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Testing the Null Hypothesis of no Cointegration against Seasonal Fractional Cointegration

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Abstract

In this article we propose a procedure for testing the null hypothesis of no cointegration against the alternative of seasonal fractional cointegration. It is a two-step procedure based on the univariate tests of Robinson (1994). Finite-sample critical values are computed, and the power properties of the tests are examined. The tests are also extended to allow seasonally fractionally cointegrated alternatives at each of the seasonal frequencies separately. An empirical application, illustrating the use of the tests, is also carried out at the end of the article.

Mathematics Subject Classification: 62P20; 91B70

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1. Introduction

Modelling the seasonal component of macroeconomic time series is a matter that still remains controversial. Seasonal dummy variables were initially employed, but they were shown to be inappropriate in many cases, especially if the seasonal component changes or evolves over time. Following the unit root approach (initia lly developed by Box & Jenkins, 1970, and widely used after the seminal paper by Nelson & Plosser, 1982), seasonal unit root models became popular and many test statistics of this type were developed by Dickey, Hasza & Fuller (DHF, 1984); Hylleberg, Engle, Granger & Yoo (HEGY, 1990); Canova & Hansen (1995) and others. Seasonal unit root models were later extended to allow for other types of long memory behaviour, in particular, allowing for a fractional degree of integration (see, e.g. Porter-Hudak, 1990; Ray, 1993; Sutcliffe, 1994 and more recently, Gil-Alana & Robinson, 2001). The idea behind the concept of seasonal fractional integration is that the number of seasonal differences required to obtain stationarity might not necessarily be an integer but a real value. Thus, assuming that s is the number of time periods within a year, the seasonal polynomial $(1 - L^s)^d$ can be expressed in terms of its Binomial expansion, such that, for all real d,

$$(1 - L^s)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j L^{s\,j} = 1 - d L^s + \frac{d(d-1)}{2} L^{2s} - \ldots,$$

and higher the d is, the higher is the level of association between the observations far apart in time.

The concept of seasonal fractional cointegration has hardly been investigated. For the purpose of the present paper, we say that a given vector Y_t is seasonally fractionally cointegrated if: a) all its components (y_{it}) are seasonally fractionally integrated of the same order, say d, i.e., $(1 - L^s)^d y_{it} = u_{it}$ for all i, where u_{it} is an I(0) process defined as in Section 2, and b) there is at least one linear combination of these components, which is seasonally fractionally integrated of order b, with $b < d$. Other more complex definitions of seasonal fractional cointegration allow us to consider different orders of integration for each of the individual series. However, Robinson & Marinucci (2001) show that, in a bivariate context, a necessary condition for cointegration is that both individual series share the same order of integration. In the context of fractional processes, this assumption may appear unrealistic because of the continuity on the real line for the orders of integration. However, in empirical work, there might be cases when, even though the orders of integration of both series are fractional and different, the tests are unable to reject the unit root null ($d = 1$). In such cases, we can proceed further with cointegration analysis.

In this article we propose a two-step procedure for testing the null hypothesis of no cointegration against the alternative of seasonal fractional cointegration, which is based on the univariate tests of Robinson (1994). The outline of the paper is as follows: Section 2 firstly describes the tests of Robinson (1994). Then, the two-step procedure against seasonally fractionally cointegrated alternatives is presented. Section 3 gives finite-sample critical values of the new tests, and the testing procedure is extended to the case of fractionally cointegrated alternatives for each of the frequencies separately. The power properties of the tests against different fractional alternatives are examined in Section 4. Section 5 contains an empirical application, and, finally, Section 6 offers some concluding remarks.

2. The tests of Robinson (1994) and seasonal fractional cointegration

We firstly describe a version of the univariate tests of Robinson (1994) for testing seasonally fractionally integrated hypotheses. Assume that y_t is the observed time series, $t = 1, 2, \ldots, T$, and consider the following model,

$$(1 - L^s)^d \, y_t = u_t, \qquad t = 1, 2, \ldots, \tag{1}$$

where L^S is the seasonal lag operator ($L^S y_t = y_{t-s}$); d is a real number and u_t is an I(0) process, defined as a covariance stationary process with spectral density function that is positive and finite at any frequency on the interval $[0, \pi]$. Clearly, if d = 0 in (1), $y_t = u_t$, and a 'weakly autocorrelated' y_t is allowed for. However, if d > 0, y_t is defined as a 'long memory' process, also called 'strongly dependent' and so-named because of the strong association (in the seasonal structure) between observations far apart in time. If $d \in (0, 0.5)$, y_t is covariance stationary, having autocovariances which decay much more slowly than those of a seasonal ARMA process, in fact, so slowly as to be non-summable; if $d \geq 0.5$ y_t is nonstationary and, as d increases beyond 0.5, can be viewed as becoming 'more nonstationary' in the sense, for example, that the variance of the partial sums increases in magnitude. Note that the variance of the partial sums is $O(T^{2d+1})$, so that stationarity implies d > 0.5. (See Hosking, 1981).

Few empirical applications can be found based on seasonal fractional models. The notion of fractional Gaussian noise with seasonality was initially suggested by Abrahams & Dempster (1979) and Jonas (1981), and extended in a Bayesian framework by Carlin, Dempster & Jonas (1985) and Carlin & Dempster (1989). Porter-Hudak (1990) applied a seasonally fractionally integrated model to quarterly U.S. monetary aggregates with the conclusion that a fractional model could be more appropriate than standard ARIMAs. The advantages of seasonally fractionally integrated models for forecasting are illustrated in Ray (1993) and Sutcliffe (1994), and another recent empirical application can be found in Gil-Alana & Robinson (2001).

In general, we want to test the null hypothesis:

$$H_o: \quad d = d_o, \tag{2}$$

for a given real number d_o, and the test statistic proposed by Robinson (1994), which is based on the Lagrange Multiplier (LM) principle, is given by:

$$\hat{r} = \left(\frac{T}{\hat{A}}\right)^{1/2} \frac{\hat{a}}{\hat{\sigma}^2}, \tag{3}$$

where

$$\hat{\sigma}^2 = \sigma^2(\hat{\tau}) = \frac{2\pi}{T}\sum_j^* g(\lambda_j;\hat{\tau})^{-1} I(\lambda_j); \quad \hat{a} = \frac{-2\pi}{T}\sum_j^* \psi(\lambda_j) g(\lambda_j;\hat{\tau})^{-1} I(\lambda_j); \quad \lambda_j = \frac{2\pi j}{T}$$

$$\hat{A} = \frac{2}{T}\left(\sum_j^* \psi(\lambda_j)^2 - \sum_j^* \psi(\lambda_j)\hat{\varepsilon}(\lambda_j)' \times \left(\sum_j^* \hat{\varepsilon}(\lambda_j)\hat{\varepsilon}(\lambda_j)'\right)^{-1} \times \sum_j^* \hat{\varepsilon}(\lambda_j)\psi(\lambda_j)\right),$$

$$\psi(\lambda_j) = \log\left|\sin\frac{\lambda_j}{2}\right| + \log\left(2\cos\frac{\lambda_j}{2}\right) + \log\left|2\cos\lambda_j\right|; \quad \hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial\tau}\log g(\lambda_j;\hat{\tau}).$$

$I(\lambda_j)$ is the periodogram of $\hat{u}_t$, where $\hat{u}_t = (1-L^s)^{d_o} y_t$, and the function g above is a known function coming from the spectral density of u_t, $f(\lambda;\tau) = (\sigma^2/2\pi)\, g(\lambda;\tau)$, evaluated at $\hat{\tau} = \arg\min_{\tau\in T^*} \sigma^2(\tau)$, where T^* is a compact subset of the R^q Euclidean space. Finally, the summation on * in the above expressions is over $\lambda \in M$ where $M = \{\lambda: -\pi < \lambda < \pi, \lambda \notin (\rho_l - \lambda_1, \rho_l + \lambda_1), l = 1, 2, \ldots, s\}$ such that $\rho_l = 0, \pi/2, -\pi/2$ and π are the distinct poles of $\psi(\lambda)$ on $(-\pi, \pi]$. Note that these tests are purely parametric and, therefore, they require specific modelling assumptions about the short memory specification of u_t. Thus, if u_t is white noise, $g \equiv 1$, (and thus, $\hat{\varepsilon}(\lambda_j) = 0$), and if u_t is an AR process of form $\phi(L)u_t = \varepsilon_t$, $g = |\phi(e^{i\lambda})|^{-2}$, with $\sigma^2 = V(\varepsilon_t)$, so that the AR coefficients are a function of τ.

Under the null hypothesis (2), Robinson (1994) showed that, under certain regularity conditions,

$$\hat{r} \;\to_d\; N(0,1) \qquad as \quad T \to \infty. \tag{4}$$

These conditions are very mild, regarding technical assumptions on $\psi(\lambda)$, which are satisfied by model (1). Thus, an approximate one-sided test of (2) against H_a: $d > d_o$ will be given by the rule: "Reject H_o (2) if $\hat{r} > z_\alpha$", where the probability that a normal variate exceeds z_α is α, and conversely, a test of (2) against H_a: $d < d_o$ will be given by the rule: "Reject H_o (2) if $\hat{r} < -z_\alpha$". As these rules indicate, we are in a classical large sample testing situation, for the reasons spelt out in Robinson (1994), who also showed that the tests are efficient in the Pitman sense, i.e., that against local alternatives of form: H_a: $d = d_o + \delta T^{-1/2}$ for $\delta \neq 0$, $\hat{r}$ has an asymptotic distribution given by a normal distribution with variance 1 and mean that cannot (when u_t is Gaussian) be exceeded in absolute value by any rival regular statistic.

The test statistic presented just above was used in Gil-Alana & Robinson (2001) to study the seasonal (quarterly) structure of the UK and Japanese consumption and income. Other versions of Robinson's (1994) tests, based on annual, seasonal (monthly) and cyclical models, were studied in Gil-Alana & Robinson (1997) and Gil-Alana (1999, 2001) respectively. Ooms (1997) also proposed tests based on seasonal fractional models. They are Wald tests, and thus require efficient estimates of the fractional differencing parameters. He used a modified periodogram regression estimation procedure due to Hassler (1994). Also, Hosoya (1997) established the limit theory for long memory processes with the singularities not restricted at the zero frequency, and proposed a set of quasi log-likelihood statistics to be applied in raw time series. Unlike these methods, the tests of Robinson (1994) do not require estimation of the long memory parameters, since the differenced series have short memory under the null.

Next, we introduce a testing procedure, based on $\hat{r}$ in (3), for testing the null hypothesis of no cointegration against the alternative of seasonal fractional cointegration. For simplicity, we consider a bivariate system of two time series (y_{1t} and y_{2t}) that might be seasonally fractionally cointegrated. In this bivariate context, a necessary condition for cointegration is that both series must have the same degree of integration (say d_o). Thus, in the first step, we can use Robinson's (1994) univariate tests described above, to test the order of integration of each of the individual series and, if both are (seasonally) integrated of the same order (say $d_o = 1$), we can go further and test the degree of integration of the residuals from the cointegrating regression. There also exist multivariate versions of the tests of Robinson (1994) for simultaneously testing the degree of integration of the individual series (e.g., Gil-Alana, 2003a). This procedure, however, has only been developed for non-seasonal cases, and the extension to the seasonal case is still in progress. It might be argued that the use of Robinson's (1994) tests on the individual series is not adequate since the two series may be dependent. In general, this is a problem that is faced by all univariate procedures. Note, however, that this is the same approach as the one used by Engle & Granger (1987) in their classical paper on cointegration, and also by Cheung & Lai (1993) and Dueker & Startz (1998) when testing for cointegration at the long run frequency. A problem occurs here, as the residuals are not actually observed but obtained from minimizing the residual variance of the cointegrating regression, and thus a bias might appear in favour of stationary residuals. Note that this problem is similar to the one noticed by Engle & Granger (1987) when testing cointegration at the long run or zero frequency with the tests of Fuller (1976) and Dickey & Fuller (1979). (See, also Phillips & Ouliaris, 1991, and Kremers, Ericsson & Dolado, 1992). In order to solve this problem, finite-sample critical values of the tests will be computed in the next section. We can consider the model:

$$(1 - L^s)^d \, e_t \;=\; v_t, \qquad t \;=\; 1, 2, \ldots,$$

where e_t are the OLS residuals from the cointegration regression of y_{1t} on y_{2t} (or viceversa) and v_t is I(0), and test H_o (2) against the alternative:

$$H_a : \quad d \;<\; d_o. \tag{5}$$

Note that if we cannot reject H_o (2) on the estimated residuals above, we will find evidence of no cointegration, since the residuals will be integrated of the same order as the individual series. On the other hand, rejections of H_o (2) against (5) ($d < d_o$) will support fractional cointegration, since the estimated residuals will be integrated of a smaller order than that of the individual series.

3. Finite-sample critical values and extensions of the tests

Table 1 reports finite-sample critical values of the tests of Robinson (1994) for testing the null hypothesis of no cointegration against seasonal fractional cointegration. We use Monte Carlo simulations based on 50,000 replications, for sample sizes T = 48, 96, 144 and 192, assuming that the true system consists of two quarterly I(d) seasonal processes of the form:

$$(1 - L^4)^{d_o} \, y_{it} \;=\; \varepsilon_{it}, \qquad t \;=\; 1, 2, \ldots, \qquad i = 1, 2, \tag{6}$$

with Gaussian independent white noise disturbances that are not cointegrated, and take values of d_o ranging from 0.6 through 1.5 with 0.1 increments. We have concentrated on values of $d > 0.5$ since most macroeconomic time series are nonstationary, though the results based on $d < 0.5$ are not substantially different from those reported in the tables.

We can see that the finite-sample critical values are smaller than those from the normal distribution, which is consistent with the earlier argument that, when testing H_o (2) against (5), the use of the standard critical values will result in the cointegrating tests rejecting the null hypothesis of no cointegration too often. We can also note that these critical values are similar around d_o and, as we increase the number of observations, they approximate the values from the normal distribution.

The seasonal structure described in the preceding section may be too restrictive, in the sense that it imposes the same degree of integration at each of the frequencies of the process. Note, for example, that the polynomial $(1 – L^4)$ can be factored as $(1 - L) (1 + L) (1 + L^2)$, containing four roots of modulus unity: one at the long run or zero frequency, one at two cycles per year, corresponding to the frequency π, and two complex pairs at one cycle per year, corresponding to frequencies $\pi/2$ and $-\pi/2$. Thus, the seasonal process (6) imposes the same degree of integration, d_o, at each of these frequencies.

Next, we consider the possibility of two time series being cointegrated at a single frequency, i.e. following the same structure at a given frequency, for a given value d_o. We examine the possibility of their being cointegrated, either at the zero frequency or, alternatively, at the seasonal frequencies π or $\pi/2$ $(-\pi/2)$.

TABLE 1

Finite-sample critical values of the tests of Robinson (1994) for testing the null hypothesis of no cointegration against seasonal fractional cointegration

T = 48

P. / d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
0.1 %	-3.16	-3.16	-3.20	-2.21	-3.21	-3.17	-3.19	-3.20	-3.20	-3.20	-3.19
0.5 %	-2.94	-2.93	-2.93	-2.94	-2.95	-2.95	-2.95	-2.95	-2.95	-2.95	-2.95
1 %	-2.81	-2.81	-2.82	-2.82	-2.84	-2.84	-2.84	-2.83	-2.84	-2.83	-2.84
2 %	-2.68	-2.69	-2.70	-2.71	-2.71	-2.71	-2.71	-2.70	-2.70	-2.70	-2.70
2.5 %	-2.64	-2.65	-2.66	-2.66	-2.66	-2.66	-2.66	-2.66	-2.66	-2.66	-2.66
5 %	-2.49	-2.50	-2.50	-2.51	-2.51	-2.51	-2.51	-2.51	-2.51	-2.51	-2.51
10 %	-2.30	-2.31	-2.33	-2.32	-2.32	-2.32	-2.32	-2.32	-2.32	-2.32	-2.32

T = 96

P. / d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
0.1 %	-2.94	-2.94	-2.93	-2.93	-2.93	-2.93	-2.93	-2.93	-2.93	-2.93	-2.93
0.5 %	-2.66	-2.68	-2.68	-2.67	-2.67	-2.67	-2.66	-2.66	-2.66	-2.66	-2.66
1 %	-2.55	-2.56	-2.56	-2.56	-2.56	-2.55	-2.56	-2.56	-2.55	-2.55	-2.55
2 %	-2.40	-2.40	-2.40	-2.40	-2.40	-2.40	-2.40	-2.40	-2.40	-2.39	-2.39
2.5 %	-2.35	-2.34	-2.35	-2.35	-2.35	-2.35	-2.35	-2.35	-2.34	-2.34	-2.34
5 %	-2.19	-2.19	-2.19	-2.18	-2.18	-2.18	-2.18	-2.18	-2.17	-2.17	-2.17
10 %	-1.98	-1.99	-1.99	-1.98	-1.98	-1.98	-1.97	-1.97	-1.97	-1.97	-1.97

T = 144

P. / d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
0.1 %	-2.80	-2.82	-2.84	-2.84	-2.82	-2.83	-2.83	-2.81	-2.80	-2.78	-2.78
0.5 %	-2.53	-2.53	-2.52	-2.52	-2.51	-2.51	-2.52	-2.51	-2.51	-2.51	-2.51
1 %	-2.39	-2.39	-2.41	-2.41	-2.41	-2.40	-2.41	-2.40	-2.41	-2.41	-2.40
2 %	-2.24	-2.25	-2.25	-2.25	-2.25	-2.24	-2.24	-2.24	-2.24	-2.24	-2.24
2.5 %	-2.20	-2.20	-2.20	-2.20	-2.19	-2.18	-2.19	-2.19	-2.19	-2.19	-2.19
5 %	-2.02	-2.03	-2.03	-2.03	-2.03	-2.03	-2.02	-2.03	-2.02	-2.02	-2.02
10 %	-1.82	-1.82	-1.82	-1.83	-1.82	-1.82	-1.81	-1.81	-1.81	-1.81	-1.81

T = 192

P. / d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
0.1 %	-2.77	-2.77	-2.76	-2.75	-2.72	-2.72	-2.73	-2.73	-2.74	-2.74	-2.74
0.5 %	-2.52	-2.51	-2.53	-2.51	-2.49	-2.48	-2.49	-2.49	-2.49	-2.50	-2.51
1 %	-2.34	-2.35	-2.35	-2.34	-2.35	-2.36	-2.36	-2.35	-2.35	-2.34	-2.34
2 %	-2.17	-2.17	-2.18	-2.18	-2.18	-2.17	-2.18	-2.18	-2.18	-2.18	-2.18
2.5 %	-2.12	-2.12	-2.13	-2.13	-2.13	-2.13	-2.13	-2.13	-2.12	-2.13	-2.12
5 %	-1.94	-1.95	-1.95	-1.95	-1.94	-1.94	-1.94	-1.94	-1.94	-1.94	-1.93
10 %	-1.71	-1.71	-1.71	-1.71	-1.71	-1.71	-1.71	-1.70	-1.70	-1.70	-1.70

50,000 replications were used in each case.

The procedure is exactly the same as before. Once we have shown that both series are integrated of the same order (d_o) at a given frequency, we test H_o (2) against (5) with the one-sided tests of Robinson (1994), in the model,

$$(1 - L)^d e_t = v_t, \qquad t = 1, 2, \ldots, \tag{7}$$

if we focus on the long run or zero frequency, or alternatively, in the models,

$$(1 + L)^d e_t = v_t, \qquad t = 1, 2, \ldots \tag{8}$$

or

$$(1 + L^2)^d e_t = v_t, \qquad t = 1, 2, \ldots, \tag{9}$$

if we concentrate on the seasonal frequencies π, or π/2 (-π/2) respectively. Based on (7), (8) and (9), the test statistics will be given by $\hat{r}_i$, i = 1, 2 and 3 respectively, where

$$\hat{r}_i = \left(\frac{T}{\hat{A}_i}\right)^{1/2} \frac{\hat{a}_i}{\hat{\sigma}_i^{\,2}}, \quad i = 1, 2, 3, \tag{10}$$

with

$$\hat{a}_i = \frac{-2\pi}{T} \sum_j^{*} \psi_i(\lambda_j)\, g(\lambda_j; \hat{\tau})^{-1}\, I_i(\lambda_j),$$

$$\hat{A} = \frac{2}{T}\left(\sum_j^{*} \psi_i(\lambda_j)^2 - \sum_j^{*} \psi_i(\lambda_j)\, \varepsilon_i(\lambda_j)' \times \left(\sum_j^{*} \hat{\varepsilon}_i(\lambda_j)\, \hat{\varepsilon}_i(\lambda_j)' \right)^{-1} \times \sum_j^{*} \hat{\varepsilon}_i(\lambda_j)\, \psi_i(\lambda_j) \right),$$

$$\psi_1(\lambda_j) = \log\left| 2 \sin\frac{\lambda_j}{2} \right|; \quad \psi_2(\lambda_j) = \log\left(2\cos\frac{\lambda_j}{2} \right); \quad \psi_3(\lambda_j) = \log\left| 2 \cos\lambda_j \right|;$$

and $\hat{\sigma}_i^2$, $\hat{\varepsilon}_i$, g_i and I_i as below (3) but for the new residuals obtained from (7), (8) and (9). Finite-sample critical values of the new versions of the tests were also computed and the results are given in Table 2.

TABLE 2

Finite-sample critical values of the tests of Robinson (1994) for testing the null hypothesis of no cointegration against seasonal fractional cointegration at a given frequency

						T = 48						
ρ(L)	P./ d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
$(1 - L)^d$	1 %	-2.56	-2.57	-2.56	-2.55	-2.55	-2.53	-2.52	-2.53	-2.53	-2.51	-2.50
	5 %	-2.10	-2.11	-2.11	-2.11	-2.11	-2.10	-2.09	-2.08	-2.08	-2.08	-2.06
$(1 + L)^d$	1 %	-2.49	-2.51	-2.52	-2.52	-2.51	-2.51	-2.51	-2.51	-2.50	-2.48	-2.47
	5 %	-2.03	-2.05	-2.05	-2.05	-2.04	-2.04	-2.04	-2.03	-2.03	-2.03	-2.02
$(1 + L^2)^d$	1 %	-2.96	-2.97	-2.97	-2.97	-2.97	-2.98	-2.97	-2.98	-2.90	-2.96	-2.96
	5 %	-2.52	-2.53	-2.54	-2.54	-2.53	-2.52	-2.53	-2.53	-2.53	-2.53	-2.53
						T = 96						
ρ(L)	P./ d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
$(1 - L)^d$	1 %	-2.50	-2.50	-2.49	-2.48	-2.48	-2.48	-2.47	-2.46	-2.45	-2.43	-2.44
	5 %	-2.03	-2.04	-2.03	-2.04	-2.02	-2.01	-2.00	-1.99	-1.99	-1.99	-1.98
$(1 + L)^d$	1 %	-2.44	-2.45	-2.47	-2.47	-2.47	-2.47	-2.48	-2.46	-2.46	-2.45	-2.46
	5 %	-1.98	-1.98	-1.99	-1.99	-1.99	-1.98	-1.97	-1.97	-1.97	-1.96	-1.96
$(1 + L^2)^d$	1 %	-2.83	-2.84	-2.85	-2.86	-2.85	-2.83	-2.82	-2.82	-2.83	-2.83	-2.83
	5 %	-2.31	-2.32	-2.33	-2.32	-2.32	-2.32	-2.32	-2.32	-2.32	-2.32	-2.33
						T = 144						
ρ(L)	P./ d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
$(1 - L)^d$	1 %	-2.46	-2.46	-2.47	-2.47	-2.45	-2.42	-2.41	-2.41	-2.42	-2.41	-2.41
	5 %	-1.95	-1.96	-1.95	-1.94	-1.94	-1.94	-1.94	-1.93	-1.93	-1.92	-1.92
$(1 + L)^d$	1 %	-2.46	-2.47	-2.45	-2.44	-2.42	-2.40	-2.39	-2.39	-2.38	-2.38	-2.37
	5 %	-1.94	-1.95	-1.96	-1.94	-1.92	-1.92	-1.91	-1.90	-1.89	-1.89	-1.89
$(1 + L^2)^d$	1 %	-2.76	-2.77	-2.77	-2.77	-2.76	-2.76	-2.76	-2.78	-2.78	-2.76	-2.77
	5 %	-2.23	-2.23	-2.23	-2.23	-2.23	-2.22	-2.22	-2.22	-2.21	-2.21	-2.21
						T = 192						
ρ(L)	P./ d_o	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
$(1 - L)^d$	1 %	-2.43	-2.43	-2.44	-2.44	-2.43	-2.41	-2.40	-2.41	-2.41	-2.40	-2.39
	5 %	-1.93	-1.94	-1.94	1.93	-1.93	-1.92	-1.90	-1.89	-1.88	-1.88	-1.88
$(1 + L)^d$	1 %	-2.45	-2.46	-2.46	-2.45	-2.43	-2.41	-2.39	-2.41	-2.40	-2.37	-2.47
	5 %	-1.93	-1.93	-1.94	-1.94	-1.94	-1.91	-1.91	-1.89	-1.88	-1.89	-1.95
$(1 + L^2)^d$	1 %	-2.71	-2.72	-2.71	-2.71	-2.70	-2.70	-2.70	-2.71	-2.71	-2.71	-2.71
	5 %	-2.16	-2.17	-2.17	-2.17	-2.16	-2.15	-2.15	-2.15	-2.14	-2.15	-2.15

50,000 replications were used in each case.

Similarly to Table 1, we see that all the critical values are smaller than those from the normal distribution, with slight differences in some cases across d_o. When increasing the sample sizes, they become higher but, even with T = 192, they are still below those corresponding to the normal distribution. These results reinforce the argument that the use of the standard critical values when testing cointegration with the tests of Robinson (1994) will lead to reject the null hypothesis of no cointegration more often than expected, suggesting that finite-sample critical values should be employed.

4. The power of the tests against fractional alternatives

In this section we examine the power properties of Robinson's (1994) tests against fractionally cointegrated alternatives, and consider a bivariate system, where y_{1t} and y_{2t} are given by:

$$y_{1t} + y_{2t} = u_{1t}, \qquad t = 1, 2, ...$$
$$y_{1t} + 2y_{2t} = u_{2t}, \qquad t = 1, 2, ..., \qquad (11)$$

where, initially,

$$(1 - L^4)\, u_{1t} = \varepsilon_{1t}, \qquad t = 1, 2, ... \qquad (12)$$

and

$$(1 - L^4)^d\, u_{2t} = \varepsilon_{2t}, \qquad t = 1, 2, ..., \qquad (13)$$

with the innovations ε_{1t} and ε_{2t}, generated as independent standard normal variates. Thus, if d = 1 in (13), the two series are quarterly I(1) and non-cointegrated, while d < 1 will imply that y_{1t} and y_{2t} are seasonally fractionally cointegrated, and (11) will be the cointegrating relationship. We also consider cases where the root occurs at a single frequency, that is, u_{1t} and u_{2t}, are generated as

$$(1 - L)\, u_{1t} = \varepsilon_{1t}, \qquad t = 1, 2, ...$$
$$(1 - L)^d\, u_{2t} = \varepsilon_{2t}, \qquad t = 1, 2, ..., \qquad (14)$$

or alternatively,

$$(1 + L)\, u_{1t} = \varepsilon_{1t}, \qquad t = 1, 2, ...$$
$$(1 + L)^d\, u_{2t} = \varepsilon_{2t}, \qquad t = 1, 2, ..., \qquad (15)$$

or

$$(1 + L^2)\, u_{1t} = \varepsilon_{1t}, \qquad t = 1, 2, ...$$
$$(1 + L^2)^d\, u_{2t} = \varepsilon_{2t}, \qquad t = 1, 2, ... \qquad (16)$$

Again, in all these cases, if d = 1 in (14) – (16), y_{1t} and y_{2t} will be non-cointegrated and if 0 < d < 1, both series will be fractionally cointegrated with the roots occurring at zero, at π, and at $\pi/2$ ($-\pi/2$) respectively. Table 3 reports the rejection frequencies of $\hat{r}$ in (3) and (10) with d = 0, (0.10), 0.90; T = 48, 96, 144 and 192 and nominal sizes of 5% and 1%, based on 50,000 replications.

We see that the rejection frequencies considerably improve as d becomes smaller and also as we increase the number of observations. These values are relatively high in all cases if d ≤ 0.60 and T = 144 or 192. Starting with the case of four seasonal roots, we observe that if T ≥ 96, the rejection probabilities are higher than 0.50 for all cases with d ≤ 0.60 even at the 1% significance level. Looking at the results for the individual frequencies, the rejection probabilities are also relatively high, especially if the sample size is 144 or 192. Thus, for example, if T = 192 and d_o = 0.70, the rejection frequencies associated to the frequencies 0,

π and $\pi/2$ ($-\pi/2$) are respectively 0.959, 0.954 and 0.964 at the 5% level. Similar experiments were also carried out based on autocorrelated disturbances. Finite-sample critical values were computed and the power properties examined. If the roots of the AR (MA) polynomials are close to the unit root circle, the results are poor. However, if they are far away from 1, they are similar to those reported here, the rejection probabilities being relatively high for $d \leq 0.6$ and $T \geq 144$.

TABLE 3

Rejection frequencies of the tests of Robinson (1994) against fractional cointegration

ρ(L)	T	Sz/ d	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
$(1-L^4)^d$	48	1%	0.021	0.049	0.106	0.197	0.329	0.471	0.619	0.735	0.825	0.886
		5%	0.088	0.159	0.287	0.437	0.600	0.733	0.844	0.913	0.952	0.974
	96	1%	0.037	0.138	0.352	0.650	0.870	0.961	0.992	0.998	0.999	1.000
		5%	0.136	0.349	0.635	0.866	0.968	0.994	0.999	1.000	1.000	1.000
	144	1%	0.057	0.267	0.659	0.918	0.991	0.999	0.999	1.000	1.000	1.000
		5%	0.186	0.523	0.859	0.981	0.999	1.000	1.000	1.000	1.000	1.000
	192	1%	0.068	0.381	0.822	0.985	0.999	1.000	1.000	1.000	1.000	1.000
		5%	0.226	0.670	0.947	0.997	1.000	1.000	1.000	1.000	1.000	1.000
ρ(L)	T	Sz/ d	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
$(1-L)^d$	48	1%	0.025	0.072	0.176	0.351	0.549	0.741	0.867	0.940	0.970	0.988
		5%	0.108	0.231	0.420	0.630	0.815	0.924	0.975	0.991	0.997	0.999
	96	1%	0.045	0.199	0.506	0.821	0.962	0.994	0.999	1.000	1.000	1.000
		5%	0.151	0.423	0.761	0.939	0.992	0.999	1.000	1.000	1.000	1.000
	144	1%	0.084	0.377	0.794	0.968	0.999	1.000	1.000	1.000	1.000	1.000
		5%	0.233	0.620	0.917	0.989	0.999	1.000	1.000	1.000	1.000	1.000
	192	1%	0.112	0.535	0.908	0.992	0.999	1.000	1.000	1.000	1.000	1.000
		5%	0.279	0.741	0.959	0.997	0.999	1.000	1.000	1.000	1.000	1.000
ρ(L)	T	Sz/ d	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
$(1+L)^d$	48	1%	0.021	0.064	0.153	0.317	0.524	0.717	0.848	0.924	0.964	0.981
		5%	0.103	0.215	0.394	0.629	0.807	0.924	0.969	0.990	0.996	0.998
	96	1%	0.045	0.177	0.492	0.804	0.952	0.993	0.999	1.000	1.000	1.000
		5%	0.159	0.423	0.753	0.938	0.992	0.999	1.000	1.000	1.000	1.000
	144	1%	0.079	0.374	0.771	0.965	0.997	0.999	1.000	1.000	1.000	1.000
		5%	0.221	0.612	0.902	0.989	0.999	1.000	1.000	1.000	1.000	1.000
	192	1%	0.104	0.508	0.893	0.990	0.999	1.000	1.000	1.000	1.000	1.000
		5%	0.268	0.719	0.954	0.996	0.999	1.000	1.000	1.000	1.000	1.000
ρ(L)	T	Sz/ d	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
$(1+L^2)^d$	48	1%	0.026	0.058	0.140	0.272	0.455	0.625	0.782	0.875	0.938	0.964
		5%	0.097	0.200	0.359	0.551	0.735	0.862	0.940	0.973	0.991	0.995
	96	1%	0.039	0.163	0.446	0.753	0.929	0.989	0.998	0.999	0.999	1.000
		5%	0.145	0.395	0.713	0.927	0.987	0.999	1.000	1.000	1.000	1.000
	144	1%	0.064	0.315	0.726	0.950	0.996	0.999	1.000	1.000	1.000	1.000
		5%	0.200	0.594	0.901	0.988	0.999	1.000	1.000	1.000	1.000	1.000
	192	1%	0.092	0.484	0.890	0.991	1.000	1.000	1.000	1.000	1.000	1.000
		5%	0.269	0.736	0.964	0.997	1.000	1.000	1.000	1.000	1.000	1.000

5. An empirical illustration

We analyse the quarterly UK and Japanese consumption and income series that were used by Hylleberg et al. (1990) and Hylleberg, Engle, Granger & Lee (1993). For the UK, the data are the log consumption expenditure on non-durables and the log personal disposable income, from 1955.1 to 1984.4, and, for Japan, the log of total real consumption and the log of real disposable income from 1961.1 to 1987.4 in 1980 prices. These data were also used by Gil-Alana & Robinson (2001) to test the presence of unit and fractional roots in univariate contexts. The results from that paper for the case of testing unit roots with white noise disturbances are summarized in Table 4. We see that, in both countries, the unit root null hypothesis cannot be rejected for any series. This is found when we impose four unit roots simultaneously (i.e., model (6) with $d_o = 1$) but also when each of the roots is considered separately.

TABLE 4

Testing the null hypothesis of a unit root (H_o: d = 1) with the tests of Robinson (1994) on the individual series

Model	Country: UNITED KINGDOM Consumption	UNITED KINGDOM Income	JAPAN Consumption	JAPAN Income
$(1 - L^4)^d x_t = u_t$	-1.00’	-1.00’	-1.02’	-1.05’
$(1 - L)^d x_t = u_t$	-0.30’	-0.31’	-0.37’	-1.07’
$(1 + L)^d x_t = u_t$	-0.92’	-1.09’	-0.98’	-1.06’
$(1 + L^2)^d x_t = u_t$	-1.19’	-1.51’	-1.12’	-1.42’

‘ and in bold: Non-rejection value of a unit rot at the 95% significance level. The results in this table have been taken from Gil-Alana and Robinson (2001).

Next we look at the possibility of both series (consumption and income) being cointegrated. The resulting OLS regressions were

$$\underset{}{c_t = \underset{(0.114)}{1.212} + \underset{(0.011)}{0.872}\, y_t} \quad \text{and} \quad y_t = \underset{(0.145)}{-1.171} + \underset{(0.011)}{1.124}\, c_t$$

for the UK, and

$$c_t = \underset{(0.100)}{0.389} + \underset{(0.017)}{0.901}\, y_t \quad \text{and} \quad y_t = \underset{(0.115)}{-0.193} + \underset{(0.017)}{1.065}\, c_t$$

for Japan.

TABLE 5

Testing the null hypothesis of no cointegration (d = 1) against seasonal fractional cointegration (d < 1) with the tests of Robinson

Model	UNITED KINGDOM		JAPAN	
	Cons. / Inc.	Inc. / Cons.	Cons. / Inc.	Inc. / Cons.
$(1 - L^4)^d x_t = u_t$	-2.58’	-2.55’	-2.11	-1.92
$(1 - L)^d x_t = u_t$	-4.69’	-4.67’	-5.37’	-5.38’
$(1 + L)^d x_t = u_t$	-4.56’	-4.72’	-4.87’	-4.41’
$(1 + L^2)^d x_t = u_t$	-6.71’	-6.77’	-6.93’	-6.95’

(Country: UNITED KINGDOM, JAPAN)

‘ and in bold: Rejection values of the null hypothesis of no cointegration against fractional cointegration at the 95% significance level.

Table 5 reports values of the tests of Robinson (1994), testing the null hypothesis of no cointegration against seasonal fractional cointegration, first imposing the same order of integration at all frequencies, and then testing each frequency separately. That is, we calculate $\hat{r}$ given by (3) and (10), testing H_o (2) against (5) with $d_o = 1$, firstly in

$$(1 - L^4)^d e_t = v_t, \qquad t = 1, 2, ... \qquad (17)$$

and then, in (7) – (9).

Starting with the case of four seasonal roots, (i.e., (17)), we see that the null hypothesis of no cointegration is clearly rejected for the UK. However, this hypothesis cannot be rejected for Japan, even at the 10% significance level. If we look at the results for each of the frequencies separately, (i.e., (7) – (9)), we observe that in both countries, all cases lead to rejections of the null in favour of cointegration. The results for Japan might seem surprising, since we find evidence of cointegration at 0, π and π/2 (3π/2) when testing these frequencies separately, but we cannot reject the null hypothesis of no cointegration when these frequencies are tested together. This may be explained by the fact that all the test statistics outlined in this section have been evaluated using white noise disturbances, and thus the lack of rejection in the case of Japan when testing all roots simultaneously might reflect the potentially autocorrelated structure underlying the I(0) disturbances in the estimated residuals of the cointegrating regressions.

6. Conclusions

In this paper we have presented a procedure for testing the null hypothesis of no cointegration against seasonal fractional cointegration. It is a two-step procedure based on the univariate tests of Robinson (1994). Initially, we test the order of integration of the individual series and, if all of them have the same degree of integration, we proceed to testing the order of integration of the estimated residuals from the cointegrating regressions. A similar procedure was proposed by Gil-Alana (2003b) in non-seasonal contexts.

We first examined the case of processes with the same degree of integration at all frequencies (i.e., at zero and the seasonal ones). Then, the procedure was extended to the case of seasonal fractional cointegration at each of the frequencies separately. Finite-sample critical values of the tests were computed and several Monte Carlo experiments were conducted in order to examine the power properties of the new tests. The results indicate that the tests behave relatively well against fractional alternatives, especially if the sample size is large.

The tests were applied to the UK and Japanese consumption and income series, and it was found that both series may be fractionally cointegrated at each of the frequencies separately. However, when testing against seasonal cointegration at all frequencies simultaneously, the null hypothesis of no cointegration was rejected for the UK but not for Japan.

The present study can be extended in several ways. First, the same methodology can be employed allowing for more than two variables, and also for weakly autocorrelated disturbances when testing both the individual series and the estimated residuals from the cointegrating regressions. However, in both cases, finite-sample critical values should be computed. ARMA structures for the I(0) disturbances have been widely used by applied researchers; however, their implementation in the context of long memory processes is still in its infancy, and the processes described here can be viewed as competing with the ARMA models in modelling the degree of association between the observations. Also, the tests of Robinson (1994) allow us to consider deterministic regressors, like an intercept, a linear trend, and/or seasonal dummies. However, once again, the inclusion of these deterministic components changes the empirical distribution of the tests. Other methods of estimating and testing the fractional differencing parameters, based on parametric or semi-parametric procedures (e.g. Robinson, 1995a,b; Ooms, 1997; Hosoya, 1997; Silvapulle, 2001; etc.) may also be applied in the second step of this procedure. An example is the approach due to Cheung & Lai (1993) for the case of the long run or zero frequency, which uses the log-periodogram estimation procedure of Geweke and Porter-Hudak (1983). In non-seasonal contexts, other more elaborate techniques on fractional cointegration (estimating and testing the fractional differencing parameters along with the coefficients of the cointegrating regression) are being developed by Robinson and his coauthors (Robinson and Marinucci, 2001, Robinson and Yajima, 2002, Robinson and Hualde, 2003). Finally, a more general procedure for simultaneously testing seasonal fractional cointegration under the null hypothesis (in a similar way to Johansen's (1988) procedure for non-seasonal contexts) would also be desirable. Work in all these directions is now in progress.

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