Applied Mathematical Sciences, Vol. 2, 2008, no. 15, 747 - 752

The Non-Truncated Bulk Arrival Queue $M^x/M/1$ with Reneging, Balking, State-Dependent and an Additional Server for Longer Queues

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Abstract

The aim of this paper is to derive the solution of the non-truncated queue: $M^x/M/1$ with reneging, balking, state-dependent and an additional server for longer queues. In this case the units arrive in batches of size X which is a random variable. At the end of this paper some special cases are deduced.

Keywords: Non-truncated queue, longer queues, Balking, Reneging concept, Probabilities generating functions

1 Introduction

Many researchers studied the problem of bulk arrival queues but without any concept. Some researchers studied the queue: $M^x/M/1$ in the homogeneous case with blocking and delays. Other discussed the queues: $M^x/M/C$ and $M^x/G/\infty$. Abou-El-Ata et al [1] discussed the bulk arrival queue: $M^x/M/1$ with both concepts of balking and reneging. El-Paoumy [4] discussed the truncated queue $M^x/M/1/N$ with the same concepts. In this paper it is aimed to treat the non-truncated bulk arrival queue: $M^x/M/1$ with balking, reneging, state-dependent and an additional server for longer queues. The discipline considered is the usual one first in first out (FIFO). In this work the researcher investigates the probability generating function of the number of units in the system. Then some special cases are deduced.

2 The equilibrium distribution

Consider the interarrival rates of the units be an exponential distribution with rate λ. The service time rate is also exponentially distributed with rate μ. The units are served according to the discipline FIFO.
Assume the group size is a random variable with distribution:

$$g_m = \rho(X = m), \; g_0 = 0, \quad m \geq 1$$

with mean:

$$\overline{g} = \sum_{m=1}^{\infty} m g_m$$

and variance:

$$\sigma^2 = \sum_{m=1}^{\infty} (m - \overline{g})^2 g_m \geq 0$$

And the balk concept with probability (1-β), i.e.:
β = prob.{a unit joins the queue}, 0 ≤ β < 1 , n > 0 and β = 1 for n = 0
Also, consider the reneging concept, which means that a unit may renege with function r(n) after joining the queue for service for a certain time t, which is a random variable with a probability density function:

$$f(t) = \alpha \overline{e}^{\alpha t}, \quad \alpha, t > 0$$

Let: $r(n) = (n-1)\alpha$ if n units are in the system, and r(1) = r(0) = 0
The service time rate in case of state-dependent and an additional server for longer queues is as follows:

$$\mu_n = \begin{cases} \mu_1 & , \quad 0 \leq n < k_1 \\ \mu_2 & , \quad k_1 \leq n < k_2 \\ \mu_2 + \mu_3 = \mu & , \quad k_2 \leq n < \infty \end{cases}$$

Now, let $p_n(t)$ be the probability that there are n units in the system at time t by:
Therefore, the differential-difference equations are:

$$p_n(t) = p\{N(t) = n\} \qquad , N(0) = 0$$

$$p_0'(t) = -\lambda p_0(t) + \mu_1 p_1(t) \qquad , \quad n = 0$$

$$p_n'(t) = -[\beta\lambda + \mu_1 + (n-1)\alpha] p_n(t) + (\mu_1 + n\alpha) p_{n+1}(t)$$
$$+ \beta\lambda \sum_{m=1}^{n-1} g_m p_{n-m}(t) + \lambda g_n p_0(t) \quad , 1 \leq n < k_1 - 1$$

$$p'_n(t) = -[\beta\lambda + \mu_1 + (k_1 - 2)\alpha]p_{k_1-1}(t) + [\mu_2 + (k_1 - 1)\alpha]p_{k_1}(t) + \beta\lambda \sum_{m=1}^{k_1-2} g_m p_{k_1-m-1}(t) + \lambda g_{k_1-1} p_0(t) \quad , n = k_1 - 1$$

$$p'_n(t) = -[\beta\lambda + \mu_2 + (n-1)\alpha]p_n(t) + (\mu_2 + n\alpha)p_{n+1}(t) + \beta\lambda \sum_{m=1}^{n-1} g_m p_{n-m}(t) + \lambda g_n p_0(t) \quad , k_1 \le n < k_2 - 1$$

$$p'_n(t) = -[\beta\lambda + \mu_2 + (k_2 - 2)\alpha]p_{k_2-1}(t) + [\mu + (k_2 - 1)\alpha]p_{k_2}(t) + \beta\lambda \sum_{m=1}^{k_2-2} g_m p_{k_2-m-1}(t) + \lambda g_{k_2-1} p_0(t) \quad , n = k_2 - 1$$

$$p'_n(t) = -[\beta\lambda + \mu + (n-1)\alpha]p_n(t) + (\mu + n\alpha)p_{n+1}(t) + \beta\lambda \sum_{m=1}^{n-1} g_m p_{n-m}(t) + \lambda g_n p_0(t) \quad , k_2 \le n < \infty$$

As $t \longrightarrow \infty$, the steady-state difference equations are

$$-\lambda p_0 + \mu_1 p_0 = 0 \quad , n = 0 \tag{1}$$

$$-[\beta\lambda + \mu_1 + (n-1)\alpha]p_n + (\mu_1 + n\alpha)p_{n+1} + \beta\lambda \sum_{m=1}^{n} g_m p_{n-m} + (1-\beta)\lambda g_n p_0 = 0 \quad , 1 \le n < k_1 - 1 \tag{2}$$

$$-[\beta\lambda + \mu_1 + (k_1 - 2)\alpha]p_{k_1-1} + [\mu_2 + (k_1 - 1)\alpha]p_{k_1} + \beta\lambda \sum_{m=1}^{k_1-1} g_m p_{k_1-m-1} + (1-\beta)\lambda g_{k_1-1} p_0 = 0 \quad , n = k_1 - 1 \tag{3}$$

$$-[\beta\lambda + \mu_2 + (n-1)\alpha]p_n + (\mu_2 + n\alpha)p_{n+1} + \beta\lambda \sum_{m=1}^{n} g_m p_{n-m} + (1-\beta)\lambda g_n p_0 = 0 \quad , k_1 \le n < k_2 \tag{4}$$

$$-[\beta\lambda + \mu_2 + (k_2 - 2)\alpha]p_{k_2-1} + [\mu + (k_2 - 1)\alpha]p_{k_2} + \beta\lambda \sum_{m=1}^{k_2-1} g_m p_{k_2-m-1} + (1-\beta)\lambda g_{k_2-1} p_0 = 0 \quad , n = k_2 - 1 \tag{5}$$

$$-[\beta\lambda + \mu + (n-1)\alpha]p_n + (\mu + n\alpha)p_{n+1} + \beta\lambda \sum_{m=1}^{n} g_m p_{n-m} + (1-\beta)\lambda g_n p_0 = 0 \quad , k_2 \le n < \infty \tag{6}$$

Let us also define the following two probabilities generating functions:

$$\mathrm{P}(z)=\sum_{n=0}^{\infty}p_nz^n \quad and \quad G(z)=\sum_{n=1}^{\infty}g_nz^n \quad , \quad g_0=0 \tag{7}$$

Multiplying relations (2), (4) and (6) by z^n ,summing over n = 1 → ∞ and adding relations (1), (3) and (5) we get:

$$\mathrm{P}'(z)+\left[\frac{s-1}{z}-\beta\, r\delta(z)\right]\mathrm{P}(z)=p_0\left[\frac{s_1-1}{z}+(1-\beta)r\delta(z)\right]-\frac{F(z)}{\alpha(1-z)} \tag{8}$$

$$\left.\begin{aligned}&\text{where: } r=\frac{\lambda}{\alpha}\ ,s=\frac{\mu_1+\mu_2+\mu}{\alpha}\ ,s_1=\frac{\mu_1}{\alpha}\ ,\delta(z)=\frac{1-G(z)}{1-z}\ ,and;\\ &F(z)=\mu_1(1-\frac{1}{z})\sum_{n=K_1}^{\infty}p_nz^n+\mu(1-\frac{1}{z})\sum_{n=0}^{k_2-1}p_nz^n+\mu_2(1-\frac{1}{z})\sum_{n=0}^{k_1-1}p_nz^n+\mu_2(1-\frac{1}{z})\sum_{n=k_2}^{\infty}p_nz^n\end{aligned}\right\} \tag{9}$$

Relation (8) is a first order differential equation in P(z) . As z → 1, it is clear that:, $\mathrm{P}(1)=\mathrm{G}(1)=1$ $\delta(1)=G'(1)=\overline{g}$, *and* ; $\mathrm{P}'(1)=E(n)$.

$$\therefore E(n)=p_0\left[s_1-1+(1-\beta)r\overline{g}\right]-(s-1-\beta\, r\overline{g})-\frac{\mu_1}{\alpha}\sum_{n=k_1}^{\infty}p_n-\frac{\mu}{\alpha}\sum_{n=0}^{k_2-1}p_n-\frac{\mu_2}{\alpha}(1-\sum_{n=k_1}^{k_2-1}p_n) \tag{10}$$

The solution of relation (8) is:

$$\mathrm{P}(z).\upsilon(z)=\int\upsilon(z)\left[p_0\left\{\frac{s_1-1}{z}+(1-\beta)r\delta(z)\right\}-\frac{F(z)}{\alpha(1-z)}\right]dz+c \tag{11}$$

Where:

$$\left.\begin{aligned}&\upsilon(z)=z^{s-1}e^{-\beta\, r\Delta(z)}\ ,and\ ;\\ &\Delta(z)=\int\delta(z)dz\end{aligned}\right\} \tag{12}$$

But as $z\longrightarrow 0$, $\mathrm{P}(0)=p_0$, $\upsilon(0)=0 \Rightarrow \mathrm{C}=0$

$$\therefore\ \mathrm{P}(z)=\upsilon^{-1}(z)\int\upsilon(z)\left[p_0\left\{\frac{s_1-1}{z}+(1-\beta)r\delta(z)\right\}-\frac{F(z)}{\alpha(1-z)}\right]dz \tag{13}$$

Some special cases

1- **Case:** let $k_1=k_2$, then we obtain this queue with an additional server for longer queues only, such that:

$\mathrm{F(z)}\to F_1(\mathrm{z})$ $\quad s\to s_2$ and $\mu_2\to\mu_1$

where $F_1(z)=\mu_1(1-\frac{1}{z})\sum_{n=k_1}^{\infty}p_nz^n+\mu(1-\frac{1}{z})\sum_{n=0}^{k_1-1}p_nz^n$

and

$$s_2 = \frac{\mu_1 + \mu}{\alpha}$$

2- <u>Case</u>: Let $k_1 = k_2 \to \infty$ we get this queue with balking and reneging only, such that:

$$F(z) \to 0 \quad \text{and} \quad s \longrightarrow s_1$$

Which are the same results as in Abou-El-Ata et al [1].

3- <u>Case</u>: Let $\alpha = 0$ $k_1 = k_2 \to \infty$ we obtain the single-channel queue: $M^x/M/1$ with balking only. And thus from relations (8) and (10) we get:

$$\left.\begin{aligned} P(z) &= \frac{p_0\left[\mu + (1-\beta)\lambda z \delta(z)\right]}{\mu - \beta\lambda z \,.\, \delta(z)} \\ p_0 &= \frac{\mu - \beta\lambda \overline{g}}{\mu + (1-\beta)\lambda \overline{g}} \end{aligned}\right\} \tag{14}$$

These results are as in [4] when $N \longrightarrow \infty$

4- <u>Case</u>: Let $\alpha = 0$, $\beta = 1$, $k1 = k2 \longrightarrow \infty$, we obtain the single-channel bulk queue: $M^x/M/1$ only. And thus relation (14) becomes:

$$P(z) = \frac{\mu p_0}{\mu - \lambda z \delta(z)} \quad , \quad p_0 = 1 - \rho\overline{g} \quad , \rho = \frac{\lambda}{\mu} < 1 \tag{15}$$

Which are the same results as in Harris [2]

5- <u>Case</u>: Let $k_1 = k_2 \longrightarrow \infty$ and the units are arrived according to the geometric distribution:

$$g_m = \begin{cases} (1 - gg)^{m-1}, & m \geq 1 \\ 0 & otherwise \end{cases}$$

Thus we get

$$G(z) = \frac{(1-g)z}{1-gz}, \ |gz| < 1$$

$$\delta(z) = \frac{1}{1-gz} \Rightarrow \Delta(z) = \frac{-1}{g} \ln |1 - gz|$$

$$\therefore \upsilon(z) = z^{s_1 - 1}(1 - zg)^{\frac{r\beta}{g}}$$

As in Dwight [3] relation (8) becomes

$$P(z) = (\frac{\beta - 1}{\beta}) p_0 + (\frac{s_1 - 1}{\beta}) p_0 \upsilon^{-1}(z) \int z^{s_1 - 2} (1 - zg)^{-\frac{r\beta}{g}} dz$$

$$= (\frac{\beta - 1}{\beta}) p_0 + (\frac{s_1 - 1}{\beta}) p_0 \frac{\upsilon^{-1}(z)}{(-g)^{s_1 - 1}} \sum_{k=0}^{s_1 - 2} \frac{(s_1 - 2)!(-1)^k (1 - zg)^{s_1 - \frac{r\beta}{g} - k - 1}}{k!(s_1 - k - 2)!(s_1 - \frac{r\beta}{g} - k - 1)!} \qquad (16)$$

Except for the case:

$$s_1 = -\frac{r\beta}{g} - k - 1 = 0, \quad then$$

$$p(z) = \left(\frac{\beta - 1}{\beta}\right) p_0 + \left(\frac{s_1 - 1}{\beta}\right) p_0 \frac{v^{-1}(z)}{(g)^{n-1}} \cdot \left(\frac{(s_1 - 2)r\beta}{g}\right) (-1)^{-\frac{r\beta}{g}} \ln|1 - zg| \qquad (17)$$

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Received: July 1, 2007