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Some Modified Extragradient Approximation Methods for Variational Inequality Problems and Equilibrium Problems of Nonexpansive Mappings[1]

Rabian Wangkeeree and Uthai Kamraksa

Department of Mathematics, Faculty of Science
Naresuan University, Phitsanulok 65000, Thailand
rabianw@nu.ac.th

Abstract. In this paper, we introduce a modified extragradient approximation method for finding the common element of the set of fixed points of nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality. We show that the sequence converges strongly to a common element of the above three sets under some parameters controlling conditions. Our results extend and improve a recent result of Bnouhachem, Aslam Noor, and Hao [2] [A. Bnouhachem, M. Aslam Noor and Z. Hao, Some new extragradient iterative methods for variational inequalities, Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.014] and many other.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively and let C be a closed convex subset of H. Let F be a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : C\times C \longrightarrow \mathbb{R}$ is to find $x\in C$ such that

$$\phi(x,y)\geq 0 \text{ for all } y\in C. \tag{1.1.1}$$

The set of solutions of (1.1.1) is denoted by $EP(\phi)$. Given a mapping $T : C \longrightarrow H$, let $\phi(x,y)=\langle Tx, y-x\rangle$ for all $x,y\in C$. Then $z\in EP(\phi)$ if and only if $\langle Tz, y-z\rangle \geq 0$ for all $y\in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1.1). In

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1997, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to initial data when $EP(\phi)$ is nonempty and proved a strong convergence theorem.

Let $A : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0 \quad \forall v \in C. \tag{1.1.2}$$

The variational inequality has been extensively studied in the literature. See, e.g. [16, 17] and the references therein. A mapping A of C into H is called α-inverse-strongly monotone [3, 7] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2 \quad \forall u, v \in C.$$

It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\| \quad \forall u, v \in C.$$

We denote by $F(S)$ the set of fixed points of S. For finding an element of $F(S) \cap VI(A, C)$, under the assumption that a set $C \subseteq H$ is nonempty, closed and convex, a mapping $S : C \longrightarrow C$ is nonexpansive and a mapping $A : C \longrightarrow H$ is α-inverse-strongly monotone, Takahashi and Toyoda [14] introduced the following iterative scheme:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \;\; n \geq 1 \end{cases} \tag{1.1.3}$$

where $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap VI(A, C) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.1.3) converges weakly to some $z \in F(S) \cap VI(A, C)$. Recently, Iiduka and Takahashi [5] proposed another iterative scheme as following

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \;\; n \geq 1 \end{cases} \tag{1.1.4}$$

where A is an α-cocoerceive map, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\lambda_n\} \subseteq (0, 2\alpha)$ satisfy some parameters controlling conditions. They showed that, if $F(S) \cap VI(A, C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.1.4) converges strongly to some $z \in F(S) \cap VI(A, C)$.

Based on the so-called extragradient method of Korpelevich [6], Nadezhkina and Takahashi [8] introduced an iterative scheme for finding an element of $F(S) \cap VI(A, C)$ and the weak convergence theorem is presented. Moreover, Zeng and Yao [18] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$ and obtained the weak convergence theorem for such schemes. Very recently, Bnouhachem, Aslam Noor, and Hao [2] introduced the following new extragradient iterative method for finding an element of $F(S) \cap VI(A, C)$. Let C be a closed convex subset of a real Hilbert space H, A an α-inverse strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\}$ be given by

$$(1.1.5) \quad \begin{cases} x_1 \in C, u \in C \text{ chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n u + (1 - \alpha_n) P_C(x_n - \lambda_n A y_n)), \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.1.5) converges strongly to a common element of $F(S) \cap VI(A, C)$.

On the other hand, Takahashi and Takahashi [13] considered the so-called viscosity approximation method for finding a common element of the solution set of the equilibrium problem (1.1.1) and the set of fixed points of a nonexpansive mapping in a real Hilbert space and also obtained a strong convergence theorem under certain appropriate conditions imposed on the parameters.

In this paper, we introduce a modified extragradient approximation method as follows:

$$\begin{cases} x_1 \in C, u \in C \text{ chosen arbitrary}, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n u + (1 - \alpha_n) P_C(u_n - \lambda_n A y_n)), \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \subseteq [0, 1]$, $\{\lambda_n\} \subseteq (0, 1)$ and $\{r_n\} \subseteq (0, \infty)$ satisfy the some appropriate conditions, A is a monotone L-Lipschitz continuous mapping of C into H. We prove that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of fixed points of a nonexpansive map, the set of solutions of the classical variational inequality (1.1.2) for a monotone L-Lipschitz continuous mapping, and the set of solutions of the equilibrium problem (1.1.1). The results obtained in this paper extend and improve the recent ones announced by Bnouhachem, Aslam Noor, and Hao [2], Takahashi and Takahashi [13], Zeng and Yao [18], Iiduka and Takahashi [5] and many others.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$(2.2.1) \qquad \langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$(2.2.2) \qquad \langle x - P_C x, y - P_C x \rangle \leq 0,$$

$$\|x-y\|^2 \geq \|x-P_Cx\|^2+\|y-P_Cx\|^2 \tag{2.2.3}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(A,C) \Leftrightarrow u = P_C(u-\lambda Au), \lambda > 0. \tag{2.2.4}$$

A set-valued mapping $T : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \longrightarrow 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let N_Cv be the normal cone to C at $v \in C$, i.e., $N_Cv = \{w \in H : \langle u-v, w\rangle \geq 0, \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_Cv, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [11].

Throughout this paper, we shall use the notations: " $\rightharpoonup$ " and "$\rightarrow$" to stand for the weak convergence and strong convergence, respectively.

It is also known that H satisfies Opia's condition [9], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| \leq \liminf_{n\longrightarrow\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(2) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.2. [10] *Let $(E, \langle ., .\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.3. ([12]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0$.*

Lemma 2.4. ([15]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,\ n \geq 1$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$
(2) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $\phi : C \times C \longrightarrow \mathbb{R}$, let us assume that ϕ satisfies the following conditions:

(A1) $\phi(x,x) = 0$ for all $x \in C$;
(A2) ϕ is monotone, i.e., $\phi(x,y) + \phi(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \longrightarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y)$;
(A4) for each $x \in C, y \mapsto \phi(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1]

Lemma 2.5. [1] *Let C be a nonempty closed convex subset of H and let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [4].

Lemma 2.6. [4] *Assume that $\phi : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(i) *T_r is single- valued;*
(ii) *T_r is firmly nonexpansive,i.e.,for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
(iii) *$F(T_r) = EP(\phi)$;*
(iv) *$EP(\phi)$ is closed and convex.*

3. Main Results

Now we are in a position to state and prove the main result in this paper.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let ϕ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), $A : C \longrightarrow H$ a monotone L-Lipschitz continuous mapping and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A,C) \cap EP(\phi) \neq \emptyset$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$(3.3.1) \quad \begin{cases} x_1, u \in C \text{ chosen arbitrary}, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S(\alpha_n u + (1-\alpha_n) P_C(u_n - \lambda_n A y_n)), \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \subseteq [0,1]$, $\{\lambda_n\} \subseteq (0,1)$ and $\{r_n\} \subseteq (0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(C2) $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$,

(C3) $\lim_{n\longrightarrow\infty}\lambda_n = 0$,
(C4) $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $P_{F(S)\cap VI(A,C)\cap EP(\phi)}u$.

Proof. We divide the proof into several steps as follows.

Step 1. $\{x_n\}$ is bounded. Indeed, put $t_n = P_C(u_n - \lambda_n Ay_n)$ for all $n \geq 1$. Let $x^* \in F(S)\cap VI(A,C)\cap EP(\phi)$. From (2.2.4) we have $x^* = P_C(x^* - \lambda_n Ax^*)$. Also it follows from from (2.2.3) that

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - \lambda_n Ay_n - x^*\|^2 - \|u_n - \lambda_n Ay_n - t_n\|^2 \\
&= \|u_n - x^*\|^2 - 2\lambda_n\langle Ay_n, u_n - x^*\rangle + \lambda_n^2\|Ay_n\|^2 - \|u_n - t_n\|^2 \\
&\quad + 2\lambda_n\langle Ay_n, u_n - t_n\rangle - \lambda_n^2\|Ay_n\|^2 \\
&= \|u_n - x^*\|^2 + 2\lambda_n\langle Ay_n, x^* - t_n\rangle - \|u_n - t_n\|^2 \\
&= \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n - Ax^*, x^* - y_n\rangle \\
&\quad + 2\lambda_n\langle Ax^*, x^* - y_n\rangle + 2\lambda_n\langle Ay_n, y_n - t_n\rangle. \qquad (3.3.2)
\end{aligned}$$

Since A is monotone and x^* is a solution of the variational inequality problem $VI(A,C)$, we have

$$\langle Ay_n - Ax^*, x^* - y_n\rangle \leq 0 \text{ and } \langle Ax^*, x^* - y_n\rangle \leq 0.$$

This together with (3.3.2) implies that

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - x^*\|^2 - \|(u_n - y_n) - (y_n - t_n)\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle \\
&\quad - \|y_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle. \qquad (3.3.3)
\end{aligned}$$

From (2.2.2), we have

$$\langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle \leq 0, \qquad (3.3.4)$$

so that

$$\begin{aligned}
\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &= \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\leq \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\leq \lambda_n\|Au_n - Ay_n\|\|t_n - y_n\| \\
&\leq \lambda_n L\|u_n - y_n\|\|t_n - y_n\|. \qquad (3.3.5)
\end{aligned}$$

Hence it follows from (3.3.3) and (3.3.5) that

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n L\|u_n - y_n\|\|t_n - y_n\| \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n L(\|u_n - y_n\|^2 + \|y_n - t_n\|^2) \\
&= \|u_n - x^*\|^2 + (\lambda_n L - 1)\|u_n - y_n\|^2 + (\lambda_n L - 1)\|y_n - t_n\|^2. \qquad (3.3.6)
\end{aligned}$$

Since $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$, we may assume, without loss of generality, that $\lambda_n \leq \frac{1}{L}$ for all $n \in \mathbb{N}$. Hence it follows from (3.3.6) that

$$\|t_n - x^*\| \leq \|u_n - x^*\|. \tag{3.3.7}$$

Observe that

$$\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \leq \|x_n - x^*\|,$$

and hence

$$\|t_n - x^*\| \leq \|x_n - x^*\|. \tag{3.3.8}$$

Setting $w_n = \alpha_n u + (1-\alpha_n)t_n$, we can calculate

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\beta_n x_n + (1-\beta_n)Sw_n - x^*\| \\
&= \|\beta_n(x_n - x^*) + (1-\beta_n)(Sw_n - x^*)\| \\
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\|w_n - x^*\| \\
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)[\alpha_n\|u - x^*\| + (1-\alpha_n)\|t_n - x^*\|] \\
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\alpha_n\|u - x^*\| + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\| \\
&\leq (1-(1-\beta_n)\alpha_n)\|x_n - x^*\| + (1-\beta_n)\alpha_n\|u - x^*\| \\
&\leq \max\{\|x_n - x^*\|, \|u - x^*\|\}.
\end{aligned}$$

It follows from induction that

$$\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \|u - x^*\|\right\}, n \geq 1. \tag{3.3.9}$$

Therefore, $\{x_n\}$ is bounded, so are $\{u_n\}, \{Au_n\}, \{y_n\}$ and $\{Ay_n\}$.

Step 2. $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$. Indeed, we observe that for any $x, y \in C$,

$$\begin{aligned}
\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 &= \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\
&= \|x-y\|^2 - 2\lambda_n\langle x-y, Ax-Ay\rangle + \lambda_n^2\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 + \lambda_n^2 L^2\|x-y\|^2 \\
&= (1+\lambda_n^2L^2)\|x-y\|^2, \qquad (3.3.10)
\end{aligned}$$

which implies that

$$\|(I-\lambda_n A)x - (I-\lambda_n A)y\| \leq (1+\lambda_n L)\|x-y\|. \tag{3.3.11}$$

Thus

$$\begin{aligned}
\|t_{n+1} - t_n\| &= \|P_C(u_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(u_n - \lambda_n Ay_n)\| \\
&\leq \|u_{n+1} - \lambda_{n+1}Ay_{n+1} - (u_n - \lambda_n Ay_n)\| \\
&= \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_{n+1}Au_n) \\
&\quad + \lambda_{n+1}(Au_{n+1} - Ay_{n+1} - Au_n) + \lambda_n Ay_n\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_{n+1}Au_n)\| \\
&\quad + \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n\|Ay_n\| \\
&\leq (1+\lambda_{n+1}L)\|u_{n+1} - u_n\| \\
&\quad + \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n\|Ay_n\|. \qquad (3.3.12)
\end{aligned}$$

On the other hand, we note that $u_n = T_{r_n} x_n$, $u_{n+1} = T_{r_{n+1}} x_{n+1}$, and

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C \tag{3.3.13}$$

and

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \text{ for all } y \in C. \tag{3.3.14}$$

Putting $y = u_{n+1}$ in (3.3.13) and $y = u_n$ in (3.3.14), we obtain

$$\phi(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$\phi(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A2) we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Hence, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\}, \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M, \end{aligned} \tag{3.3.15}$$

where $M = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. It follows from (3.3.12) and the last inequality that

$$\begin{aligned} \|t_{n+1} - t_n\| &\leq (1 + \lambda_{n+1}L)\|x_{n+1} - x_n\| + (1 + \lambda_{n+1}L)\frac{1}{c}|r_{n+1} - r_n|M \\ &\quad + \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n\|Ay_n\|. \end{aligned} \tag{3.3.16}$$

Using (3.3.16), we have

$$\begin{aligned} \|Sw_{n+1} - Sw_n\| &\leq \|w_{n+1} - w_n\| \\ &= \|\alpha_{n+1}u + (1 - \alpha_{n+1})t_{n+1} - \alpha_n u - (1 - \alpha_n)t_n\| \\ &\leq |\alpha_{n+1} - \alpha_n|\|u\| + \alpha_{n+1}\|t_{n+1}\| + \alpha_n\|t_n\| + \|t_{n+1} - t_n\| \\ &\leq |\alpha_{n+1} - \alpha_n|\|u\| + \alpha_{n+1}\|t_{n+1}\| + \alpha_n\|t_n\| \end{aligned}$$

$$+ (1+\lambda_{n+1}L)\|x_{n+1}-x_n\| + (1+\lambda_{n+1}L)\frac{1}{c}|r_{n+1}-r_n|M$$
$$+ \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n\|Ay_n\|,$$

which implies that

$$\begin{aligned}\|Sw_{n+1}-Sw_n\| - \|x_{n+1}-x_n\| \leq\ & |\alpha_{n+1}-\alpha_n|\|u\| + \alpha_{n+1}\|t_{n+1}\| + \alpha_n\|t_n\| \\ & + \lambda_{n+1}L\|x_{n+1}-x_n\| + (1+\lambda_{n+1}L)\frac{1}{c}|r_{n+1}-r_n|M \\ & + \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n\|Ay_n\|.\end{aligned}$$

This together with (C1)-(C3) imply that

$$\limsup_{n\longrightarrow\infty}(\|Sw_{n+1}-Sw_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Hence, by Lemma 2.3, we obtain

$$\lim_{n\longrightarrow\infty}\|Sw_n - x_n\| = 0. \tag{3.3.17}$$

It then follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|Sw_n - x_n\| = 0. \tag{3.3.18}$$

By (3.3.15) and (3.3.16), we also have

$$\lim_{n\longrightarrow\infty}\|t_{n+1}-t_n\| = \lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| = 0. \tag{3.3.19}$$

Step 3. $\lim_{n\longrightarrow\infty}\|Sy_n - y_n\| = 0$. Indeed, pick any $x^* \in F(S)\cap VI(A,C)\cap EP(\phi)$, to obtain

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \leq \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle = \langle u_n - x^*, x_n - x^*\rangle \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2).\end{aligned}$$

Therefore, $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. Then, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n(x_n - x^*) + (1-\beta_n)(Sw_n - Sx^*)\|^2 \\ &= \beta_n^2\|x_n - x^*\|^2 + (1-\beta_n)^2\|Sw_n - Sx^*\|^2 + 2\beta_n(1-\beta_n)\langle x_n - x^*, Sw_n - Sx^*\rangle \\ &\leq \beta_n^2\|x_n - x^*\|^2 + (1-\beta_n)^2\|Sw_n - Sx^*\|^2 \\ &\quad + \beta_n(1-\beta_n)(\|x_n - x^*\|^2 + \|Sw_n - Sx^*\|^2) \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|w_n - x^*\|^2 \\ &= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\alpha_n(u - x^*) - (1-\alpha_n)(t_n - x^*)\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|t_n - x^*\|^2\} \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)\|t_n - x^*\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)\{\|x_n - x^*\|^2 - \|x_n - u_n\|^2\} \\ &= \|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 - (1-\beta_n)\|x_n - u_n\|^2, \qquad (3.3.20)\end{aligned}$$

and hence

$$\begin{aligned}(1-\beta_n)\|x_n - u_n\|^2 &\leq (1-\beta_n)\alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq (1-\beta_n)\alpha_n\|u - x^*\|^2 + \|x_n - x_{n+1}(\|x_n - x^*\| + \|x_{n+1} - x^*\|). \qquad (3.3.21)\end{aligned}$$

It now follows from the last inequality, (C1), (C2) and (3.3.18) that

$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0. \tag{3.3.22}$$

Consequently,

$$\begin{aligned}\|y_n-x_n\| &\le \|P_C(u_n-\lambda_n Au_n)-x_n\|\le\|u_n-x_n\|\\ &\quad+\lambda_n\|Au_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty,\end{aligned} \tag{3.3.23}$$

and

$$\begin{aligned}\|y_n-t_n\| &\le \|P_C(u_n-\lambda_n Au_n)-P_C(u_n-\lambda_n Ay_n)\|\\ &\le \lambda_n\|Au_n-Ay_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned} \tag{3.3.24}$$

Thus

$$\|y_n-u_n\| \le \|y_n-x_n\|+\|x_n-u_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty, \tag{3.3.25}$$

$$\|t_n-x_n\| \le \|t_n-y_n\|+\|y_n-x_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.3.26}$$

and

$$\|w_n-t_n\| \le \|\alpha_n+(1-\alpha_n)t_n-t_n\|=\alpha_n\|u-t_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.3.27}$$

Using (3.3.17) and the above inequality, we obtain

$$\begin{aligned}\|Sy_n-y_n\| &\le \|Sy_n-St_n\|+\|St_n-Sw_n\|+\|Sw_n-x_n\|+\|x_n-y_n\|\\ &\le \|y_n-t_n\|+\|t_n-w_n\|+\|Sw_n-x_n\|\\ &\quad+\|x_n-y_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned} \tag{3.3.28}$$

Consequently,

$$\|Sy_n-x_n\| \le \|Sy_n-y_n\|+\|y_n-x_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.3.29}$$

Step 4. $\limsup_{n\longrightarrow\infty}\langle u-q,x_n-q\rangle\le 0$, where $q=P_{F(S)\cap VI(A,C)\cap EP(\phi)}(u)$. Indeed, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle u-q,Sy_n-q\rangle=\lim_{i\longrightarrow\infty}\langle u-q,Sy_{n_i}-q\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup z$. From $\|Sy_n-y_n\|\longrightarrow 0$, we obtain $Sy_{n_i}\rightharpoonup z$. Let us show $z\in EP(\phi)$. Since $u_n=T_{r_n}x_n$, we have

$$\phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\forall y\in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge\phi(y,u_n)$$

and hence

$$\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle\ge\phi(y,u_{n_i}).$$

From $\|u_n - x_n\| \longrightarrow 0, \|x_n - Sy_n\| \longrightarrow 0$, and $\|Sy_n - y_n\| \longrightarrow 0$, we get $u_{n_i} \rightharpoonup z$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$, it follows by (A4) that $0 \geq \phi(y, z)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $\phi(y_t, z) \leq 0$. So, from (A1) and (A4) we have

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, z) \leq t\phi(y_t, y)$$

and hence $0 \leq \phi(y_t, y)$. From (A3), we have $0 \leq \phi(z, y)$ for all $y \in C$ and hence $z \in EP(\phi)$. By the opial's condition, we can obtain that $z \in F(S)$. Next we will show that $z \in VI(A, C)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is the maximal monotone (see [11]). Let $(v, w) \in G(T)$. Since $w - Av \in N_C(v)$ and $y_n \in C$, we have $\langle v - y_n, w - Av \rangle \geq 0$. On the other hand, from $y_n = P_C(u_n - \lambda_n Au_n)$, we have

$$\langle v - y_n, y_n - (u_n - \lambda_n Au_n) \rangle \geq 0 \tag{3.3.30}$$

that is,

$$\langle v - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \rangle \geq 0. \tag{3.3.31}$$

Therefore, we obtian

$$\begin{aligned} \langle v - y_{n_i}, w \rangle &\geq \langle v - y_{n_i}, Av \rangle \geq \langle v - y_{n_i}, Av \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \rangle \\ &= \langle v - y_{n_i}, Av - Au_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &= \langle v - y_{n_i}, Av - Ay_{n_i} \rangle + \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle v - y_{n_i}, Ay_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \rangle \\ &= \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle. \end{aligned} \tag{3.3.32}$$

Noting that $\|y_{n_i} - u_{n_i}\| \longrightarrow 0$ as $n \longrightarrow \infty$ and A is Lipschitz continuous, hence from (3.3.32), we obtain

$$\langle v - z, w \rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in VI(A, C)$. Hence $z \in F(S) \cap VI(A, C) \cap EP(\phi)$. The property of the metric projection implies that

$$\limsup_{n \longrightarrow \infty} \langle u - q, Sy_n - q \rangle = \lim_{i \longrightarrow \infty} \langle u - q, Sy_{n_i} - q \rangle = \langle u - q, z - q \rangle \leq 0. \tag{3.3.33}$$

It follows from the last inequality, (3.3.24), (3.3.27) and (3.3.28) that

$$\limsup_{n \longrightarrow \infty} \langle u - q, w_n - q \rangle \leq 0. \tag{3.3.34}$$

Step 5. $\lim_{n\longrightarrow\infty}\|x_n-q\|=0$. Indeed, using (3.3.8) and Lemma 2.1, we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\beta_n(x_n-q)+(1-\beta_n)(Sw_n-q)\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\|w_n-q\|^2\\ &= \beta_n\|x_n-q\|^2+(1-\beta_n)\|\alpha_n(u-q)+(1-\alpha_n)(t_n-q)\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\left[(1-\alpha_n)\|t_n-q\|^2+2\alpha_n\langle u-q,w_n-q\rangle\right]\\ &\le [1-(1-\beta_n)\alpha_n]\|x_n-q\|^2+2(1-\beta_n)\alpha_n\langle u-q,w_n-q\rangle. \quad (3.3.35)\end{aligned}$$

Setting $\gamma_n=(1-\beta_n)\alpha_n$ and $\delta_n=2(1-\beta_n)\alpha_n$. It is easily seen that $\sum_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\longrightarrow\infty}\frac{\delta_n}{\gamma_n}=\limsup_{n\longrightarrow\infty}2\langle u-q,w_n-q\rangle\le 0$. Applying Lemma 2.4 to (3.3.35), we conclude that $\{x_n\}$ converges strongly to q. Consequently, $\{u_n\}$ and $\{y_n\}$ converge strongly to q. This completes the proof. □

Setting A is an α-inverse strongly monotone of C into H, $\phi(x,y)=0$ for all $x,y\in C$ and $\{r_n\}=1$ in Theorem 3.1 , we have the following result.

Corollary 3.2. [2, Theorem 4.2]) *Let C be a closed convex subset of a real Hilbert space H. Let $A:C\longrightarrow H$ be an α-inverse strongly monotone of C into H, S a nonexpansive mapping of C into itself such that $F(S)\cap VI(A,C)\neq\emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1,u\in C \text{ chosen arbitrary},\\ y_n=P_C(x_n-\lambda_nAx_n),\\ x_{n+1}=\beta_nx_n+(1-\beta_n)S(\alpha_nu+(1-\alpha_n)P_C(x_n-\lambda_nAy_n)),\forall n\ge 1,\end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\subseteq[0,1]$, $\{\lambda_n\}\subseteq(0,1)$ and $\{r_n\}\subseteq(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty}\alpha_n=0,\sum_{n=1}^{\infty}\alpha_n=\infty$,
(C2) $0<\liminf_{n\longrightarrow\infty}\beta_n\le\limsup_{n\longrightarrow\infty}\beta_n<1$,
(C3) $\{\frac{\lambda_n}{\alpha}\}\subset(\tau,1-\delta)$ *for some* $\tau,\delta\in(0,1)$ *and* $\lim_{n\longrightarrow\infty}\lambda_n=0$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $P_{F(S)\cap VI(A,C)}u$.

4. Applications

Next we give the applications of Theorem 3.1.

Theorem 4.1. *Let H be a real Hilbert space. Let ϕ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4), $A:C\longrightarrow H$ a monotone L-Lipschitz continuous mapping and let S be a nonexpansive mapping of C into itself such that $F(S)\cap A^{-1}0\cap EP(\phi)\neq\emptyset$. Let the sequences $\{x_n\},\{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1,u\in H \text{ chosen arbitrary},\\ \phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\ \ \forall y\in H,\\ y_n=u_n-\lambda_nAu_n,\\ x_{n+1}=\beta_nx_n+(1-\beta_n)S(\alpha_nu+(1-\alpha_n)(u_n-\lambda_nAy_n)),\forall n\ge 1,\end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \subseteq [0,1]$, $\{\lambda_n\} \subseteq (0,1)$ *and* $\{r_n\} \subseteq (0,\infty)$ *satisfy the following conditions:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,
(C3) $\lim_{n\longrightarrow\infty} \lambda_n = 0$,
(C4) $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $P_{F(S)\cap A^{-1}0\cap EP(\phi)}u$.

Proof. It is clear that $A^{-1}0 = VI(A,H)$ and $P_H = I$ the identity mapping of H. By Theorem 3.1 we obtain the desired result. □

Theorem 4.2. *Let H be a real Hilbert space. Let ϕ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), $A : C \longrightarrow H$ a monotone L-Lipschitz continuous mapping and $B : H \longrightarrow 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \cap EP(\phi) \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1, u \in C \text{ chosen arbitrary}, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = u_n - \lambda_n A u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) J_r^B(\alpha_n u + (1-\alpha_n)(u_n - \lambda_n A y_n)), \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \subseteq [0,1]$, $\{\lambda_n\} \subseteq (0,1)$ *and* $\{r_n\} \subseteq (0,\infty)$ *satisfy the following conditions:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,
(C3) $\lim_{n\longrightarrow\infty} \lambda_n = 0$,
(C4) $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $P_{A^{-1}0\cap B^{-1}0\cap EP(\phi)}u$.

Proof. It is clear that $A^{-1}0 = VI(A,H)$ and $F(J_r^B) = B^{-1}0$. Putting $P_H = I$ the identity mapping of H, by Theorem 3.1 we obtain the desired result. □

References

1. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems. Math. Student 63:123-145 (1994).
2. A. Bnouhachem, M. Aslam Noor and Z. Hao, Some new extragradient iterative methods for variational inequalities, Nonlinear Analysis Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.014.
3. F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. 20:197-228 (1967).
4. P.L. Combettes, S.A. Hirstoaga, Equilibrium programming using proximal-like algorithms. Math. Program 78: 29-41 (1997).
5. H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal. 61 (2005) 341-350.

6. G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekonomika i Matematicheskie Metody 12 (1976) 747-756.
7. F. Liu, M.Z. Nashed, Regularization of nonlinear Ill-posed variational inequalities and convergence rates. Set-Valued Anal. 6:313-344 (1998).
8. N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. 128:191-201 (2006).
9. Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.
10. M.O. Osilike, D.I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudo-contractions and solutions of monotone type operator equations. Comp. Math. Appl. 40:559-567 (2000).
11. R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75-88.
12. T. Suzuki, Strong convergence of krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without bochner integrals. J. Math. Anal. Appl. 305:227-239 (2005).
13. S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces . J. Math. Anal. Appl. 311(1):506-515 (2007).
14. W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. 118: 417-428 (2003).
15. H.K. Xu, Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. 298:279-291 (2004).
16. J.-C. Yao, O. Chadli, Pseudomonotone complementarity problems and variational inequalities, in: J.P. Crouzeix, N. Haddjissas, S. Schaible (Eds.), Handbook of Generalized Convexity and Monotonicity, 2005, pp. 501-558.
17. L.C. Zeng, S. Schaible, J.C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities. J. Optim. Theory Appl. 124:725-738 (2005).
18. L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems. Taiwanese J. Math. 10:1293-1303 (2006).

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