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New Stable Numerical Solutions of Singular Integral Equations of Abel Type by Using Normalized Bernstein Polynomials

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Abstract

A new numerical method, based on the normalized Bernstein polynomials for solving singular integral equations of Abel type is presented here in this paper. We construct an othonormal family $\{b_{in}\}_{i=0}^{n}$ of polynomials of degree n from the n^{th} degree Bernstein polynomials B_{in} and use them as a basis to approximate the known and unknown functions $f(x)$ and $\varphi(x)$ respectively in the Abel's integral equations. Then orthogonality is used to reduce the integral equation to a system of algebraic equations which can be solved easily. The method is quite accurate and stable even when the approximations are performed by orthonormal Bernstein polynomials b_{in} of degree as low as 5, as illustrated by the given numerical examples with varying degree of noise terms ε added to $f(x)$.

Mathematics Subject Classification: 45E10; 45D05

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1. Introduction

Abel's integral equation [1] occurs in many branches of science. Usually, physical quantities accessible to measurement are quite often related to physically important but experimentally inaccessible ones by Abel's integral equation. Some of the examples are: microscopy [2] , seismology [3-4] , radio astronomy [5], satellite photometry of airglows [6] , electron emission [7] , atomic scattering [8], radar ranging [9] , optical fiber evaluation [10-12] . But it is most extensively used in flame and plasma diagnostics [13-15] and X-ray radiography [16-19]. In flame and plasma diagnostics the Abel's integral equation relates the emission coefficient distribution function of optically thin cylindrically symmetric extended radiation source to the line-of-sight radiance measured in the laboratory. Obtaining the physically relevant quantity from the measured one requires, therefore, the inversion of the Abel's integral.
Abel's integral equation can be written as

$$f(x) = \int_0^x \frac{\varphi(t)}{\sqrt{x-t}}\,dt \qquad 0 \le x \le 1, \tag{1}$$

where $f(x)$, is the data function and $\varphi(t)$ is the unknown function. The exact solution is given by

$$\varphi(x) = \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{x-t}}\,\frac{df(t)}{d(t)}\,dt, \quad 0 \le x \le 1, \quad [20] \tag{2}$$

assuming, without loss of generality, $f(0) = 0$.

As the process of estimating the solution function $\varphi(t)$, if the data function $f(x)$ is given approximately and only at a discrete set of data points, is ill-posed since small perturbations in the data $f(x)$ might cause large errors in the computed solution $\varphi(x)$. In fact, two explicit analytic inversion formulae were given by Abel [1], but their direct application amplifies the experimental noise inherent in the radiance data significantly. This is due to the fact these formulae require differentiating the measured data. In 1982, a third, analytic but derivative free inversion formula was obtained by Deutsch and Beniaminy [21] to avoid this problem. In addition, many numerical inversion methods [22-35] have been developed with varying degree of success with the inherent limitations of all measured data. Consequently, the direct use of (2) is restricted and stable numerical methods become important.

The aim of the present paper is to propose a new stable (with respect to small perturbations in the data function $f(x)$) algorithm based on orthonormal Bernstein polynomials to invert the Abel's integral equation (1), also known as singular Volterra integral equation of first kind and

$$\varphi(x) = f(x) + \int_0^x \frac{\varphi(t)}{\sqrt{x-t}}\,dt \ , \ 0 \le x \le 1, \tag{3}$$

the Volterra integral equation of second kind.

2. The Bernstein polynomials

A Bernstein polynomial, named after Sergei Natanovich Bernstein, is a polynomial in the Bernstein form, that is a linear combination of Bernstein basis polynomials.
The Bernstein basis polynomials of degree n are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \ , \qquad for\ \ i = 0,1,2,\cdots,n\,. \tag{4}$$

There are $(n+1)$ n^{th} degree Bernstein basis polynomials forming a basis for the linear space V_n consisting of all polynomials of degree less than or equal to n in **R**[x]-the ring of polynomials over the field **R.** For mathematical convenience, we usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$.
Any polynomial $B(x)$ in V_n may be written as

$$B(x) = \sum_{i=0}^{n} \beta_i\, B_{i,n}(x)\,. \tag{5}$$

Then $B(x)$ is called a polynomial in Bernstein form or Bernstein polynomial of degree n. The coefficients β_i are called Bernstein or Bezier coefficients. But several mathematicians call Bernstein basis polynomials $B_{i,n}(x)$ as the Bernstein polynomials. We will follow this convention as well. These polynomials have the following properties:

(i) $B_{i,n}(0) = \delta_{i0}$ and $B_{i,n}(1) = \delta_{in}$, where δ is the Kronecker delta function.
(ii) $B_{i,n}(t)$ has one root, each of multiplicity i and $n-i$, at $t = 0$ and $t = 1$ respectively.
(iii) $B_{i,n}(t) \ge 0$ for $t \in [0,1]$ and $B_{i,n}(1-t) = B_{n-i,n}(t)$.
(iv) For $i \ne 0$, $B_{i,n}$ has a unique local maximum in $[0,1]$ at $t = i/n$ and the maximum value $i^i n^{-n} (n-i)^{n-i} \binom{n}{i}$.
(v) The Bernstein polynomials form a partition of unity i.e. $\sum_{i=0}^{n} B_{i,n}(t) = 1$.
(vi) It has a degree raising property in the sense that any of the lower-degree polynomials (degree $< n$) can be expressed as a linear combinations of polynomials of degree n. We have,

$$B_{i,n-1}(t) = \left(\frac{n-i}{n}\right) B_{i,n}(t) + \left(\frac{i+1}{n}\right) B_{i+1,n}(t) .$$

(vii) Let $f(x) \in C\,[0,1]-$ (the class of continuous functions on $[0,1]$), then $B_n(f)(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(x)$ converges to $f(x)$ uniformly on $[0,1]$ as $n \to \infty$.

$(viii)$ Let $f(x) \in C^{(k)}[0,1]-$ (the class of $k-$ times differentiable function with $f^{(k)}$ continuous), then

$\left\| B_n(f)^{(k)} \right\|_\infty \le \frac{(n)_k}{n^k} \left\| f^{(k)} \right\|_\infty$ and $\left\| f^{(k)} - B_n(f)^{(k)} \right\|_\infty \to 0$ as $k \to \infty$,

where $\|.\|_\infty$ is the sup. norm and $\frac{(n)_k}{n^k} = \left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$ is an eigen value of B_n; the corresponding eigen function is a polynomial of degree k.

3. The orthonormal polynomials

Using Gram- Schmidt orthonormalization process on $B_{i,n}$, we obtain a class of orthonormal polynomials from Bernstein polynomials. We call them orthonormal Bernstein polynomials of order n and denote them by $b_{on}, b_{1n}, \cdots, b_{nn}$. For $n = 5$ the five orthonormal polynomials are given by

$$b_{05}(t) = \sqrt{11}(1-t)^5$$

$$b_{15}(t) = 6\left[5(1-t)^4 t - \frac{1}{2}(1-t)^5\right]$$

$$b_{25}(t) = \frac{18\sqrt{7}}{5}\left[10(1-t)^3 t^2 - 5(1-t)^4 t + \frac{5}{18}(1-t)^5\right]$$

$$b_{35}(t) = \frac{28}{\sqrt{5}}\left[10(1-t)^2 t^3 - 15(1-t)^3 t^2 + \frac{30}{7}(1-t)^4 t - \frac{5}{28}(1-t)^5\right]$$

$$b_{45}(t) = 7\sqrt{3}\left[5(1-t)t^4 - 20(1-t)^2 t^3 + 18(1-t)^3 t^2 - 4(1-t)^4 t + \frac{1}{7}(1-t)^5\right]$$

$$b_{55}(t) = 6\left[t^5 - \frac{25}{2}(1-t)t^4 + \frac{100}{3}(1-t)^2 t^3 - 25(1-t)^3 t^2 + 5(1-t)^4 t - \frac{1}{6}(1-t)^5\right] \qquad (6)$$

4. Function approximation

A function $f \in L^2[0,1]$ may be written as

$$f(t) = \lim_{n\to\infty} \sum_{i=0}^{n} c_{in}\, b_{in}(t), \tag{7}$$

where, $c_{in} = \left\langle f, b_{in} \right\rangle$ and $\langle , \rangle$ is the standard inner product on $L^2[0,1]$.
If the series (7) is truncated at $n = m$, then we have

$$f \cong \sum_{i=0}^{m} c_{im}\, b_{im} = C^T B(t), \tag{8}$$

where, C and $B(t)$ are $(m+1)\times 1$ matrices given by

$$C = [c_{0m}, c_{1m}, \cdots, c_{mm}]^T \tag{9}$$

and

$$B(t) = [b_{0m}(t), b_{1m}(t), \cdots, b_{mm}(t)]^T. \tag{10}$$

5. Solution of Abel's integral equation

In this section we solve Abel's integral equation (1) and singular Volterra integral equations (3) by using orthonormal Bernstein polynomials.
Using Eq.(8), we approximate $\varphi(x)$ and $f(x)$ as

$$\varphi(x) = C^T B(x), \quad f(x) = F^T B(x) \tag{11}$$

where the matrix F is known. Then from equations (1), (3) and (11) we have

for the first kind: $$F^T B(x) = \int_0^x \frac{C^T B(x)}{\sqrt{x-t}}\, dt \tag{12}$$

and

for the second kind: $$C^T B(x) = F^T B(x) + \int_0^x \frac{C^T B(x)}{\sqrt{x-t}}\, dt. \tag{13}$$

From equations (6), (10) and the well known formula

$$\int_0^x \frac{t^n}{\sqrt{x-t}}\, dt = \frac{\sqrt{\pi}\, x^{(\frac{1}{2}+n)}\, \Gamma(n+1)}{\Gamma(n+\frac{3}{2})}, \tag{14}$$

it is obvious that

$$\int_0^x \frac{B(t)}{\sqrt{x-t}}\, dt = P\, B(x), \tag{15}$$

where P is a $(m+1)\times(m+1)$ matrix, which we call as modified Bernstein operational matrix of integration for singular Volterra integral equations with Abel kernel.
Substituting (15) in (12) and (13), we get

$$C^T = F^T P^{-1} \quad \text{(for the first kind)} \tag{16}$$

and

$$C^T = F^T (I-P)^{-1} \quad . \text{ (for the second kind).} \tag{17}$$

Hence, the approximate solutions $\varphi(t)$ for the Abel's integral equation (1) and second kind singular Volterra integral equation (3) are obtained by putting the values of C^T from (16) and (17) in (11) respectively.

6. Illustrative Examples

The following examples are solved with and without noise terms to illustrate the efficiency and stability of our method. Note that in all the examples to follow, the series (8) is truncated at level $m=5$ and hence the modified operational matrix P in (15) is of order 6×6.

Example.1

Consider the following singular Volterra integral equation:

$$y(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt\,, \quad [29] \tag{18}$$

which has the exact solution $y(x) = x^2$. Equations (11) and (17) give the approximate solution $y(x) = C^T B(x)$,

where $C^T = [0.019742, 0.089286, 0.185833, 0.252889, 0.255684, 0.166667]$.

Now, we introduce a perturbation term ε in $f(x)$ and denote the new function by $f^{\varepsilon}(x)$, so that the new function $f^{\varepsilon}(x) = x^2 + \frac{16}{15}x^{5/2} + \varepsilon$.

We study the behaviour of the solution by taking $\varepsilon = 0.001,\ 0.002$ and 0.01. Fig. 3. denotes the error Ei(t)= Approximate. solution obtained with perturbation ε_i - the exact solution, $i = 1, 2, 3$.

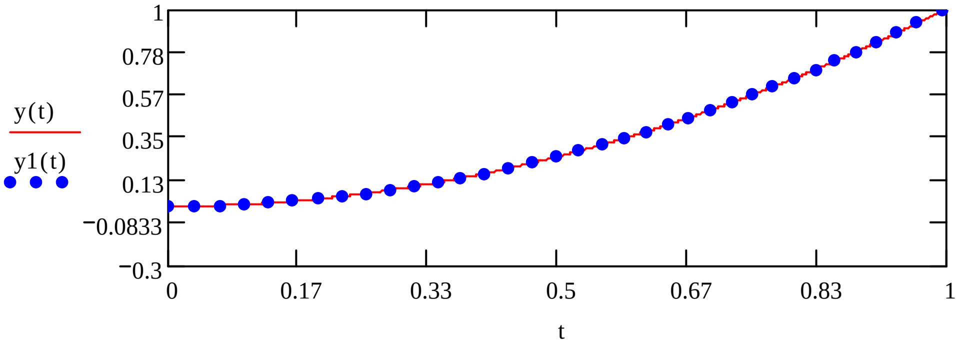

Fig.1. The exact and the approximate solutions of the singular Volterra integral equation(18) in Example 1 are represented by $y(t)$ (solid line) and $y1(t)$ (dotted line) respectively.

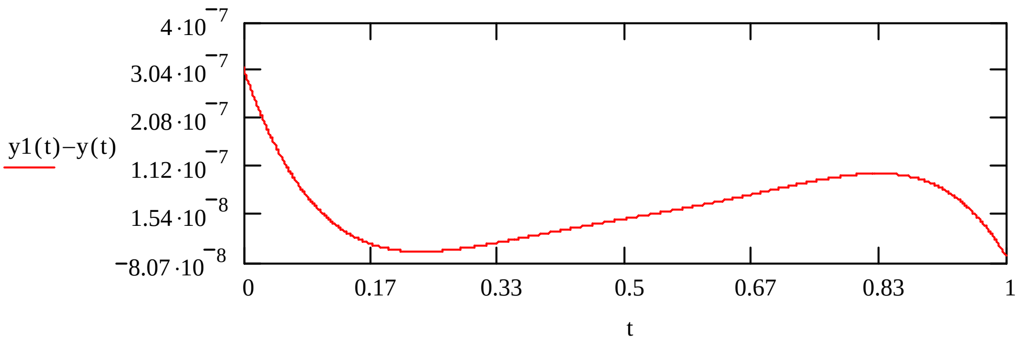

Fig.2 The error for the singular Volterra integral equation (18) in Example 1.

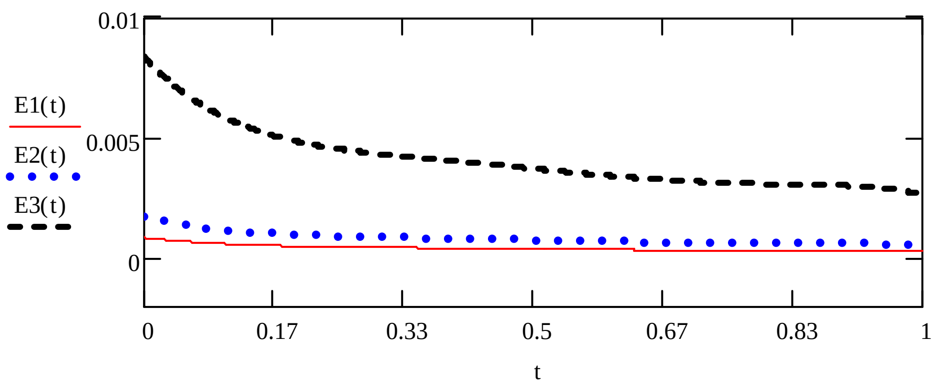

Fig.3. The errors for the singular Volterra integral equation (18) in Example 1 with different perturbations.

Example.2

Consider the singular Volterra integral equation:

$$y(x) = x + \frac{4}{3}x^{3/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt\,, \quad [36] \tag{19}$$

which has the exact solution $y(x) = x$. The approximate solution is given by

$$y(x) = C^T B(x)\,,$$

where C^T =[0.078967,0.214286,0.289773,0.30879,0.272179,0.166667].

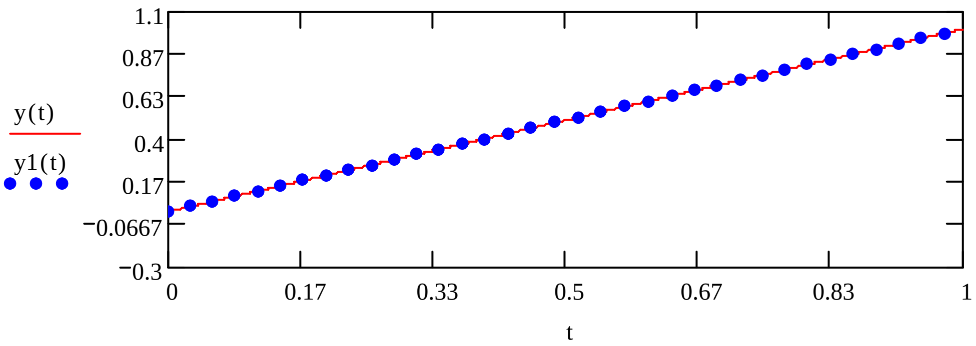

Fig.4. The exact and the approximate solutions of the singular Volterra integral equation (19) in Example 2 are represented by $y(t)$ (solid line) and $y1(t)$ (dotted line) respectively.

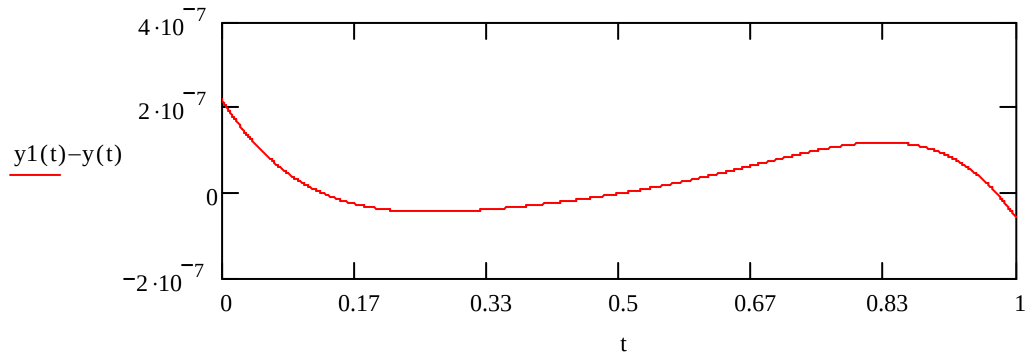

Fig.5. The error for the singular Volterra integral equation (19) in Example 2.

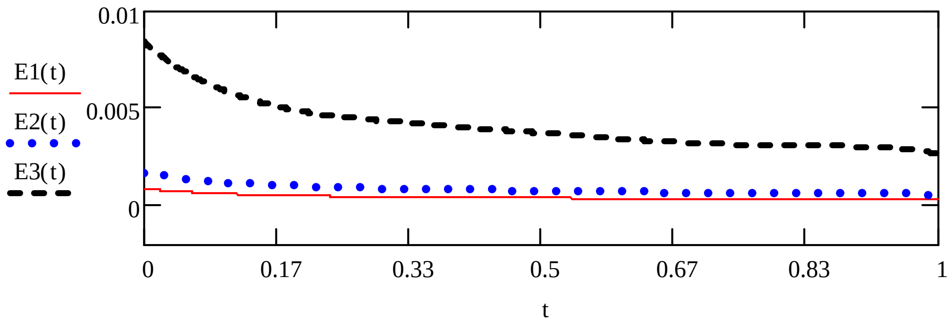

Fig.6. The errors for the singular Volterra integral equation (19) in Example 2 with different perturbations.

Example.3

Next, we consider the Abel integral equation given by,

$$\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt = x^r, \qquad 0 < x < 1 \quad [33] \tag{20}$$

where r is any positive number. This is a first kind Volterra integral equation with weak singularity.
The exact solution of the integral equation (20) is given by,

$$y(x) = \frac{2^{2r-1}}{\pi} r \frac{(\Gamma(r))^2}{\Gamma(2r)} x^{r-\frac{1}{2}}. \tag{21}$$

We apply our proposed method to solve the above Abel's integral equation by taking $r = 1, 1.5, 5$, and study the solution with noise term ε added to the observable data $f(x)$.
When $r = 1$, the data function $f(x) = x$ with exact inverse Abel transform $y(x) = \frac{2}{\pi}\sqrt{x}$.
Fig.7 shows the approximate solution (dotted line) and the exact solution (solid line). The next Fig. 8 shows the error between exact and approximate solutions. Now, we add noise term ε to the data function $f(x)$ so that it becomes $f^{\varepsilon}(x) = f(x) + \varepsilon$ and compute the inverse function $y(x)$ using the proposed method. Fig. 9 shows errors at three different noise levels, $\varepsilon = 0.001, 0.002$ and 0.01. This problem was studied by Murio et. al. [32] with $\varepsilon = 0.001$. The stability of our method is better than [32].
The other two cases for the data function $f(x)$ with $r = 1.5$ and 5 having exact inversions as $y(x) = \frac{3}{4}x$ and $y(x) = \frac{1280}{315\pi}x^{9/2}$ respectively, have been treated in exactly the same way as was done for $r = 1$. The Figures 10 to 15 corresponds to figures 7 to 9 for the other two cases pertaining to $r = 1.5$ and 5 respectively.

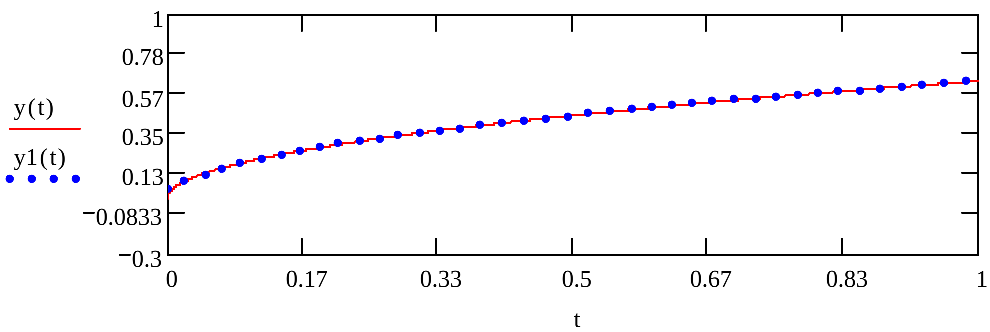

Fig.7. The exact and the approximate solutions of the Abel integral equation (20) in Example 3 are represented by $y(t)$ (solid line) and $y1(t)$ (dotted line) respectively, $r = 1$.

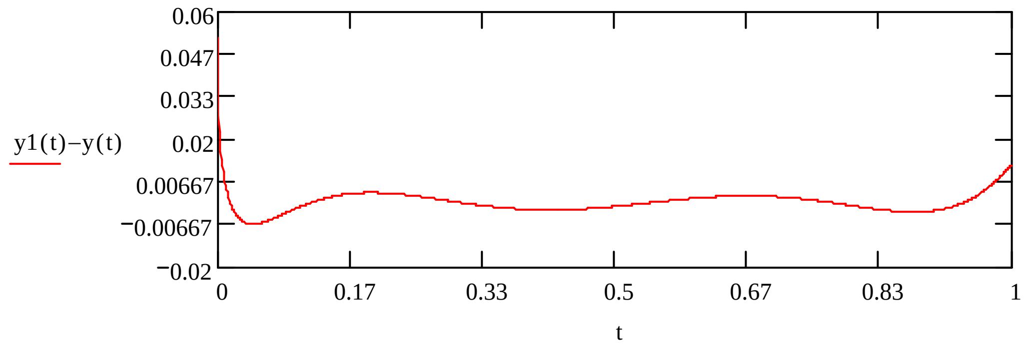

Fig.8. The error for the Abel's integral equation (20) in Example 3, $r = 1$.

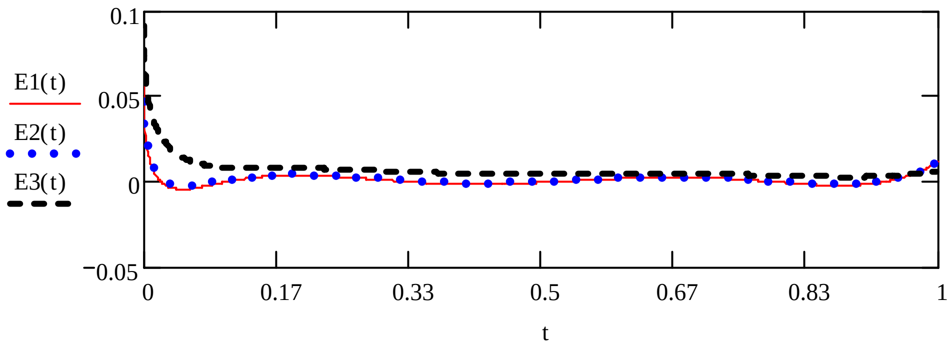

Fig.9. The errors for the Abel's integral equation (20) in Example 3, $r = 1$.

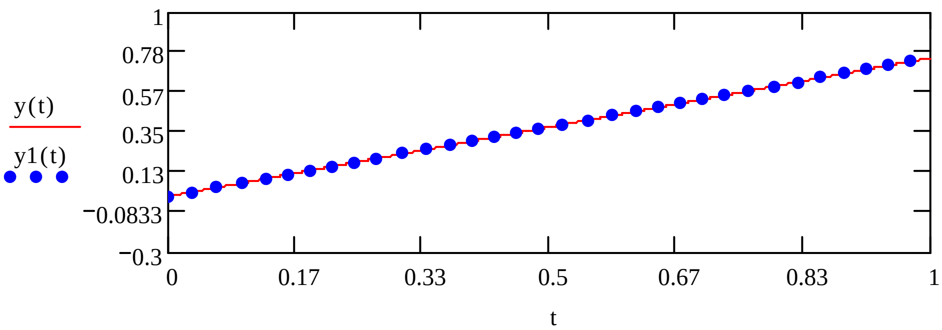

Fig.10 The exact and the approximate solutions of the Abel's integral equation (20) in Example 3 are represented by $y(t)$ (solid line) and $y1(t)$ (dotted line) respectively, $r = 1.5$.

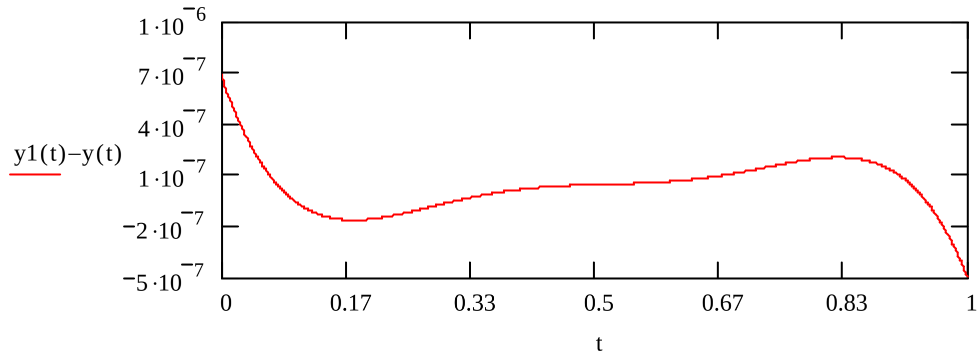

Fig.11. The error for the Abel's integral equation (20) in Example 3, $r = 1.5$.

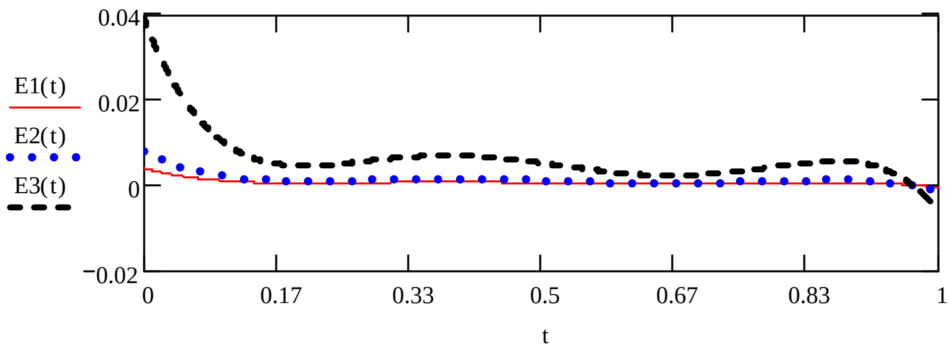

Fig.12. The errors for the Abel's integral equation (20) in Example 3, $r = 1.5$.

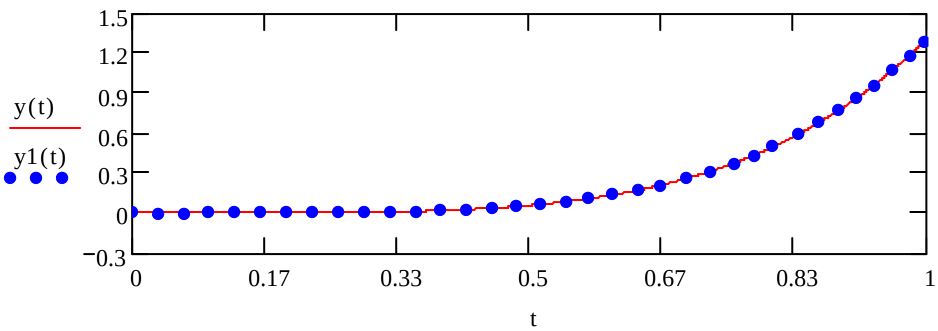

Fig.13 The exact and the approximate solutions of the Abel's integral equation (20) in Example 3 are represented by $y(t)$ (solid line) and $y1(t)$ (dotted line) respectively, $r = 5$.

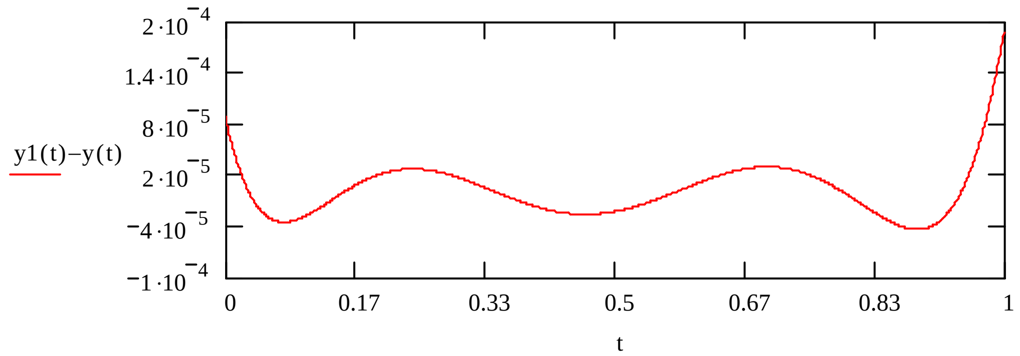

Fig.14. The error for the Abel's integral equation (20) in Example 3, $r = 5$.

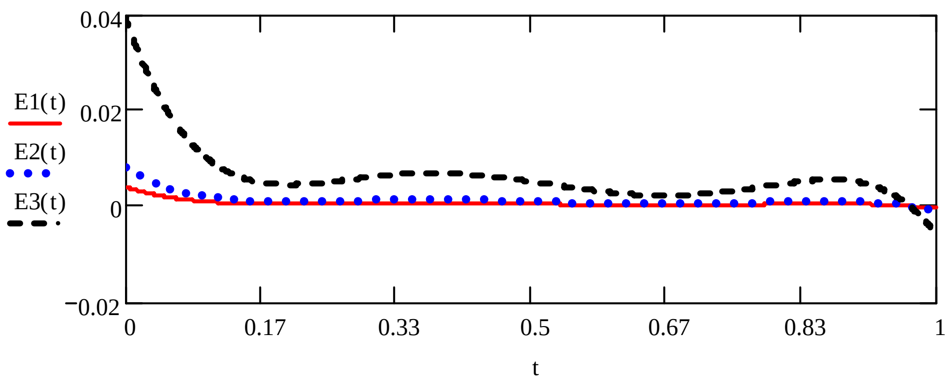

Fig.15. The errors for the Abel's integral equation (20) in Example 3, $r = 5$.

7. Conclusions

We have introduced a modified Bernstein operational matrix of integration to propose a new and stable algorithm for numerical solution of Abel's and singular Volterra integral equations. It is found that the method is accurate, easy to use and stable as shown by the given numerical examples.

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