1 and 2 show the values of $|y - \tilde{y}|$ corresponding to linear Legendre multi-wavelets method (LLMW) and other two methods in[2, 3] for example 1 and 2 respectively.

Example 5.1. *Consider the integro-differential equation given in [2]*

$$\begin{cases} y'(t) = \int_0^1 tsy(s)ds + y(t) - \frac{4}{3}t + 1, \\ y(0) = 0, \end{cases} \tag{19}$$

The exact solution for this problem is $y(t) = t$.
We solve (19) by using the method with $M = 1$ in section 4.

Table 1
Comparison of Linear Legendre multi-wavelets method with the methods in[2, 3]

x	$\|y - \tilde{y}\|$		
	LLMW	Method in [2]	Method in [3]
0.1	2.77555756e-17	2.17942375e-04	1.66666667e-03
0.2	5.55111512e-17	6.38548213e-04	6.09388620e-03
0.3	0.00000000e+00	7.91370487e-04	1.32017875e-02
0.4	0.00000000e+00	2.15586005e-02	2.29140636e-02
0.5	0.00000000e+00	4.99358429e-03	3.51578404e-02
0.6	0.00000000e+00	2.21728815e-02	6.69648304e-02
0.7	0.00000000e+00	1.05645449e-04	7.12430514e-02
0.8	0.00000000e+00	1.43233681e-03	8.63983845e-02
0.9	0.00000000e+00	2.07747461e-02	1.08103910e-01

Example 5.2. *Consider the integro-differential equation*

$$\begin{cases} y'(t) = \int_0^1 ty(s)ds + y(t) + e^t - t \\ y(0) = 0, \end{cases} \tag{20}$$

The exact solution for this problem is $y(t) = te^t$.
We solve (5.2) by using the method with $M = 2$ in section 4.

Table 2
Comparison of Linear Legendre multi-wavelet method with the methods in[2, 3]

x	$\|y - \tilde{y}\|$		
	LLMW	Method in [2]	Method in [3]
0.1	5.29804602e-03	1.34917637e-03	1.00118319e-02
0.2	1.88094590e-04	1.15960044e-03	2.78651355e-02
0.3	3.12963308e-04	5.67152531e-03	5.08730892e-02
0.4	5.57099653e-03	5.593105645e-02	7.5535631e-02
0.5	5.88044588e-03	1.32330751e-02	9.71888592e-02
0.6	1.60682888e-04	4.3928772e-02	1.09551714e-01
0.7	3.09086926e-03	1.41201624e-02	1.04133232e-01
0.8	3.52966666e-03	1.34514117e-02	6.94512700e-02
0.9	5.90031210e-04	1.32045209e-02	1.00034260e-02

6 Conclusion

The linear Legendre multi-wavelets has been applied for solving integro-differential equations by reducing an integral equation into a system of algebraic equations. The examples expressed that the accuracy of this method is better than the CAS wavelets and the method in [3].

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