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On Fekete-Szegö Problems for Certain Subclass of Analytic Functions

K. Al-Shaqsi and *M. Darus[1]

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
ommath@hotmail.com
*maslina@pkrisc.cc.ukm.my

Abstract

In this present investigation, the authors obtain Fekete-Szegö inequality for certain normalized analytic function $f(z)$ defined on the open unit disc for which $\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}$, $(n \in \mathbb{N}_0,\ \lambda \geq 0)$ lines in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by Hadamard product (convolution) are given. As a special case of this result, Fekete-Szegö inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to give a generalization of the Fekete- Szegö inequalities obtained by Srivastava and Mishra by making use of D_λ^n the generalized Ruscheweyh derivatives operator introduced by authors [6].

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1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f defined on $\mathbf{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{A}_0$ be the family of functions $f \in \mathcal{A}$ normalized by the

[1]Corresponding author

conditions $f(0) = 0, f'(0) = 1$. Such functions $f \in \mathcal{A}_0$ have the Taylor series expansion given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbf{U}) \tag{1.1}$$

Let $\mathcal{S}$ be the family of functions $f \in A_0$ which are univalent. Let $\phi(z)$ be an analytic function with positive real part on $\mathcal{A}$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\mathbf{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \qquad (z \in \mathbf{U}),$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \qquad (z \in \mathbf{U}),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [9]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete- Szegö problem for class of starlike, convex, and close-to convex functions, see the recent paper by Srivastava et al. [4].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_\lambda^n(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through Hadamard product (or convolution) and in particular we consider a class $M_\lambda^{n,\gamma}(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivastava and Mishra [3].

Definition 1.1 *Let $\phi(z)$ be a univalent starlike function with respect to* 1 *which maps the unit disc $\boldsymbol{U}$ onto a region in the right half plane which is symmetric with respect to the real axis, $phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the class $M_\lambda^n(\phi)$ if*

$$\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} \prec \phi(z) \tag{1.2}$$

where $n \in \mathbb{N}_0$ and D_λ^n denote the operator introduced by authors [6] and is given by

$$D_\lambda^0 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z),$$
$$D_\lambda^1 f(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))', \qquad \lambda \geq 0.$$

Note that if f is given by (1.1), then we see that

$$D_\lambda^n f(z) = z + \sum_{k=m+1}^{\infty} \big[1+\lambda(k-1)\big] C(n,k) a_k z^k, \tag{1.3}$$

where

$$C(n,k) = \binom{k+n-1}{n} \qquad k = 2,3,4....$$

To prove our main result, we need the following:

Lemma 1.2 *If $p_1(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function with positive real part in $\boldsymbol{U}$, then*

$$|c_2 - \upsilon c_1^2| \leq \begin{cases} -4\upsilon + 2 & if \ \upsilon \leq 0, \\ 2 & if \ 0 \leq \upsilon \leq 1, \\ 4\upsilon - 2 & if \ \upsilon \geq 1. \end{cases}$$

When $\upsilon < 0$ or $\upsilon > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \upsilon < 1$, then equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\upsilon = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1)$$

or one of its rotations. If $\upsilon = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\upsilon = 0$. Also the above upper bound is sharp, it can be improved as follows when $0 < \upsilon < 1$:

$$|c_2 - \upsilon c_1^2| + \upsilon |c_1|^2 \leq 2 \qquad (0 < \upsilon \leq 1/2)$$

and

$$|c_2 - \upsilon c_1^2| + (1-\upsilon)|c_1|^2 \leq 2 \qquad (1/2 < \upsilon \leq 1).$$

2 Fekete-Szegö problem

Our main result is the following:

Theorem 2.1 *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$. *If* $f(z)$ *given by (1.1) belongs to* $M_\lambda^n(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{B_2}{(n+2)(n+1)(1+2\lambda)} - \frac{\mu B_1^2}{(n+1)^2(1+\lambda)^2} + \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)} & if \ \ \mu \leq \sigma_1 \ ; \\ \frac{B_1}{(n+2)(n+1)(1+2\lambda)} & if \ \ \sigma_1 \leq \mu \leq \sigma_2 \ ; \\ -\frac{B_2}{(n+2)(n+1)(1+2\lambda)} + \frac{\mu B_1^2}{(n+1)^2(1+\lambda)^2} - \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)} & if \ \ \mu \geq \sigma_2, \end{cases} \quad (2.4)$$

where

$$\begin{aligned} \sigma_1 &:= \frac{(n+1)^2(1+\lambda)^2\big\{(B_2 - B_1) + B_1^2\big\}}{(n+2)(n+1)(1+2\lambda)B_1^2}, \\ \sigma_2 &:= \frac{(n+1)^2(1+\lambda)^2\big\{(B_2 + B_1) + B_1^2\big\}}{(n+2)(n+1)(1+2\lambda)B_1^2}. \end{aligned}$$

The result is sharp.

Proof. For $f(z) \in M_\lambda^n(\phi)$, let

$$p(z) = \frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} = 1 + b_1 z + b_2 z^2 + \dots \ . \quad (2.5)$$

From (2.2), we obtain

$(n+1)(1+\lambda)a_2 = b_1$ and $(n+2)(n+1)(1+2\lambda)a_3 = (n+1)^2(1+\lambda)^2 a_2^2 + b_2$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \dots,$$

is analytic and has a positive real part in $\mathbf{U}$. Also we have

$$p(z) = \phi\Big(\frac{p_1(z) - 1}{p_1(z) + 1}\Big),$$

and from this equation (2.2),

$$\begin{aligned} 1 + b_1 z + b_2 z^2 + \dots &= \phi\Big(\frac{c_1 z + c_2 z^2 + \dots}{2 + c_1 z + c_2 z^2 + \dots}\Big) \\ &= \phi\Big[\frac{1}{2}c_1 z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \dots\Big] \\ &= 1 + B_1\frac{1}{2}c_1 z + B_1\frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \dots + B_2\frac{1}{4}c_1^2 z^2 + \dots \end{aligned}$$

we obtain

$$b_1 = \frac{1}{2}B_1c_1 \text{ and } b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2.$$

Therefore we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{B_1}{2(n+2)(n+1)(1+2\lambda)}\Big\{c_2 - c_1^2\Big[\frac{1}{2}\Big(1 - \frac{B_2}{B_1} \\ &\qquad + \frac{(n+2)(n+1)(1+2\lambda)\mu - (n+1)^2(1+\lambda)^2}{(n+1)^2(1+\lambda)^2}B_1\Big)\Big]\Big\} \\ &= \frac{B_1}{2(n+2)(n+1)(1+2\lambda)}[c_2 - \upsilon c_1^2] \end{aligned}$$

where

$$\upsilon = \frac{1}{2}\Big(1 - \frac{B_2}{B_1} + \frac{(n+2)(n+1)(1+2\lambda)\mu - (n+1)^2(1+\lambda)^2}{(n+1)^2(1+\lambda)^2}B_1\Big).$$

If $\mu \leq \sigma_1$, then by applying Lemma 1.2, we get

$$|a_3 - \mu a_2^2| \leq \frac{B_2}{(n+2)(n+1)(1+2\lambda)} - \frac{\mu B_1^2}{(n+1)^2(1+\lambda)^2} + \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)},$$

which is the first part of assertion (2.1).

Similarly, if $\mu \geq \sigma_2$, we get

$$|a_3 - \mu a_2^2| \leq -\frac{B_2}{(n+2)(n+1)(1+2\lambda)} + \frac{\mu B_1^2}{(n+1)^2(1+\lambda)^2} - \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)},$$

If $\mu = \sigma_1$, then equality holds if and only if

$$p_1(z) = \Big(\frac{1+\gamma}{2}\Big)\frac{1+z}{1-z} + \Big(\frac{1-\gamma}{2}\Big)\frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1; z \in \mathbf{U})$$

or one of its rotations.
Also, if $\mu = \sigma_2$, then

$$\frac{1}{2}\Big(1 - \frac{B_2}{B_1} + \frac{(n+2)(n+1)(1+2\lambda)\mu - (n+1)^2(1+\lambda)^2}{(n+1)^2(1+\lambda)^2}B_1\Big) = 0.$$

Therefore,

$$\frac{1}{p_1(z)} = \Big(\frac{1+\gamma}{2}\Big)\frac{1+z}{1-z} + \Big(\frac{1-\gamma}{2}\Big)\frac{1-z}{1+z} \quad (0 < \gamma < 1; z \in \mathbf{U})$$

Finally, we see that

$$|a_3-\mu a_2^2| \;=\; \frac{B_1}{2(n+2)(n+1)(1+2\lambda)}\left|c_2-c_1^2\Big[\frac{1}{2}\Big(1-\frac{B_2}{B_1}+\frac{(n+2)(n+1)(1+2\lambda)\mu-(n+1)^2(1+\lambda)^2}{(n+1)^2(1+\lambda)^2}B_1\Big)\Big]\right|$$

and

$$\max\left|\frac{1}{2}\Big(1-\frac{B_2}{B_1}+\frac{(n+2)(n+1)(1+2\lambda)\mu-(n+1)^2(1+\lambda)^2}{(n+1)^2(1+\lambda)^2}B_1\Big)\right| \quad (\sigma_1\le\mu\le\sigma_2).$$

Therefore using Lemma 1.2, we get

$$|a_3-\mu a_2^2| \;=\; \frac{B_1|c_1|}{2(n+2)(n+1)(1+2\lambda)}\le\frac{B_1}{(n+2)(n+1)(1+2\lambda)}, \quad (\sigma_1\le\mu\le\sigma_2).$$

If $\sigma_1<\mu<\sigma_2$, then we have

$$p_1(z)=\frac{1+\lambda z^2}{1-\lambda z^2}, \qquad (0\le\lambda\le 1).$$

Our result now follows by an application of Lemma 1.2. To show that these bounds are sharp, we define the functions $K_\delta^\phi(\delta=2,3,...)$ by

$$\frac{z(D_\lambda^n K_\delta^\phi(z))'}{D_\lambda^n K_\delta^\phi(z)}=\phi(z^{\delta-1}), \quad K_\delta^\phi(0)=0=(K_\delta^\phi(0))'-1$$

and the function F_γ and G_γ $(0\le\gamma\le 1)$ by

$$\frac{z(D_\lambda^n F_\gamma(z))'}{D_\lambda^n F_\gamma(z)}=\phi\Big(\frac{z(z+\gamma)}{1+\gamma z}\Big), \quad F_\gamma(0)=0=(F_\gamma(0))'-1$$

and

$$\frac{z(D_\lambda^n G_\gamma(z))'}{D_\lambda^n G_\gamma(z)}=\phi\Big(-\frac{z(z+\gamma)}{1+\gamma z}\Big), \quad G_\gamma(0)=0=(G_\gamma(0))'-1$$

Clearly the functions $K_\delta^\phi, F_\gamma, G_\gamma\in M_\lambda^n(\phi)$. Also we write $K^\phi:=K_2^\phi$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality holds if and only if f is K^ϕ or one of its rotations. When $\sigma_1<\mu<\sigma_2$, the equality holds if and only if f is K_3^ϕ or one of its rotations. If $\mu=\sigma_1$ then the equality holds if and only if f is F_γ or one of its rotations. If $\mu=\sigma_2$ then the equality holds if and only if f is G_γ or one of its rotations.

Remark 2.2 *If $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 1.2, Theorem 2.1 can be improved. Let σ_3 be given by*

$$\sigma_3 := \frac{(n+1)^2(1+\lambda)^2\{B_1^2 + B_2\}}{(n+2)(n+1)(1+2\lambda)B_1^2}$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(n+1)^2(1+\lambda)^2}{(n+2)(n+1)(1+2\lambda)B_1^2}\Bigg[B_1 - B_2 + \frac{(n+2)(n+1)(1+2\lambda)\mu - (n+1)^2(1+\lambda)^2}{(n+1)^2(1+\lambda)^2}B_1^2\Bigg]|a_2|^2 \le \frac{B_1}{(n+2)(n+1)(1+2\lambda)}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(n+1)^2(1+\lambda)^2}{(n+2)(n+1)(1+2\lambda)B_1^2}\Bigg[B_1 + B_2 - \frac{(n+2)(n+1)(1+2\lambda)\mu - (n+1)^2(1+\lambda)^2}{(n+1)^2(1+\lambda)^2}B_1^2\Bigg]|a_2|^2 \le \frac{B_1}{(n+2)(n+1)(1+2\lambda)}.$$

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\begin{aligned}
& |a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 \\
= \; & \frac{B_1}{2(n+2)(n+2)(1+2\lambda)}|c_2 - \upsilon c_1^2| + (\mu - \sigma_1)\frac{B_1^2}{4(n+1)^2(1+\lambda)^2}|c_1|^2 \\
= \; & \frac{B_1}{2(n+2)(n+2)(1+2\lambda)}|c_2 - \upsilon c_1^2| \\
& + \left(\mu - \frac{(n+1)^2(1+\lambda)^2\{(B_2 - B_1) + B_1^2\}}{(n+2)(n+1)(1+2\lambda)B_1^2}\right)\frac{B_1^2}{4(n+1)^2(1+\lambda)^2}|c_1|^2 \\
= \; & \frac{B_1}{(n+2)(n+2)(1+2\lambda)}\left\{\frac{1}{2}\big[|c_2 - \upsilon c_1^2| + \upsilon|c_1|^2\big]\right\} \\
\le \; & \frac{B_1}{(n+2)(n+2)(1+2\lambda)}.
\end{aligned}$$

Similarly, for the values of $\sigma_3 \le \mu \le \sigma_2$, we write

$$\begin{aligned}
&|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2\\
= \;& \frac{B_1}{2(n+2)(n+2)(1+2\lambda)}|c_2-\upsilon c_1^2|+(\sigma_2-\mu)\frac{B_1^2}{4(n+1)^2(1+\lambda)^2}|c_1|^2\\
= \;& \frac{B_1}{2(n+2)(n+2)(1+2\lambda)}|c_2-\upsilon c_1^2|\\
&+\left(\frac{(n+1)^2(1+\lambda)^2\{(B_2+B_1)+B_1^2\}}{(n+2)(n+1)(1+2\lambda)B_1^2}-\mu\right)\frac{B_1^2}{4(n+1)^2(1+\lambda)^2}|c_1|^2\\
= \;& \frac{B_1}{(n+2)(n+2)(1+2\lambda)}\left\{\frac{1}{2}\left[|c_2-\upsilon c_1^2|+(1-\upsilon)|c_1|^2\right]\right\}\\
\le \;& \frac{B_1}{(n+2)(n+2)(1+2\lambda)}.
\end{aligned}$$

Thus, the proof of Remark 2.2 is evidently completed.

3 Applications to functions defined by fractional derivatives

For two analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, their convolution (or Hadamard product) is defined to be the function $(f * g)(z)$ given by $(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. For fixed $g \in \mathcal{A}_0$, let $M_\lambda^{n,g}(\phi)$ be the class of functions $f \in \mathcal{A}_0$ for which $(f * g) \in M_\lambda^n(\phi)$.

Definition 3.1 *(see [5], [8]). Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order γ is defined by*

$$D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\gamma}d\zeta \qquad (0 \le \gamma < 1).$$

where the multiplicity of $(z-\zeta)^\gamma$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$. Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [7] introduced the operator $\Omega^\gamma : \mathcal{A}_0 \to \mathcal{A}_0$ defined by

$$\Omega^\gamma f(z) = \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z) \qquad (\gamma \ne 2, 3, 4, ...)$$

The class $M_\delta^{n,\gamma}(\phi)$ consists of functions $f \in \mathcal{A}_0$ for which $\Omega^\gamma f \in M_\delta^n(\phi)$. Note that $M_\delta^{n,\gamma}(\phi)$ is the special case of the class $M_\lambda^{n,g}(\phi)$ when

$$g(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} z^k.$$

Let

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k \qquad (g_k > 0).$$

Since $D_\lambda^n f(z) \in M_\lambda^{n,g}(\phi)$ if and only if $D_\lambda^n f(z) * g(z) \in M_\lambda^n(\phi)$, we obtain the coefficient estimate for functions in the class $M_\lambda^{n,g}(\phi)$, from the corresponding estimate for functions in the class $M_\lambda^n(\phi)$.
Applying Theorem 2.1 for the function $D_\lambda^n f(z) * g(z) = z + (1+\lambda)(n+1)a_2 g_2 z^2 +$ ... we get the following Theorem 3.2 after an obvious change of the parameter μ:

Theorem 3.2 *Let $g(z) = z + \sum\limits_{n=2}^{\infty} g_n z^n, (g_n > 0)$ and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum\limits_{k=1}^{\infty} B_k z^k$. If $D_\lambda^n f(z)$ given by (1.3) belongs to $M_\lambda^{n,g}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{g_3}\left[\frac{B_2}{(n+2)(n+1)(1+2\lambda)} - \frac{\mu g_3 B_1^2}{(n+1)^2(1+\lambda)^2 g_2^2} + \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)}\right] & if \;\; \mu \leq \sigma_1 \; ; \\ \frac{1}{g_3}\left[\frac{B_1}{(n+2)(n+1)(1+2\lambda)}\right] & if \;\; \sigma_1 \leq \mu \leq \sigma_2 \; ; \\ \frac{1}{g_3}\left[-\frac{B_2}{(n+2)(n+1)(1+2\lambda)} + \frac{\mu g_3 B_1^2}{(n+1)^2(1+\lambda)^2 g_2^2} - \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)}\right] & if \;\; \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2 (n+1)^2 (1+\lambda)^2 \{(B_2 - B_1) + B_1^2\}}{g_3 (n+2)(n+1)(1+2\lambda) B_1^2},$$

$$\sigma_2 := \frac{g_2^2 (n+1)^2 (1+\lambda)^2 \{(B_2 + B_1) + B_1^2\}}{g_3 (n+2)(n+1)(1+2\lambda) B_1^2}.$$

The result is sharp.

Since

$$(\Omega^\gamma D_\lambda^n f)(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} [1 + \lambda(k-1)] C(n,k) z^k,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(3-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}.$$

For g_2 and g_3 given by above inequalites, Theorem 3.2 reduces to the following:

Theorem 3.3 *Let* $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$. *If* $D_\lambda^n f(z)$ *given by (1.3) belongs to* $M_\lambda^{n,\gamma}(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_2}{(n+2)(n+1)(1+2\lambda)} - \frac{3(2-\gamma)\mu B_1^2}{2(3-\gamma)(n+1)^2(1+\lambda)^2} + \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)}\right] & if\ \ \mu \leq \sigma_1\ ; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_1}{(n+2)(n+1)(1+2\lambda)}\right] & if\ \ \sigma_1 \leq \mu \leq \sigma_2\ ; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[-\frac{B_2}{(n+2)(n+1)(1+2\lambda)} + \frac{3(2-\gamma)\mu B_1^2}{2(3-\gamma)(n+1)^2(1+\lambda)^2} - \frac{B_1^2}{(n+2)(n+1)(1+2\lambda)}\right] & if\ \ \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{2(3-\gamma)(n+1)^2(1+\lambda)^2\{(B_2 - B_1) + B_1^2\}}{3(2-\gamma)(n+2)(n+1)(1+2\lambda)B_1^2},$$

$$\sigma_2 := \frac{2(3-\gamma)(n+1)^2(1+\lambda)^2\{(B_2 + B_1) + B_1^2\}}{3(2-\gamma)(n+2)(n+1)(1+2\lambda)B_1^2}.$$

The result is sharp.

Remark 3.4 *When* $\lambda = 0$, $n = 0$, $B_1 = 8/\pi^2$ *and* $B_2 = 16/3\pi^2$ *the above Theorem 3.3 reduces to a recent result of Srivastava and Mishra ([3], Theorem 8, p.64) for a class of functions for which* $\Omega^\gamma f(z)$ *is a parabolic starlike functions [1], [2].*

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