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Application of fast multi-pole boundary element method to 2D acoustic scattering problem

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Abstract: The fast multi-pole method (FMM) is a very effective approach to accelerating numerical solutions of the boundary element method (BEM) for the problems requiring large scale computation. An application of the FMM to two-dimensional boundary integral equation method for the acoustic scattering problem was discussed. We seek the solution of Helmholtz equation $\Delta u+k^2u=0$ in the form of a combined single- and double-layer potential. The boundary integral equation is discretized with Nyström method. It is obvious that the kernel of integral operator is unsymmetrical. If the resulting linear system is solved by the conjugate gradient method of unsymmetrical linear system, both the products of matrix $\boldsymbol{A}$ with vector $\boldsymbol{x}$ and $\boldsymbol{A}^{\mathrm{T}}$ with $\boldsymbol{x}$ should be repeatedly evaluated. The hierarchical cell structures of FMM with two different methods was constructed, and the multi-pole expansion, local expansion and translations of the coefficients were given for the second integral operator $\boldsymbol{A}$ and its conjugate operator $\boldsymbol{A}^{\mathrm{T}}$. The boundary integral equation was solved by FMM. The numerical results show that FMM is more efficient than the direct computation approach.

Key words: Helmholtz equation; acoustic scattering; boundary element method; fast multi-pole method

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快速多极边界元方法在二维声散射问题中的应用

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摘要:快速多极算法(FMM)是求解边界元方法(BEM)在大尺度情况下的一种非常有效的算法.研究了快速多极算法在二维声散射问题的边界积分方程求解中的应用.给出了积分核函数以及其共轭积分算子核函数的多极展开式,局部展开式以及相应展开系数之间的转化关系.分别应用两种不同的层级树结构的FMM来进行求解,并对两种树结构下的求解效率进行了对比.数值算例表明用快速多极算法求解该问题时在存储量

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和计算量上比直接求解方法效率更高.
关键词:Helmholtz 方程;声散射;边界元方法;快速多极算法

0 Introduction

The fast multi-pole method (FMM) pioneered by Refs. [1, 2] can be used to accelerate the solutions of particle interaction problems and boundary integral equations. By means of the FMM, both computing amount and memory requirement are reduced to $O(N)$. In the last decade, the fast multi-pole accelerated BEM (FMM BEM) has been developed to solve a variety of the problems with large-scale computation. Some of the works on the FMM BEM can be found in Refs. [4～10], which show great promises of the FMM BEM for solving large-scale computing problems.

It is obvious that the conventional BEM in general produce dense and non-symmetric matrices. It requires $O(N^2)$ operations to compute the coefficients and another $O(N^3)$ operations to solve the system by direct solvers. Then, the efficiency of solving the boundary integral equations has become a serious problem for large-scale problems. While the finite element method (FEM) had been routinely used to solve the problems with near million of degrees of freedom (DOF's), the BEM was limited to solve the problems with thousands DOF's for many years due to its lower efficiency in computation. FMM overcomes this issue, and makes it possible for BEM to solve large-scale problems due to dramatically reduced computing amount and lower memory requirement.

The basic structure used by FMM, in operation and storage, is a quad-tree for 2D problems or an octal-tree for 3D problems. During the procedure of iterative solutions for BEM, the matrix-vector multiplication in each iterative step is operated based on the tree structure with $O(N)$ storage and $O(N)$ operations, that is, the tree stores information of matrix-vector multiplication, not that of matrix and vector separately. Therefore, it is more efficient and faster than the traditional BEM for large-scale computing problem.

In this paper, a 2D acoustic scattering problem will be computed by FMM BEM. The solution of Helmholtz equation $\Delta u + k^2 u = 0$ was denoted by the form of a combined single- and double-layer potential. The conventional BEM scheme can be found in Ref. [3]. The boundary integral equation is discretized with Nyström method. It is obvious that the kernel of integral operator is unsymmetrical. The resulting linear system is solved by the conjugate gradient method of unsymmetrical linear system, then it is necessary to calculate the products of matrix $\boldsymbol{A}$ with vector $\boldsymbol{x}$ and $\boldsymbol{A}^{\mathrm{T}}$ with $\boldsymbol{x}$. In this paper, from the Graf's addition theorem, the multi-pole expansion, local expansion and translations of the coefficients are given for the second integral operator $\boldsymbol{A}$ and its conjugate operator $\boldsymbol{A}^{\mathrm{T}}$.

This paper is organized as follows. In Section 1, the conventional BEM formulation for 2D acoustic scattering problems is reviewed. In Section 2, the FMM to solve the boundary integral equation is presented, the fast multi-pole expansions, local expansions and the translation of coefficients and also the algorithm of the FMM are constructed. In Section 3, a numerical example by using FMM BEM algorithm developed in this paper is shown, the comparison of CPU times and memory requirement of FMM ($O(N)$) and the conventional BEM ($O(N^2)$) is given. The result shows the FMM is more efficient for large-scale computing problems.

1 Boundary integral equation to 2D acoustic scattering problem

Consider an acoustic wave propagating in a homogeneous and isotropic medium and impinging on an infinitely long cylinder having uniform cross section $D \subset R^2$. Suppose that the axis of the cylinder is parallel to the z-axis and the incident wave is the plane wave $u^i(x) = e^{ikx\cdot\alpha}$, where $x \in R^2$, $k > 0$ is the wave number, α is a fixed unit vector. Clearly, the cylinder will scatter the incident plane wave u^i. We describe u^s as the scattered wave and u^∞ as the far field pattern of u^s. Suppose that ∂D is of class C^2, $u = u^i + u^s$ is the total field. The acoustic scattering problem is to find a solution $u \in C^2(R^2 \backslash \overline{D}) \cap C(R^2 \backslash D)$ satisfies

$$\Delta u + k^2 u = 0 \quad \text{in} \quad R^2 \backslash \overline{D},$$

and the exterior Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial D.$$

The scattered wave u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \ r = |x|.$$

It is easy to show that u^s has the asymptotic behavior[3]

$$u^s(r,\theta) = \frac{e^{ikr}}{\sqrt{r}} F(\theta; k, \alpha) + O(r^{-\frac{3}{2}}),$$

where (r,θ) is the polar representation of a point x in the plane and F is known as the far field pattern for the scattered wave u^s. From Green's formula and the asymptotic of the first Hankel function $H_0^{(1)}$, we can easily conclude that

$$F(\theta; k, \alpha) = -\frac{e^{i\pi/4}}{\sqrt{8k\pi}} \int_{\partial D} \frac{\partial u^s}{\partial \nu}(y) e^{-ik\rho\cos(\theta - \varphi)} ds(y),$$

where $x = re^{i\theta}$, $y = \rho e^{i\varphi}$, ν is the unit outward normal to ∂D.

We seek the scattering solution u^s of the previous Helmholtz equation in the form of a combined acoustic double- and single-layer potential

$$\left.\begin{aligned} u^s(x) = \int_{\partial D} \left\{\frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta\Phi(x,y)\right\} \varphi(y) ds(y), \\ x \in R^2 \backslash \partial D, \end{aligned}\right\} \tag{1}$$

with some positive coupling parameter η. Here Φ is the fundamental solution of the Helmholtz equation in 2D expressed as

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \ x \neq y,$$

where $H_0^{(1)}$ is the 1st-kind Hankel function of order zero. From the jump-relation of double-layer potential[3], we see that solution (1) solves the exterior Dirichlet problem, provided that the density $\varphi(y)$ is a solution of the boundary integral equation

$$\varphi + K\varphi - i\eta S\varphi = 2f, \ x \in \partial D, \tag{2}$$

where $f = -u^i(x)$, K, $S: C(\partial D) \to C(\partial D)$ denote the integral operators defined by

$$(K\varphi)(x) = 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y), \ x \in \partial D,$$

$$(S\varphi)(x) = 2\int_{\partial D} \Phi(x,y)\varphi(y) ds(y), \ x \in \partial D.$$

These integral operators are compact since they can be shown to have weakly singular kernels. Therefore, the existence of solution to (2) can be established by the Riesz theory for equations of the second kind with a compact operator.

It is obvious that $\Phi(x,y)$ and $\partial\Phi(x,y)/\partial\nu(y)$ have logarithmic singularities at $x = y$. Hence, their proper numerical treatment can be found in Ref. [3].

2 Fast multi-pole method

The main idea of the FMM is to translate the point-to-point interactions to cell-to-cell interactions by using multi-pole expansions and translations, where cells can have a hierarchical tree structure. The conjugate gradient method will be used in the FMM to solve the resulting linear system, where matrix-vector multiplications are calculated using fast multi-pole expansions. Using the FMM, the solution time of a problem can be reduced to order $O(N)$. The memory requirement can also be reduced to $O(N)$ since iterative solvers do not need to store the entire matrix in the memory.

In this section, a fast multi-pole method for the boundary integral equation (2) will be presented. The fast multi-pole expansions, local expansions and the translations of coefficients will be provided.

2.1 Expansions and the translations of coefficients

Consider the boundary integral equation

$$\frac{\varphi(x)}{2}+\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\mathrm{d}s(y)-i\eta\int_{\partial D}\Phi(x,y)\varphi(y)\mathrm{d}s(y)=-\mathrm{e}^{ikx\cdot\alpha},\ x\in\partial D,\tag{3}$$

where x and y denote the source and field points on the boundary ∂D. The multi-pole expansions, local expansions and the translations of coefficients of the following two integral operators should be formulated.

$$(S\varphi)(x)=\int_{\partial D}\Phi(x,y)\varphi(y)\mathrm{d}s(y),$$

$$(K\varphi)(x)=\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\mathrm{d}s(y),$$

Firstly, introduce the Graf's Addition Theorem[6]:

Theorem 2.1 Bessel function and Hankel function have the following properties:

$$\mathrm{H}_{2n}^{1,2}=\mathrm{H}_{-2n}^{1,2},\quad \mathrm{H}_{2n+1}^{1,2}=-H_{-2n-1}^{1,2}.$$

For each integer n, the expansion of Bessel function J_n and Hankel function $\mathrm{H}_n^{(1)}$ are given by:

$$\mathrm{J}_n(k\mid x-y\mid)\mathrm{e}^{\pm in\theta_{x-y}}=\sum_{m=-\infty}^{\infty}\mathrm{J}_{m+n}(k\mid x\mid)\mathrm{e}^{\pm i(m+n)\theta_x}\mathrm{J}_m(k\mid y\mid)\mathrm{e}^{\mp im\theta_y},\quad \mid x\mid>\mid y\mid;$$

$$\mathrm{H}_n^{(1)}(k\mid x-y\mid)\mathrm{e}^{\pm in\theta_{x-y}}=\sum_{m=-\infty}^{\infty}\mathrm{H}_{m+n}^{(1)}(k\mid x\mid)\mathrm{e}^{\pm i(m+n)\theta_x}\mathrm{J}_m(k\mid y\mid)\mathrm{e}^{\mp im\theta_y},\quad \mid x\mid>\mid y\mid.$$

And from $\mathrm{H}_n^{(1)}=\overline{\mathrm{H}_n^{(2)}}$, the expansion of $\mathrm{H}_n^{(2)}$ can be easily concluded.

According to the previous theorem, the kernel $\Phi(x,y)$ of integral operator S have the following expansion:

$$\Phi(x,y)=\frac{i}{4}\mathrm{H}_0^{(1)}(k\mid x-y\mid)=\frac{i}{4}\sum_{m=-\infty}^{\infty}\mathrm{H}_m^{(1)}(k\mid x\mid)\mathrm{J}_m(k\mid y\mid)\mathrm{e}^{im(\theta_x-\theta_y)},\quad \mid x\mid>\mid y\mid,$$

where x is source points and y is field points on the boundary ∂D.

Suppose that y_0 is a point of ∂D and $|y-y_0| < |x-y_0|$, then the following multi-pole expansion can be derived:

$$\int_{\partial D}\Phi(x,y)\varphi(y)\mathrm{d}s(y)=\frac{i}{4}\sum_{m=-\infty}^{\infty}\mathrm{H}_m^{(1)}(k\mid x-y_0\mid)\mathrm{e}^{im\theta_{x-y_0}}M_m(y_0),\tag{4}$$

where

$$M_m(y_0)=\int_{\partial D}\mathrm{J}_m(k\mid y-y_0\mid)\mathrm{e}^{-im\theta_{y-y_0}}\varphi(y)\mathrm{d}s(y),\tag{5}$$

are called moments about y_0, which are independent of the source point x_0 and only need to be computed once.

When point y_0 is moved to a new local point y_1, the moments of y_1 can be written as

$$M_n(y_1)=\int_{\partial D}\mathrm{J}_n(k\mid y-y_1\mid)\mathrm{e}^{-in\theta_{y-y_0}}\varphi(y)\mathrm{d}s(y).$$

According to Theorem 2.1 we can evaluate the following translation from $M_m(y_0)$ to $M_n(y_1)$, M2M translation for short,

$$M_n(y_1)=\sum_{m=-\infty}^{\infty}M_m(y_0)\mathrm{J}_{n-m}(k\mid y_0-y_1\mid)\mathrm{e}^{-i(n-m)\theta_{y_0-y_1}}.\tag{6}$$

Suppose that x_0 is a point close to the source point x, that is $|x-x_0|<|y_0-x_0|$. Then we can derive the local expansion as

$$\int_{\partial D}\Phi(x,y)\varphi(y)\mathrm{d}s(y)=\frac{i}{4}\sum_{m=-\infty}^{\infty}\mathrm{H}_m^{(1)}(k\mid x-y_0\mid)\mathrm{e}^{im\theta_{x-y_0}}M_m(y_0)=\frac{i}{4}\sum_{l=-\infty}^{\infty}L_l(x_0)\mathrm{J}_l(k\mid x-x_0\mid)\mathrm{e}^{il\theta_{x-x_0}},$$

where the coefficients are given by the following moment-to-local translation (M2L translation)

$$L_l(x_0)=\sum_{m=-\infty}^{\infty}M_m(y_0)\mathrm{H}_{m-l}^{(1)}(k\mid x_0-y_0\mid)\mathrm{e}^{i(m-l)\theta_{x_0-y_0}},\tag{7}$$

If the point for local expansion is moved from x_0 to x_1, then have the following local expansion about x_1:

$$\int_{\partial D}\Phi(x,y)\varphi(y)\mathrm{d}s(y)=\frac{i}{4}\sum_{m=-\infty}^{\infty}L_m(x_1)\mathrm{J}_m(k\mid x-x_1\mid)\mathrm{e}^{im\theta_{x-x_1}}.$$

According to Theorem 2.1 and $|x_0-x_1|>|x-x_1|$, the local-to-local translation (L2L translation) can be obtained

$$L_m(x_1)=\sum_{n=-\infty}^{\infty}L_n(x_0)\mathrm{J}_{n-m}(k\mid x_1-x_0\mid)\mathrm{e}^{i(n-m)\theta_{x_1-x_0}}. \tag{8}$$

Consider the multi-pole expansion of $\partial\Phi(x,y)/\partial\nu(y)$, from the formulation

$$\frac{\partial\Phi(x,y)}{\partial\nu(y)}=\hat{\nu}(y)\frac{\partial\Phi(x,y)}{\partial y},$$

we can derive that

$$\frac{\partial \mathrm{H}_0^{(1)}(k\mid x-y\mid)}{\partial\nu(y)}=\sum_{m=-\infty}^{\infty}\mathrm{H}_m^{(1)}(k\mid x-y_0\mid)\cdot \mathrm{e}^{im\theta_{x-y_0}}\hat{\nu}(y)\frac{\partial \mathrm{J}_m(k\mid y-y_0\mid)\mathrm{e}^{-im\theta_{y-y_0}}}{\partial y},\quad |x|>|y|.$$

And further more, we have the following multi-pole expansion

$$\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\mathrm{d}s(y)=\frac{i}{4}\sum_{m=-\infty}^{\infty}\mathrm{H}_m^{(1)}(k\mid x-y_0\mid)\mathrm{e}^{im\theta_{x-y_0}}M_m(y_0),$$

where the multi-pole moment $M_m(y_0)$ is given by

$$M_m(y_0)=\int_{\partial D}\hat{\nu}(y)\frac{\partial \mathrm{J}_m(k\mid y-y_0\mid)\mathrm{e}^{-im\theta_{y-y_0}}}{\partial y}\varphi(y)\mathrm{d}s(y).$$

It is obvious that $\partial\Phi(x,y)/\partial\nu(y)$ and $\Phi(x,y)$ have the same Eqs. (5)～(8). Although their multi-pole moments are different, they have the same multipole expansion, local expansion and translations of coefficients.

The derivatives of Bessel and Hankel function are given by the following theorem[3]:

Theorem 2.2 Suppose $f_n=\mathrm{J}_n$ or $\mathrm{H}_n^{(1,2)}$ is the Bessel function or Hankel function. From the relationship

$$f_{n+1}(t)+f_{n-1}(t)=\frac{2n}{t}f_n(t),$$

$$f_{n+1}(t)=-t^n\frac{\mathrm{d}}{\mathrm{d}t}\{t^{-n}f_n(t)\},$$

$$t^nf_{n-1}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\{t^nf_n(t)\},\ n\in Z,$$

we can conclude that

$$f_n'(t)=\frac{1}{2}(f_{n-1}(t)-f_{n+1}(t)),$$

for each integer n.

Assume that the boundary curve ∂D is analytic, with a regular parametric representation of the form

$$x(t)=(x_1(t),x_2(t)),$$
$$y(\tau)=(x_1(\tau),x_2(\tau)),\ t,\tau\in[0,2\pi],$$

the outward normal $\nu(x)=(x_2',-x_1')$. Then

$$\int_{\partial D}\Phi(x,y)\varphi(y)\mathrm{d}s(y)=\int_0^{2\pi}\Phi(x(t),y(\tau))\varphi(y(\tau))s(y(\tau))\mathrm{d}\tau,$$

$$s(y(\tau))=\sqrt{x_1'^2(\tau)+x_2'^2(\tau)}.$$

The integrals are discretized with Nyström method, i.e., straight forward approximate the integrals by quadrature formulas. That is

$$\int_0^{2\pi}\Phi(x(t),y(\tau))\varphi(y(\tau))s(y(\tau))\mathrm{d}\tau=\frac{\pi}{n}\sum_{j=1}^{2n}\Phi(x(t_i),y(\tau_j))\varphi(y(\tau_j))s(y(\tau_j)),$$

$$t_i=i\pi/n,\ \tau_j=j\pi/n.$$

It is obvious that the resulting linear system $A\varphi=B$ to Eq. (3) is unsymmetrical. If it is solved by the conjugate gradient method of unsymmetrical linear system, the results of $A^{\mathrm{T}}\varphi$ should also be evaluated. Then the fast multi-pole expansion, local expansion and translations of coefficients for the following operators should also be formulated.

$$(S^*\varphi)(x)=s(x)\int_{\partial D}\Psi(x,y)\varphi(y)\mathrm{d}s(y),$$

$$(K^*\varphi)(x)=s(x)\int_{\partial D}\frac{\partial\Psi(x,y)}{\partial\nu(x)}\varphi(y)\mathrm{d}s(y),$$

where $\Psi(x,y)=\overline{\Phi(x,y)}$, $s(x)=\sqrt{x_1'^2(t)+x_2'^2(t)}$.

Theorem 2.3 The integral operators S^* and K^* have the following multi-pole expansion:

$$s(x)\int_{\partial D}\Psi(x,y)\varphi(y)\mathrm{d}s(y)=-\frac{i}{4}s(x)\cdot\sum_{m=-\infty}^{\infty}\mathrm{H}_m^{(2)}(k\mid x-y_0\mid)\mathrm{e}^{-im\theta_{x-y_0}}M_m(y_0),$$

$$s(x)\int_{\partial D}\frac{\partial\Psi(x,y)}{\partial\nu(x)}\varphi(y)\mathrm{d}s(y)=-\frac{i}{4}s(x)\cdot\sum_{m=-\infty}^{\infty}\hat{\nu}(x)\frac{\partial \mathrm{H}_m^{(2)}(k\mid x-y_0\mid)\mathrm{e}^{-im\theta_{x-y_0}}}{\partial x}M_m(y_0),$$

where the multi-pole moments and M2M translation is given by

$$M_m(y_0) = \int_{\partial D} \mathrm{J}_m(k \mid y - y_0) \mathrm{e}^{im\theta_{y-y_0}} \varphi(y) \mathrm{d}s(y),$$

$$M_n(y_1) = \sum_{m=-\infty}^{\infty} M_m(y_0) \mathrm{J}_{n-m}(k \mid y_0 - y_1) \mathrm{e}^{i(n-m)\theta_{y_0-y_1}}.$$

The operators S^* and K^* have the following local expansion:

$$s(x)\int_{\partial D} \Psi(x,y)\varphi(y)\mathrm{d}s(y) = -\frac{i}{4}s(x)\sum_{l=-\infty}^{\infty} L_l(x_0)\mathrm{J}_l(k \mid x - x_0 \mid)\mathrm{e}^{-il\theta_{x-x_0}},$$

$$s(x)\int_{\partial D} \frac{\partial \Psi(x,y)}{\partial \nu(x)}\varphi(y)\mathrm{d}s(y) = -\frac{i}{4}s(x)\sum_{l=-\infty}^{\infty} L_l(x_0)\hat{\nu}(x)\frac{\partial \mathrm{J}_l(k \mid x - x_0 \mid)\mathrm{e}^{-il\theta_{x-x_0}}}{\partial x},$$

where the M2L and L2L translations are given by

$$L_l(x_0) = \sum_{m=-\infty}^{\infty} M_m(y_0)\mathrm{H}_{m-l}^{(2)}(k \mid x_0 - y_0 \mid)\mathrm{e}^{-i(m-l)\theta_{x_0-y_0}},$$

$$L_m(x_1) = \sum_{n=-\infty}^{\infty} L_n(x_0)\mathrm{J}_{n-m}(k \mid x_1 - x_0 \mid)\mathrm{e}^{-i(n-m)\theta_{x_1-x_0}}.$$

It is obvious that we can derive the same formulae as the above operators S and K.

2.2 The fast multi-pole algorithms

The algorithm of the fast multi-pole method goes as follows:

Step 1 Discretization.

For a given domain D, discretize the boundary ∂D in the same way the conventional BEM. If the boundary curve ∂D is analytic, we can discretize the parametric representation $x(t)=(x_1(t),x_2(t))$ of ∂D with $t_i=2\pi i/n$, $i=1,2,\cdots,n$.

Step 2 Determine the hierarchical cell structures.

Consider a square that covers the domain D and call this square the cell of level 0. Take this cell (parent cell) and divide it into four equal cells (child cells) of level 1. Continue dividing in this way, that is, take a parent cell of level l and divide it into four child cells of level $l+1$. Stop dividing a cell if it only includes one point. A cell having no child cells is called a leaf cell. Another method to determine the hierarchical cell structures can be used, that is, divide the square to l level straightly, where l is determined by the number of the discretized points.

Step 3 Upward pass.

Firstly, compute the multi-pole moments (p terms) at all leaves with Eq. (5) for every field point y, where the point y_0 is the centroid of the leaf containing the field point y. Secondly, we calculate the moments (p terms) of the parent cell with the M2M translation, that is, Eq. (6), in which y_1 is the centroid of the parent cell and y_0 the centroid of a child cell. Continue calculating the multi-pole moments of the cells up to level 2.

Step 4 Downward pass.

We first introduce some definitions. Two cells at the same level sharing at least one common vertex are said to be adjacent cells. When cell C is a leaf cell, its adjacent cells also include the cells at different level sharing at least one common vertex with C, and the children of C's adjacent cells are also C's adjacent cells. Two cells are said to be well separated at level l if they are not adjacent at level l but their parent cells are adjacent at level $l-1$. A cell C's well separated cells also include the cells which are leaves and adjacent to C's parent. The list of all the well separated cells of cell C is called the interaction list of C. Cells are called far cells of C if their parent cells are not adjacent to the parent cell of C.

Now compute the local expansion coefficients on all cells starting from level 2 and tracing the tree structure downward to all the leaves. The local expansion coefficients associated with a cell C is the sum of the contributions from the cells in the interaction list of cell C and from all the far cells. The former is calculated by using the M2L translation, with multi-pole moments associated with cells in the interaction list, and the latter is calculated by using the L2L translation, for the parent cell of C with the expansion point being shifted from the centroid of C's parent cell to that of C. For a cell C at level 2, M2L translation can be used to compute the coefficients of the local expansion, because his parent has no well separated cell.

Step 5 Evaluation of the integrals in Eq. (3).

Suppose that the values of $\varphi(y_j)$ are given. For a leaf cell C, we compute the contributions from its adjacent cells directly as in the conventional BEM. Contributions from all other cells (cells in the interaction list of C and far cells) are computed by using the local expansion, where the local expansion coefficients for cell C have been computed in Step 4.

Step 6 Iterations of the solution.

Update the iterative vector $\varphi(y_j)$ in the resulting system $A\varphi=b$, and continue at Step 3 for the matrix and unknown vector multiplication until the solution converges within a given tolerance using the conjugate gradient solver.

In these steps, all the infinite expansions are truncated with p terms.

3 Numerical example

In this section, a two-dimensional acoustic scattering problem is computed, based on the FMM-BEM method described above. Consider the scattering of a plane wave u^i by a cylinder with a non-convex kite-shaped cross section with boundary ∂D illustrated in Fig. 1 and described by the parametric representation

$$\partial D:\rho(\theta)=(\cos\theta+0.65\cos 2\theta-0.65,1.5\sin\theta),\quad 0\leqslant\theta\leqslant 2\pi,$$

where the direction of the incident wave u^i is $\alpha=(1,0)$.

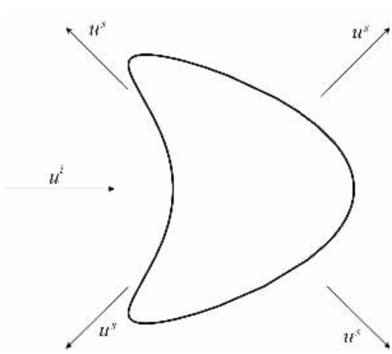

Fig. 1 Kite-shaped domain and its scattering problem

Here we construct the hierarchical cell structures with two different methods described in Step 2 of the algorithms. The boundary integral equation was discretized with Nyström method. The boundary parametric representation is discretized to $2N$ points, where the parameter $\theta_i=\pi/N$, $i=1,2,\cdots,2N$. In this example, let the wave number $k=1$ and the parameter $\eta=1$. The numbers of terms for both moments and local expansions were set to 13 and the tolerance for convergence of the solution to 10^{-10}. All computations were run on a Windows PC computer equipped with 3.02 GHz Pentium 4 CPU unit and 1 GB of core memory.

Method 1 The scattering boundary is discretized to $2N$ points. Then construct $L=\log_2 N$ levels hierarchical tree structures, i. e. , the square covering the domain is divided into $2^L\times 2^L$ sub squares. The cells containing the point at each level are in the tree structure. It is obvious that a cell and its well separated cells are at the same level, and so are all leaves at the same level. So it is very easy to find the interaction list of each cell and the neighbors of leaves. But this hierarchical tree structure has more branches, so the M2M, M2L and L2L translations should be computed more times. According to the fast multi-pole algorithms we can solve the boundary integral equation.

The average iterations of this method are 15. The CPU time used for both matrix-vector method and FMM in these calculations are plotted in Fig. 2.

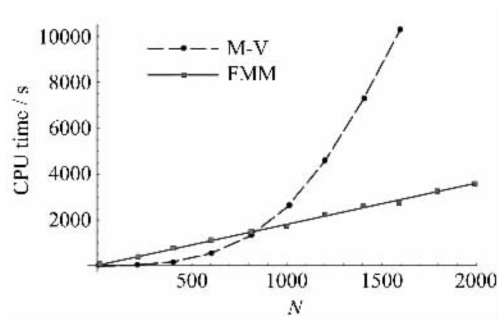

Fig. 2 The comparison of CPU time for matrix-vector method and FMM

Method 2 Firstly, divide the square that covers all the points into 4 equal child cells of level 1 . If these cells of level 1 include more than one

Tab. 1 Computed far field pattern at direction $d=(1,0)$

N	matrix-vector		FMM(Method 1)		FMM(Method 2)	
	Re $u_\infty(d)$	Im $u_\infty(d)$	Re $u_\infty(d)$	Im $u_\infty(d)$	Re $u_\infty(d)$	Im $u_\infty(d)$
8	−1.626 423 961 8	0.602 926 152 2	−1.625 452 869 0	0.611 054 226 8	−1.627 614 308 2	0.603 697 024 1
200	−1.627 457 479 4	0.602 225 888 8	−1.627 543 664 4	0.602 965 500 6	−1.627 924 353 3	0.602 070 081 1
400	−1.627 457 482 2	0.602 225 909 0	−1.627 513 983 4	0.602 596 043 9	−1.627 377 561 3	0.602 213 336 7
600	−1.627 457 483 6	0.602 225 916 8	−1.627 487 894 0	0.602 502 075 5	−1.627 329 896 6	0.602 227 148 7
800	−1.627 457 483 8	0.602 225 919 0	−1.627 502 564 9	0.602 419 850 4	−1.626 975 447 2	0.602 348 750 7
1 000	−1.627 457 484 1	0.602 225 920 1	−1.627 497 724 1	0.602 346 729 9	−1.627 048 984 9	0.602 417 041 8
1 200	−1.627 457 484 5	0.602 225 922 9	−1.627 488 026 0	0.602 368 205 1	−1.627 695 331 5	0.602 147 946 9
1 400	−1.627 457 484 7	0.602 225 922 5	−1.627 500 223 9	0.602 349 178 8	−1.627 295 592 0	0.602 245 759 8
1 600	−1.627 457 484 6	0.602 225 923 2	−1.627 492 782 5	0.602 328 452 6	−1.627 421 860 9	0.602 279 060 9
2 000			−1.627 486 604 8	0.602 293 616 2	−1.627 215 049 7	0.602 303 966 2

point, then divide it into 4 child cells of level 2. Continue dividing in this way, that is, if a cell of level L includes more than one point, it should be divided into four child cells of level $L+1$. Stop dividing a cell if it only includes one point. The cells containing the point at each level are in the tree structure.

It is obvious that this hierarchical tree structure is compact, so the M2M, M2L and L2L translations are less than Method 1. But it is difficult to find the interaction list of each cell and the neighbors of leaves, since they are may be at different levels. According to the fast multi-pole algorithms the boundary integral equation can also be solved.

The average iterations of this method are 20. The CPU time used for both matrix-vector method and FMM in these calculations are plotted in Fig. 3.

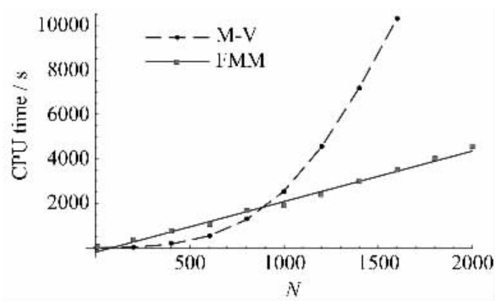

Fig. 3 The comparison of CPU time for matrix-vector method and FMM

From this example, we can see that the FMM of Method 1 consume less times than Method 2.

The results of $u_\infty(d)$ at direction $d=(1,0)$ for this problem using both the FMM of these two hierarchical tree structures and matrix-vector method as the total number of points increase from 16 to 4 000 are shown in Tab. 1. Here Re $u_\infty(d)$ and Im $u_\infty(d)$ denote the real and imaginary part of $u_\infty(d)$ respectively. The memory requirement of FMM and matrix-vectors method in these calculations is plotted in Fig. 4. The memory dispersion between Method 1 and Method 2 is less than 1 MB.

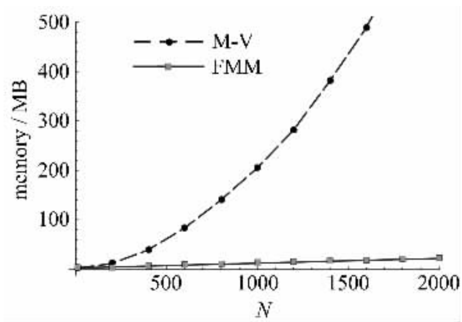

Fig. 4 The comparison of memory requirement for matrix-vector method and FMM

This example shows the computing time and memory requirement of FMM is $O(N)$, and it is more efficient than the matrix-vector method for large-scale problem. The results should be more accurate if we increase the terms of moments and local expansions.

4 Conclusion

The application of the FMM to two-dimensional boundary integral equation method for acoustic scattering problem was discussed in this paper. The relatively expansions and translations with the algorithm of FMM are constructed. Two different hierarchical tree structures of FMM are shown. The efficiency of these two tree structures was shown in the figures. It can conclude that Structure 1 consume less times than Structure 2. A numerical example by using FMM BEM algorithm developed in this paper is shown, the comparison of CPU time and memory requirement of FMM ($O(N)$) and the conventional BEM ($O(N^2)$) is given. It shows that FMM is more efficient than direct computation approach for large-scale computing problems.

Fast multi-pole method have been widely used, it can also be applied to solve the multiple and multilayered obstacles scattering problem for acoustic or electromagnetic wave. In future work, the application of FMM for multiple and multilayered obstacles scattering problem will be considered.

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