第38卷第9期 中国科学技术大学学报 Vol. 38, No. 9
2008年9月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Sep. 2008

Article ID: 0253-2778(2008)09-1017-03

Edge-fault-tolerant bipanconnectivity of hypercubes

JING Jin[1], DU Zheng-zhong[1], MA Mei-jie[2], XU Jun-ming[1]

(1. *Department of Mathematics, University of Science and Technology of China, Hefei* 230026, *China*;
2. *Department of Mathematics, Zhejiang Normal University, Jinhua* 321004, *China*)

Abstract: It was shown that for any two vertices u and v of the hypercube Q_n ($n\geqslant 4$) with at most $n-1$ faulty edges, which are not incident with the same vertex if they are exactly $n-1$, there exists a fault-free uv-path of length l with $d_{Q_n}(u,v)+4\leqslant l\leqslant 2^n-1$ and $2|(l-d_{Q_n}(u,v))$. This improves some known results.

Key words: Hamiltonian path; fault-tolerance; hypercube; bipanconnectivity

CLC number: O157.5; TP302.1 **Document code**: A

AMS Subject Classification (2000): Primary 05C38; Secondary 90B10

超立方体网络的边容错二部泛连通度

经　紟[1],杜正中[1],马美杰[2],徐俊明[1]

(1.中国科学技术大学数学系,安徽合肥 230026;2.浙江师范大学数学系,浙江金华 321004)

摘要:证明了对于至多有$n-1$条故障边的容错超立方体网络Q_n,如果它正好有$n-1$条故障边但不关联于同一个顶点,那么对于Q_n中任意两点u和v,存在一条长为l的uv非故障路,路长l满足$d_{Q_n}(u,v)+2\leqslant l\leqslant 2^n-1$且$2|(l-d_{Q_n}(u,v))$.这改进了许多已知结果.

关键词:哈密尔顿路;容错;超立方体网络;二部泛连通性

0 Introduction

It is well-known that when the underlying topology of an interconnection network is modelled by a connected graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network, study of the structure of G is of quite great interest.

A graph G is panconnected if for any two different vertices u and v in G there exists a uv-path of length l with $d_G(u,v)\leqslant l\leqslant |V(G)|-1$, where $d_G(u,v)$ is the distance between u and v in G. A bipartite graph G, since it contains no cycles of odd length, is bipanconnected if for any two different vertices u and v in G there exists a uv-path of length l with $d_G(u,v)\leqslant l\leqslant |V(G)|-1$ and $2|(l-d_G(u,v))$.

A graph G is k-edge-fault-tolerant panconnected if the resulting graph by deleting any k edges from G is panconnected. A subgraph of G is fault-free if it contains no faulty edges in G.

Received: 2007-09-14; **Revised**: 2007-12-14

Foundation item: Supported by NNSF of China (10671191).

Biography: JING Jin, male, born in 1980. Research field: graphs and combinatorics. E-mail: jingmath@mail.ustc.edu.cn

Corresponding author: XU Jun-ming, Prof. E-mail: xujm@ustc.edu.cn

1 Main results

In this paper, we consider the hypercube Q_n and show that for any two different vertices u and v of Q_n ($n\geqslant 4$) with at most $n-1$ faulty edges, which are not incident with the same vertex if they are exactly $n-1$, there exists a fault-free uv-path of length l with $d_{Q_n}(u,v)+4\leqslant l\leqslant 2^n-1$ and $2|(l-d_{Q_n}(u,v))$.

As consequences of our results, we immediately obtain Li et al's result that Q_n is bipanconnected and ($n-2$)-edge-fault-tolerant edge-bipancyclic[1], and Wang's result that FQ_n is ($n-1$)-edge-fault-tolerant Hamiltonian[3].

The proof of our main theorem is in Section 2. Throughout this paper, we follow Xu[4] for graph-theoretical terminologies and notations not defined here.

2 Edge-fault-tolerant bipanconnectivity of Q_n

The n-dimensional hypercube Q_n is a graph with 2^n vertices, each vertex with a distinct binary string $u_n\cdots u_2u_1$ on the set $\{0,1\}$. Two vertices are linked by an edge if and only if their strings differ in exactly one bit. As a topology for an interconnection network of a multiprocessor system, the hypercube structure is a widely used and well-known interconnection model since it possesses many attractive properties[2,4]. In particular, Q_n is vertex-transitive and edge-transitive with diameter n.

By definition, for any $k\in\{1,2,\cdots,n\}$, Q_n can be expressed as $Q_n=L_k\odot R_k$, where L_k and R_k are the two ($n-1$)-subcubes of Q_n induced by the vertices with the k position being 0 and 1, respectively. We call edges between L_k and R_k k-dimensional, which form a perfect match of Q_n. Use u_L and u_R to denote the two vertices in L_k and R_k, respectively, linked by the k-dimensional edge u_Lu_R in Q_n. For a subset F of $E(Q_n)$ and any $k\in\{1,2,\cdots,n\}$, we always express Q_n as $Q_n=L_k\odot R_k$, and let $F_L=F\cap E(L_k)$, $F_R=F\cap E(R_k)$. Clearly, for any edge e of Q_n, there is some $k\in\{1,2,\cdots,n\}$ such that e is k-dimensional. Let E_k be all k-dimensional edges in Q_n. Clearly,

$$E(Q_n)=E_1\cup E_2\cup\cdots\cup E_n.$$

Lemma 2.1[5] If Q_n ($n\geqslant 2$) has at most $n-2$ faulty edges, then for any two different vertices u and v there exists a fault-free uv-path of length l with $d_{Q_n}(u,v)+2\leqslant l\leqslant 2^n-1$ and $2|(l-d_{Q_n}(u,v))$.

Theorem 2.2 Let F be a set of faulty edges in Q_n ($n\geqslant 4$). If $|F|\leqslant n-1$, and all edges in F are not incident with the same vertex if $|F|=n-1$, then for any two different vertices u and v there exists a fault-free uv-path of length l with $d(u,v)+4\leqslant l\leqslant 2^n-1$ and $2|(l-d(u,v))$.

Proof We prove this theorem by induction on $n\geqslant 4$. For $n=4$, we have verified this conclusion with a computer by the depth first search method within a polynomial time. Suppose that the theorem is true for m with $4\leqslant m<n$. Let F be a subset of $E(Q_n)$, and without loss of generality, $|F|=n-1$ and suppose that all edges in F are not incident with the same vertex. Let u and v be the two vertices in Q_n and we need to construct a uv-path of length l in Q_n-F with $d(u,v)+4\leqslant l\leqslant 2^n-1$ and $2|(l-d(u,v))$.

Let $F_k=F\cap E_k$. We choose $k\in\{1,2,\cdots,n\}$ according to the following rules:

(Ⅰ) If it is possible, we choose any k such that $|F_k|\geqslant 2$.

(Ⅱ) Otherwise, the $n-1$ faulty edges are on different dimensions. We choose k such that the k-dimensional faulty edge is incident with maximum faulty edges.

We express $Q_n=L_k\odot R_k$. Let $F_L=F\cap L_k$, $F_R=F\cap R_k$. Then $F=F_k\cup F_L\cup F_R$, $|F_k|\geqslant 1$, $|F_L|\cup|F_R|\leqslant n-2$ and there are no $n-2$ faulty edges in L or R which are incident with the same vertex.

Case 1 $u,v\in L$ (or R).

Since $|F_L|\leqslant n-2$, by the induction hypothesis, there is a uv-path P_{uv} of length l in L with

$$d(u,v)+4\leqslant l\leqslant 2^{n-1}-1,$$
$$2\mid(l-d(u,v)).$$

Let P_L be a uv-path of length l_L in L with $2^{n-1}-6\leqslant l_L\leqslant 2^{n-1}-1$ and $2|(l_L-d(u,v))$. Since $2^{n-1}-6>2(n-1)$ when $n\geqslant 5$, there exists an edge xy of P_L such that $\{xx_R, yy_R, x_Ry_R\}\cap F=\emptyset$. By the induction hypothesis, there is an x_Ry_R-path $P_{x_Ry_R}$ of odd length l_R in R with $5\leqslant l_R\leqslant 2^{n-1}$. Then $P_L-xy+xx_R+yy_R+P_{x_Ry_R}$ is a uv-path of length $l=l_L+l_R+1$ in Q_n-F with $2^{n-1}\leqslant l\leqslant 2^n-1$ and $2|(l-d(u,v))$.

Case 2 $u\in L$, $v\in R$ (or $u\in R$, $v\in L$).

Subcase 2.1 Assume $d_{Q_n}(u,v)\geqslant 2$.

Suppose that one of the two edges $\{uu_R, vv_L\}$ is fault-free. Without loss of generality, uu_R is fault-free. It is clear that $d(u,v)=1+d(u_R,v)$.

By the induction hypothesis, there is a u_Rv-path P_{u_Rv} of length l_L with $d(u_R,v)+4\leqslant l_L\leqslant 2^{n-1}-1$ and $2|(l_L-d(u_R,v))$. Thus $uu_R+P_{u_Rv}$ is a uv-path of length l with $d(u,v)+4\leqslant l\leqslant 2^{n-1}$ and $2|(l-d(u,v))$.

We construct a uv-path of length l with $2^{n-1}+1\leqslant l\leqslant 2^n$ and $2|(l-d(u,v))$ as follows. Select a fault-free k-dimensional edge x_Lx_R in Q_n. By the induction hypothesis, there is a ux_L-path P_{ux_L} of length l_L with $(n-1)+4\leqslant l_L\leqslant 2^{n-1}-1$ and $2|\,l_L-d(u,x_L)$ in L since $d(u,x_L)\leqslant n-1$. Similarly, there is an x_Rv-path P_{x_Rv} of length l_R with $(n-1)+4\leqslant l_R\leqslant 2^{n-1}-1$ and $2|(l_R-d(x_R,v))$ in R. Thus $P_{ux_L}+x_Lx_R+P_{x_Rv}$ is a uv-path of length $l=l_L+l_R+1$ in Q_n-F with $2^{n-1}+1\leqslant l\leqslant 2^n-1$ and $2|(l-d(u,v))$.

When the two edges $\{uu_R, vv_L\}$ are both faulty, then $|F_k|\geqslant 2$, $|F_L|\cup|F_R|\leqslant n-3$.

There exists a neighbor vertex x_L of u in L such that $\{ux_L, x_Lx_R\}\cap F=\emptyset$ and $x_R\neq v$. It is clear that $d(u,v)\geqslant d(x_R,v)$. By Lemma 2.1, there is an x_Rv-path P_{x_Rv} of length l_R with $d(x_R,v)+2\leqslant l_R\leqslant 2^{n-1}-1$ and $2|\,l_R-d(x_R,v)$ in R. Since the edge ux_L is fault-free and $|F_L|\leqslant n-3$, there is a ux_L-path P_{ux_L} of odd length $l_L=1,3,5,\cdots,2^{n-1}-1$ in L. Thus $P_{ux_L}+x_Lx_R+P_{x_Rv}$ is a uv-path of length $l=l_L+l_R+1$ in Q_n-F with $d(u,v)+4\leqslant l\leqslant 2^n-1$ and $2|(l-d(u,v))$.

Subcase 2.2 Assume $d_{Q_n}(u,v)=1$. Without loss of generality, let $|F_L|\geqslant|F_R|$.

Assume $|F_k|\geqslant 2$. Note that every neighbor x of u is adjacent to neighbor x_R of v. If there exists a vertex x adjacent to v in R such that $\{xv, xx_L\}\cap F=\emptyset$, by Lemma 2.1, there is a vx-path P_R of odd length l_R in R with $1\leqslant l_R\leqslant 2^{n-1}-1$ and there is a ux_L-path P_L of odd length l_L in L with $3\leqslant l_L\leqslant 2^{n-1}-1$. Then $P_L+xx_L+P_R$ is a uv-path of odd length l in Q_n-F with $5\leqslant l\leqslant 2^n-1$.

Notice that $|F_L|\geqslant|F_R|$, if we can not choose a vertex x adjacent to v in R such that $\{xv, xx_L\}\cap F=\emptyset$, then every $xx_L\in F_k$ and $|F_k|=n-1$, $|F_L|=|F_R|=0$. There exists a vertex y in L such that $uy+yy_R+y_Rv$ is a uv-path of length 5 in Q_n-F. There is a uy-path P_L of even length l_L in L with $2\leqslant l_L\leqslant 2^{n-1}-1$ and there is a vy_R-path P_R of even length l_R in R with $2\leqslant l_R\leqslant 2^{n-1}-1$. Then $P_L+yy_R+P_R$ is a uv-path of odd length l in Q_n-F with $5\leqslant l\leqslant 2^n-1$.

Assume $|F_k|=1$. At least two incident edges with u in L are fault-free. There exists a neighbor x of u in L such that $xx_R\notin F_k$. By the induction hypothesis, there is a ux-path P_L of odd length $l_L=1, 2^{n-1}-1$ in $L-F_L$. Since $|F_R|<n-3$, by Lemma 2.1, there is a vx_R-path P_R of odd length l_R in $R-F_R$ with $3\leqslant l_R\leqslant 2^{n-1}-1$. Then $P_L+xx_R+P_R$ is a uv-path of odd length l in Q_n-F with $5\leqslant l\leqslant 2^n-1$.

This completes the proof of Theorem 2.2. □

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