第38卷第5期 中国科学技术大学学报 Vol. 38, No. 5
2008年5月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA May 2008

Article ID: 0253-2778(2008)05-0466-09

Topological sequence entropy of the space of measures

HU Po[1], ZHANG Guo-hua[1,2]

(1. *Department of Mathematics, University of Science and Technology of China, Hefei* 230026, *China*;
2. *School of Mathematical Sciences, Fudan University, Shanghai* 200433, *China*)

Abstract: Let (X, T) be a TDS and $\mathscr{M}(X)$ the space of all Borel probability measures on X equipped with the weak* topology. (X, T) is topo-null if (X, T) has zero topological sequence entropy. Given a pseudo-metric space and a self-map, the topological sequence entropy was studied for a special class of pseudo-metrics induced by continuous real-valued functions on the space. As an application, it was proved that, given a sequence $\mathscr{A}\subseteq\mathbb{Z}_+$, if X is zero-dimensional then (X, T) has zero topological entropy along $\mathscr{A}$ if and only if $(\mathscr{M}(X), T)$ has zero topological entropy along $\mathscr{A}$. In particular, if X is zero-dimensional then (X, T) is topo-null if and only if $(\mathscr{M}(X), T)$ is topo-null.

Key words: topological sequence entropy; topo-null; pseudo-metric

CLC number: O189.1; O192 **Document code**: A

AMS Subject Classification (2000): Primary 37A35; Secondary 37B40

测度空间的拓扑序列熵

胡 泊[1],张国华[1,2]

(1.中国科学技术大学数学系,安徽合肥 230026;2.复旦大学数学科学学院,上海 200433)

摘要:给定一个拓扑动力系统(X,T),记$\mathscr{M}(X)$为X上Borel概率测度的全体,其上的拓扑由弱拓扑所诱导.如果系统(X,T)具有零拓扑序列熵,则它称为拓扑-null的.对于给定的一个伪度量空间以及其上的一个自映射(不必连续),引入并研究沿着给定序列的拓扑熵,包括由空间上连续实值函数所诱导的伪度量.作为应用可以证明,给定一个序列$\mathscr{A}\subseteq\mathbb{Z}_+$,如果$X$为零维的,那么,系统$(X,T)$沿着$\mathscr{A}$具有零拓扑熵当且仅当$(\mathscr{M}(X),T)$沿着$\mathscr{A}$具有零拓扑熵.特别的,当$X$为一个零维空间时,系统$(X,T)$为拓扑-null的当且仅当$(\mathscr{M}(X),T)$为拓扑-null的.

关键词:拓扑序列熵;拓扑-null;伪度量

0 Introduction

By a topological dynamical system (TDS) (X, T) we mean that X is a compact metric space and T is a homeomorphism from X onto X. Let (X, T) be a TDS and $\mathscr{M}(X)$ the space of all Borel probability measures on X equipped with the weak* topology, then $\mathscr{M}(X)$ is a compact metric space. That is, for

Received: 2006-11-08; **Revised**: 2007-03-10

Foundation item: Supported by NNSF of China (No. 10401031).

Biography: HU Po, female, born in 1984, master. Research field: ergodic theory and dynamical system.

Corresponding author: ZHANG Guo-hua, PhD. E-mail: ghzhang@mail.ustc.edu.cn

$\mu, \mu_1, \mu_2, \cdots \in \mathcal{M}(X)$, $\mu_n \to \mu$ if and only if (iff) $\mu_n(f) \to \mu(f)$ for each $f \in C(X)$, where $C(X)$ denotes the space of all continuous real-valued functions on X. Then T induces naturally an action on $\mathcal{M}(X)$ (denoted still by T) such that $(\mathcal{M}(X), T)$ forms a TDS, and (X, T) may be viewed as a sub-system of $(\mathcal{M}(X), T)$ by a canonical mapping $x \mapsto \delta_x$.

Entropy is defined in both ergodic theory and topological dynamics. Since the introduction of measure-theoretical entropy for an invariant measure in 1958[1] and topological entropy in 1965[2], a lot of attention has been paid to these two kinds of entropy and the relationship between them, named the classical variational principle has been obtained. Viewing the canonical mapping $x \mapsto \delta_x$, it is clear that if $(\mathcal{M}(X), T)$ has zero topological entropy then (X, T) also has zero topological entropy. The converse of the statement also holds, which was proved in Ref. [3] using two different ideas. In fact, it has been studied in Refs. [3～5] that certain dynamical properties of (X, T) need not be enjoyed by $(\mathcal{M}(X), T)$, such as minimality, unique ergodicity and so on; whereas, besides zero entropy, there are some other dynamical properties which do carry over.

In 1967 measure-theoretical sequence entropy was introduced and measure-theoretical null systems were characterized to be equivalent to the systems having a discrete spectrum[6]. Then in 1974 topological sequence entropy was introduced[7], but according to Ref. [7] there is no variational principle for sequence entropy. Let (X, T) be a TDS. We say that (X, T) is topo-null if (X, T) has zero topological sequence entropy, i.e. it has zero topological entropy along any given sequence $\mathcal{A} \subseteq \mathbb{Z}_+$. When we consider topological sequence entropy, a natural question arises that, for a given sequence $\mathcal{A} \subseteq \mathbb{Z}_+$ whether $(\mathcal{M}(X), T)$ must have zero topological entropy along $\mathcal{A}$ if (X, T) has zero topological entropy along $\mathcal{A}$; moreover, whether $(\mathcal{M}(X), T)$ must be topo-null if (X, T) is topo-null. The question is addressed in this paper. In fact, in view of results obtained in the paper, it seems possible that some dynamical behavior of a TDS may be obtained by studying some special pseudo-metrics on the space.

First, given a pseudo-metric space and a self-map we introduce the topological sequence entropy and give a systematic description of it, including a special class of pseudo-metrics induced by continuous real-valued functions on the space. We prove that, given a TDS and a sequence $\mathcal{A} \subseteq \mathbb{Z}_+$, the system has zero topological entropy along $\mathcal{A}$ iff all pseudo-metrics induced by continuous real-valued functions on the space have zero topological entropy along $\mathcal{A}$. Then, inspired by the geometric idea in Ref. [3], as an application we prove that, for any given sequence $\mathcal{A} \subseteq \mathbb{Z}_+$, if X is zero-dimensional then (X, T) has zero topological entropy along $\mathcal{A}$ iff $(\mathcal{M}(X), T)$ has zero topological entropy along $\mathcal{A}$, thus if X is zero-dimensional then (X, T) is topo-null iff $(\mathcal{M}(X), T)$ is topo-null. This gives an affirmative answer to our question in the case of zero-dimensional spaces.

The question remains open that in the general case whether $(\mathcal{M}(X), T)$ must have zero topological entropy along $\mathcal{A}$ if (X, T) has zero topological entropy along $\mathcal{A}$ for any sequence $\mathcal{A} \subseteq \mathbb{Z}_+$. It should be mentioned that we were in the process of completing the first version of the paper when we were informed of Ref. [8, Theorem 5.10] by Huang, which states that (X, T) is topo-null iff $(\mathcal{M}(X), T)$ is topo-null using a completely different method.

The paper is organized as follows. In Section 1 we introduce the topological sequence entropy of a pseudo-metric space with a self-map and give it a systematic description. Then in Section 2, on any given TDS we study the topological sequence entropy of a special class of pseudo-metrics induced by continuous real-valued functions on the space, and prove that given a TDS and a sequence of non-negative integers, the TDS has zero topological entropy along the sequence iff any pseudo-metric in this special class has zero topological entropy along the sequence. As an application, in Section 3 we prove that, given a zero-dimensional TDS (X, T) and a sequence of non-negative integers, (X, T) has zero topological entropy along the sequence iff $(\mathcal{M}(X), T)$ has zero topological entropy along the sequence, which implies that for a zero-dimensional TDS (X, T), (X, T) is topo-null iff $(\mathcal{M}(X), T)$ is topo-null.

1 Topological sequence entropy of a pseudo-metric

In this section, as the main tool of the following sections, we introduce the concept of topological entropy of a pseudo-metric space with a self-map for any given sequence $\mathcal{A}\subseteq\mathbb{Z}_+$, and discuss some basic properties of it.

Let (X,ρ) be a pseudo-metric space and T: $X\to X$ a self-map. Let $n\in\mathbb{N}$, $\varepsilon>0$ and $\mathcal{A}\subseteq\mathbb{Z}_+$ a given sequence. Denote $\mathcal{A}=\{t_i\}_{i\in\mathbf{N}}$ and fix it throughout the paper. A set $E\subseteq X$ is said to be $(\rho,n,\varepsilon,\mathcal{A})$-separated with respect to (w. r. t.) T if for every $x_1,x_2\in E$, $x_1\neq x_2$ implies

$$\max_{1\leqslant i\leqslant n}\rho(T^{t_i}x_1,\ T^{t_i}x_2)>\varepsilon.$$

A set $F\subseteq X$ is said to be $(\rho,n,\varepsilon,\mathcal{A})$-spanning w. r. t. T if for every $x\in X$ there exists $x'\in F$ such that $\max_{1\leqslant i\leqslant n}\rho(T^{t_i}x,\ T^{t_i}x')\leqslant\varepsilon$. Denote by $\operatorname{sep}_n(\rho,T,\varepsilon,\mathcal{A})$ (resp. $\operatorname{span}_n(\rho,T,\varepsilon,\mathcal{A})$) the largest (resp. smallest) cardinality of a $(\rho,n,\varepsilon,\mathcal{A})$-separated set (resp. $(\rho,n,\varepsilon,\mathcal{A})$-spanning set) w. r. t. T. Note that they may be infinite. Then we set

$$h_\rho^{\mathcal{A}}(T)=\sup_{\varepsilon>0}\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{sep}_n(\rho,T,\varepsilon,\mathcal{A}).\tag{1}$$

We call $h_\rho^{\mathcal{A}}(T)$ the topological entropy of (X,ρ,T) along $\mathcal{A}$. Sometimes we write it as $h_\rho^{\mathcal{A}}(X,T)$. When $\mathcal{A}=\mathbb{Z}_+$, we shall omit the restriction $\mathcal{A}$. And when (X,T) is a TDS with ρ the metric on the space X, we shall omit the restriction ρ. Obviously, it accords with the definition of topological sequence entropy of a TDS. It's not hard to check that

$$\operatorname{sep}_n(\rho,T,2\varepsilon,\mathcal{A})\leqslant\operatorname{span}_n(\rho,T,\varepsilon,\mathcal{A})\leqslant\operatorname{sep}_n(\rho,T,\varepsilon,\mathcal{A}),$$

which implies that

$$h_\rho^{\mathcal{A}}(T)=\sup_{\varepsilon>0}\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{span}_n(\rho,T,\varepsilon,\mathcal{A}).\tag{2}$$

Let X be a set and T: $X\to X$ a self-map. Let ρ_1 and ρ_2 be two pseudo-metrics on X. We say that ρ_1 dominates ρ_2 (denoted by $\rho_1\geq\rho_2$) if for each $\varepsilon>0$ there exists $\delta>0$ such that $\rho_1(x_1,x_2)\leqslant\delta$ implies $\rho_2(x_1,x_2)\leqslant\varepsilon$. We say that ρ_1 is equivalent to ρ_2 (denoted by $\rho_1\approx\rho_2$) if $\rho_1\geq\rho_2$ and $\rho_2\geq\rho_1$. The following fact is obvious.

Lemma 1.1 Let X be a set, T: $X\to X$ a self-map and $\mathcal{A}\subseteq\mathbb{Z}_+$ a given sequence. Let ρ_1 and ρ_2 be two pseudo-metrics on X. If $\rho_1\geq\rho_2$ then $h_{\rho_1}^{\mathcal{A}}(T)\geqslant h_{\rho_2}^{\mathcal{A}}(T)$. Moreover, if $\rho_1\approx\rho_2$ then $h_{\rho_1}^{\mathcal{A}}(T)=h_{\rho_2}^{\mathcal{A}}(T)$.

Let (X_i,ρ_i) be a pseudo-metric space, $i=1,2$. The pseudo-metric $\rho_1\oplus\rho_2$ on $X_1\times X_2$ is given by $\rho_1\oplus\rho_2((x_1,x_2),(x_1',x_2'))=\rho_1(x_1,x_1')+\rho_2(x_2,x_2')$. Then

Proposition 1.2 Let (X_i,ρ_i) be a pseudo-metric space, T_i: $X_i\to X_i$ a self-map $(i=1,2)$ and $\mathcal{A}\subseteq\mathbb{Z}_+$ a given sequence. Then

$$\max\{h_{\rho_1}^{\mathcal{A}}(T_1),h_{\rho_2}^{\mathcal{A}}(T_2)\}\leqslant h_{\rho_1\oplus\rho_2}^{\mathcal{A}}(T_1\times T_2)\leqslant h_{\rho_1}^{\mathcal{A}}(T_1)+h_{\rho_2}^{\mathcal{A}}(T_2).$$

As a direct application we have

Corollary 1.3 Let X be a set, T: $X\to X$ a self-map and $\mathcal{A}\subseteq\mathbb{Z}_+$ a given sequence. Let ρ_1 and ρ_2 be two pseudo-metrics on X. The pseudo-metric $\rho_1+\rho_2$ on X is given by

$$(\rho_1+\rho_2)(x_1,x_2)=\rho_1(x_1,x_2)+\rho_2(x_1,x_2).$$

Then

$$\max\{h_{\rho_1}^{\mathcal{A}}(T),h_{\rho_2}^{\mathcal{A}}(T)\}\leqslant h_{\rho_1+\rho_2}^{\mathcal{A}}(T)\leqslant h_{\rho_1}^{\mathcal{A}}(T)+h_{\rho_2}^{\mathcal{A}}(T).$$

Let X be a set and ρ_1, ρ_2 two pseudo-metrics on X. Put

$$\operatorname{dist}(\rho_1,\rho_2)=\sup\{|\rho_1(x_1,x_2)-\rho_2(x_1,x_2)|:x_1,x_2\in X\}.$$

Then

Proposition 1.4 Let (X,ρ) be a pseudo-metric space, T: $X\to X$ a self-map and $\mathcal{A}\subseteq\mathbb{Z}_+$ a given sequence. Let $\{\rho_i\}_{i\in\mathbf{N}}$ be a sequence of pseudo-metrics on X satisfying $\operatorname{dist}(\rho_i,\rho)\to 0$.

Then $h_\rho^{\mathscr{A}}(T)\leqslant\liminf_{i\to\infty} h_{\rho_i}^{\mathscr{A}}(T)\leqslant\sup_{i\in\mathbb{N}} h_{\rho_i}^{\mathscr{A}}(T)$.

Proof Let $\varepsilon>0$. There exists $N\in\mathbb{N}$ such that if $i\geqslant N$ then $\mathrm{dist}(\rho_i,\rho)<\frac{\varepsilon}{2}$, which implies $\rho(x_1,x_2)<\rho_i(x_1,x_2)+\frac{\varepsilon}{2}$ for any $x_1,x_2\in X$. So for each $n\in\mathbb{N}$, if $E\subseteq X$ is $(\rho,n,\varepsilon,\mathscr{A})$-separated w. r. t. T, then when $i\geqslant N$ it is $(\rho_i,n,\frac{\varepsilon}{2},\mathscr{A})$-separated w. r. t. T which implies

$$\mathrm{sep}_n(\rho,T,\varepsilon,\mathscr{A})\leqslant \mathrm{sep}_n(\rho_i,T,\frac{\varepsilon}{2},\mathscr{A})$$

and

$$\limsup_{n\to\infty}\frac{1}{n}\log \mathrm{sep}_n(\rho,T,\varepsilon,\mathscr{A})\leqslant \sup_{j\in\mathbb{N}}\inf_{i\geqslant j}\limsup_{n\to\infty}\frac{1}{n}\log \mathrm{sep}_n\Big(\rho_i,T,\frac{\varepsilon}{2},\mathscr{A}\Big).$$

Let $\varepsilon\to 0+$, we obtain

$$h_\rho^{\mathscr{A}}(T)\leqslant\sup_{j\in\mathbb{N}}\inf_{i\geqslant j}h_{\rho_i}^{\mathscr{A}}(T)=\liminf_{i\to\infty}h_{\rho_i}^{\mathscr{A}}(T)\leqslant\sup_{i\in\mathbb{N}}h_{\rho_i}^{\mathscr{A}}(T). \tag{3}$$

□

Remark 1.5 The inequality $h_\rho^{\mathscr{A}}(T)\leqslant\liminf_{i\to\infty}h_{\rho_i}^{\mathscr{A}}(T)$ may hold strictly. For example, let (X,ρ) be any pseudo-metric space containing infinitely many points. For each $i\in\mathbb{N}$, we set $\rho_i(x_1,x_2)=\max\{\rho(x_1,x_2),\frac{1}{i}\}$ if $x_1\neq x_2$ and $\rho_i(x_1,x_2)=0$ if $x_1=x_2$, which implies $\mathrm{dist}(\rho,\rho_i)\leqslant\frac{2}{i}\to 0$. Whereas, from the construction, if $x_1\neq x_2$ then $\rho_i(x_1,x_2)\geqslant\frac{1}{i}$, thus for any self-map T: $X\to X$ and any sequence $\mathscr{A}\subseteq\mathbb{Z}_+$ containing 0, $\mathrm{sep}_n(\rho_i,T,\frac{1}{2i},\mathscr{A})=\infty$ if only n is large enough. Then $h_{\rho_i}^{\mathscr{A}}(T)=\infty$ for each $i\in\mathbb{N}$.

Remark 1.6 In particular, let X be a set, T: $X\to X$ a self-map and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. Assume that $\{\rho_i\}_{i\in\mathbb{N}}$ is a sequence of pseudo-metrics on X satisfying $\sup_{i\in\mathbb{N}}\sup_{x_1,x_2\in X}\rho_i(x_1,x_2)<\infty$. Then $h_\rho^{\mathscr{A}}(T)=0$ iff $h_{\rho_i}^{\mathscr{A}}(T)=0$ for each $i\in\mathbb{N}$, where $\rho=\sum_{i\in\mathbb{N}}\frac{\rho_i}{2^i}$. In fact, set $\rho'_i=\sum_{1\leqslant j\leqslant i}\frac{\rho_j}{2^j}$ for each $i\in\mathbb{N}$. We have $\mathrm{dist}(\rho'_i,\rho)\to 0$, so $h_\rho^{\mathscr{A}}(T)\leqslant\liminf_{i\to\infty}h_{\rho'_i}^{\mathscr{A}}(T)$ (using Proposition 1.4). We also have $\rho\geq\rho'_{i+1}\geq\rho'_i$, thus $h_\rho^{\mathscr{A}}(T)\geqslant h_{\rho'_{i+1}}^{\mathscr{A}}(T)\geqslant h_{\rho'_i}^{\mathscr{A}}(T)$ (using Lemma 1.1). That is, $h_{\rho'_i}^{\mathscr{A}}(T)\nearrow h_\rho^{\mathscr{A}}(T)$. Consequently, $h_\rho^{\mathscr{A}}(T)=0$ iff $h_{\rho'_i}^{\mathscr{A}}(T)=0$ for each $i\in\mathbb{N}$, iff $h_{\rho_i/2^i}^{\mathscr{A}}(T)=0$ for each $i\in\mathbb{N}$ (using Corollary 1.3), iff $h_{\rho_i}^{\mathscr{A}}(T)=0$ (using Lemma 1.1, as $\rho_i/2^i\approx\rho_i$) for each $i\in\mathbb{N}$.

2 Topological sequence entropy of a continuous function

In this section we shall study the topological sequence entropy of a special class of pseudo-metrics on any given TDS (X,T) induced by continuous real-valued functions on X.

Let (X,T) be a TDS and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. Now for each $f\in C(X)$ we define a pseudo-metric d_f on X by setting $d_f(x_1,x_2)=|f(x_1)-f(x_2)|$. We write $h^{\mathscr{A}}(T,f)=h_{d_f}^{\mathscr{A}}(T)$, and also call it the f-topological entropy of (X,T) along $\mathscr{A}$. Thus

Lemma 2.1 Let (X,T) be a TDS and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. Then $h^{\mathscr{A}}(T,|f|)\leqslant h^{\mathscr{A}}(T,f)$ for each $f\in C(X)$, where $|f|$ denotes the absolute value of f.

Equip $C(X)$ with the maximum norm $\|\cdot\|$ and denote by $\mathrm{cl}(\mathscr{M})$ the closure of $\mathscr{M}$ in the space $C(X)$ for each $\mathscr{M}\subseteq C(X)$. Note that if $f,f_1,f_2,\cdots\in C(X)$ satisfy $\|f-f_i\|\to 0$ then $\mathrm{dist}(d_f,d_{f_i})\to 0$, thus we have (using Proposition 1.4)

Proposition 2.2 Let (X,T) be a TDS, $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence and $\mathscr{M}\subseteq C(X)$. Then $h^{\mathscr{A}}(T,f)=0$ for each $f\in\mathrm{cl}(\mathscr{M})$ iff $h^{\mathscr{A}}(T,f)=0$ for each $f\in\mathscr{M}$. Moreover,

$$\sup_{f\in\mathscr{M}}h^{\mathscr{A}}(T,f)=\sup_{f\in\mathrm{cl}(\mathscr{M})}h^{\mathscr{A}}(T,f). \tag{4}$$

The following basic facts are easy to obtain.

Proposition 2.3 Let (X,T) be a TDS, $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence and $f,f_1,f_2\in C(X)$. The functions $f^{\otimes}$, $f^{\oplus}\in C(X\times X)$ are defined as $f^{\otimes}(x_1,x_2)=f_1(x_1)f_2(x_2)$ and $f^{\oplus}(x_1,x_2)=f_1(x_1)+f_2(x_2)$. Then we have

(Ⅰ) $h^{\mathscr{A}}(T,c)=0$, where c is any constant real function on X.

(Ⅱ) $h^{\mathscr{A}}(T, c+f)=h^{\mathscr{A}}(T,f)$, where c is any real constant.

(Ⅲ) $h^{\mathscr{A}}(T, cf)=h^{\mathscr{A}}(T,f)$, where c is any non-zero real constant.

(Ⅳ) $h^{\mathscr{A}}(T\times T,f^{\otimes})\leqslant h^{\mathscr{A}}(T,f_1)+h^{\mathscr{A}}(T,f_2)$.

(Ⅴ) $h^{\mathscr{A}}(T\times T,f^{\oplus})\leqslant h^{\mathscr{A}}(T,f_1)+h^{\mathscr{A}}(T,f_2)$.

(Ⅵ) $\max\{h^{\mathscr{A}}(T,f_1f_2),h^{\mathscr{A}}(T,f_1+f_2)\}\leqslant h^{\mathscr{A}}(T,f_1)+h^{\mathscr{A}}(T,f_2)$.

proof Parts (Ⅰ), (Ⅱ) and (Ⅲ) are obvious from the definitions and Lemma 1.1.

(Ⅳ) The inequality holds clearly if $\|f_1\|\cdot\|f_2\|=0$. Now we assume $\|f_1\|\cdot\|f_2\|>0$.

Let $\varepsilon>0$ and $n\in\mathbb{N}$. Let $F_i\subseteq X$ be any $(d_{f_i},n,\frac{\varepsilon}{\|f_1\|+\|f_2\|},\mathscr{A})$-spanning subset w. r. t. T, $i=1, 2$. Set $F=F_1\times F_2\subseteq X\times X$. If $(x_1,x_2)\in X\times X$, there exists $x_i'\in F_i(i=1, 2)$ such that

$$\max_{1\leqslant j\leqslant n} d_{f_i}(T^{t_j}x_i,T^{t_j}x_i')=\max_{1\leqslant j\leqslant n}|f_i(T^{t_j}x_i)-f_i(T^{t_j}x_i')|\leqslant\frac{\varepsilon}{\|f_1\|+\|f_2\|}. \tag{5}$$

Then we have

$$\begin{aligned}&\max_{1\leqslant j\leqslant n} d_{f^{\otimes}}((T^{t_j}x_1,T^{t_j}x_2),(T^{t_j}x_1',T^{t_j}x_2'))=\\&\max_{1\leqslant j\leqslant n}|f^{\otimes}(T^{t_j}x_1,T^{t_j}x_2)-f^{\otimes}(T^{t_j}x_1',T^{t_j}x_2')|\leqslant\\&\max_{1\leqslant j\leqslant n}(|f^{\otimes}(T^{t_j}x_1,T^{t_j}x_2)-f^{\otimes}(T^{t_j}x_1',T^{t_j}x_2)|)+\\&\max_{1\leqslant j\leqslant n}(|f^{\otimes}(T^{t_j}x_1',T^{t_j}x_2)-f^{\otimes}(T^{t_j}x_1',T^{t_j}x_2')|)\leqslant\\&\|f_2\|\cdot\max_{1\leqslant j\leqslant n}|f_1(T^{t_j}x_1)-f_1(T^{t_j}x_1')|+\\&\|f_1\|\cdot\max_{1\leqslant j\leqslant n}|f_2(T^{t_j}x_2)-f_2(T^{t_j}x_2')|\leqslant\\&(\|f_2\|+\|f_1\|)\cdot\frac{\varepsilon}{\|f_1\|+\|f_2\|}(\text{by Eq. }(5))=\varepsilon.\end{aligned}$$

That is, F is $(d_{f^{\otimes}},n,\varepsilon,\mathscr{A})$-spanning w. r. t. T. So

$$\mathrm{span}_n(d_{f^{\otimes}},T,\varepsilon,\mathscr{A})\leqslant\prod_{j=1}^{2}\mathrm{span}_n\left(d_{f_j},T,\frac{\varepsilon}{\|f_1\|+\|f_2\|},\mathscr{A}\right),$$

hence

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathrm{span}_n(d_{f^{\otimes}},T,\varepsilon,\mathscr{A})\leqslant h^{\mathscr{A}}(T,f_1)+h^{\mathscr{A}}(T,f_2). \tag{6}$$

Then we claim the inequality by letting $\varepsilon\to 0+$.

(Ⅴ) We deduce it by a similar procedure as in Eq. (6), if only we notice that

$$\begin{aligned}&\max_{1\leqslant j\leqslant n} d_{f^{\oplus}}((T^{t_j}x_1, T^{t_j}x_2),(T^{t_j}x_1',T^{t_j}x_2'))\leqslant\\&\max_{1\leqslant j\leqslant n}(|f^{\oplus}(T^{t_j}x_1,T^{t_j}x_2)-f^{\oplus}(T^{t_j}x_1',T^{t_j}x_2)|)+\\&\max_{1\leqslant j\leqslant n}(|f^{\oplus}(T^{t_j}x_1',T^{t_j}x_2)-f^{\oplus}(T^{t_j}x_1',T^{t_j}x_2')|)=\\&\max_{1\leqslant j\leqslant n}|f_1(T^{t_j}x_1)-f_1(T^{t_j}x_1')|+\\&\max_{1\leqslant j\leqslant n}|f_2(T^{t_j}x_2)-f_2(T^{t_j}x_2')|\leqslant\\&\frac{2\varepsilon}{\|f_1\|+\|f_2\|}(\text{by Eq. }(5)).\end{aligned}$$

(Ⅵ) Let R denote the restriction of action $T\times T$ on Δ_X, the diagonal $\{(x,x):x\in X\}$ of X. Note that $h^{\mathscr{A}}(R,(g)^*)\leqslant h^{\mathscr{A}}(T\times T,g)$ for each $g\in C(X\times X)$, where $(g)^*\in C(\Delta_X)$ denotes the restriction of g over Δ_X. Then we have

$$\begin{aligned}&h^{\mathscr{A}}(T,f_1f_2)=h^{\mathscr{A}}(R,(f^{\otimes})^*)\\&(\text{via the canonical mapping }(x,x)\mapsto x)\leqslant\\&h^{\mathscr{A}}(T\times T,f^{\otimes})\leqslant\\&h^{\mathscr{A}}(T,f_1)+h^{\mathscr{A}}(T,f_2)\ (\text{by (Ⅳ)}).\end{aligned}$$

By the same reasoning we obtain

$$h^{\mathscr{A}}(T,f_1+f_2)\leqslant h^{\mathscr{A}}(T,f_1)+h^{\mathscr{A}}(T,f_2).\qquad\square$$

Denote by $C^+(X)$ the collection of all non-negative functions in $C(X)$. We have

Lemma 2.4 Let (X,T) be a TDS and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. Then the following statements are equivalent:

(Ⅰ) $h^{\mathscr{A}}(T)=0$.

(Ⅱ) $h^{\mathscr{A}}(T,f)=0$ for all $f\in C(X)$.

(Ⅲ) $h^{\mathscr{A}}(T,f)=0$ for all $f\in C^+(X)$.

(Ⅳ) $h^{\mathscr{A}}(T,f)=0$ for all $f\in\mathscr{M}$, where $\mathscr{M}$ is any dense subset of $C(X)$.

(Ⅴ) $h^{\mathscr{A}}(T,f)=0$ for all $f\in\mathscr{M}$, where $\mathscr{M}$ is any dense subset of $C^+(X)$.

Proof ((Ⅱ)⇔(Ⅳ)) and ((Ⅲ)⇔(Ⅴ)) follow from Proposition 2.2, ((Ⅱ)⇔(Ⅲ)) follows from Proposition 2.3 (Ⅱ). Now let's turn to the proof of ((Ⅰ)⇔(Ⅱ)). Let d be the metric on X.

((Ⅰ)⇒(Ⅱ)): Let $f\in C(X)$ and $\varepsilon>0$. Since X is compact, there exists $\delta>0$ such that $d(x_1,x_2)\leqslant\delta$ implies $d_f(x_1,x_2)=|f(x_1)-f(x_2)|\leqslant\varepsilon$. That is $d\geq d_f$, hence $h^{\mathscr{A}}(T,f)\leqslant h^{\mathscr{A}}(T)=0$ (using Lemma 1.1).

((Ⅱ)⇒(Ⅰ)): For the proof we shall follow the idea of Ref. [9, Lemma 4.2]. Assume the contrary that there exists an open cover $\{U_1, U_2\}$ of X such that $h^{\mathscr{A}}(T, \{U_1, U_2\}) > 0$ and $X\setminus U_1$ (resp. $X\setminus U_2$) has a non-empty interior containing x_1 (resp. x_2). Then by the known Urysohn Lemma there exists $f\in C^+(X)$ such that $f(x)=0$ if $x\in X\setminus U_1$ and $f(x)=1$ if $x\in X\setminus U_2$. Thus for each $x\in X$, $\{z\in X: d_f(x,z)\leqslant\frac{1}{3}\}$ is contained in either U_1 or U_2.

Now for each $n\in\mathbb{N}$, if E is $(d_f, n, \frac{1}{3}, \mathscr{A})$-spanning w. r. t. T then

$$\bigcup_{x\in E}\left\{z\in X: d_f(T^{t_i}x, T^{t_i}z)\leqslant\frac{1}{3}, 1\leqslant i\leqslant n\right\}=X.$$

Note that for all $x\in X$, $\{z\in X: d_f(T^{t_i}x, T^{t_i}z)\leqslant \frac{1}{3}, 1\leqslant i\leqslant n\}$ is contained in some elements of $\bigvee_{j=1}^{n}T^{-t_j}\{U_1, U_2\}$, we have

$$N(\bigvee_{j=1}^{n}T^{-t_j}\{U_1,U_2\})\leqslant \mathrm{span}_n(d_f, T, \frac{1}{3}, \mathscr{A}).$$

So

$$\limsup_{n\to\infty}\frac{1}{n}\log \mathrm{span}_n(d_f, T, \frac{1}{3}, \mathscr{A})\geqslant h^{\mathscr{A}}(T, \{U_1, U_2\})>0.$$

In particular, $h^{\mathscr{A}}(T, f)>0$, a contradiction with the assumption. □

For $\mathscr{M}\subseteq C(X)$ denote by $\mathrm{span}(\mathscr{M})$ the set $\{\sum_{1\leqslant i\leqslant n}c_if_i: n\in\mathbb{N}, f_1,\cdots,f_n\in\mathscr{M}, c_1,\cdots,c_n\in\mathbb{R}\}$. Then by Proposition 2.3 and Lemma 2.4 we have

Corollary 2.5 Let (X, T) be a TDS, $\mathscr{M}\subseteq C(X)$ and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. If $\mathrm{span}(\mathscr{M})$ is dense in $C(X)$, then $h^{\mathscr{A}}(T)=0$ iff $h^{\mathscr{A}}(T, f)=0$ for all $f\in\mathscr{M}$.

Let (X, T) be a TDS and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. We say that (X, T) has uniformly positive entropy (u. p. e.) along $\mathscr{A}$ if $h^{\mathscr{A}}(T, \mathscr{U})>0$ when $\mathscr{U}=\{U_1, U_2\}$ is a standard open cover of X (i. e. both $X\setminus U_1$ and $X\setminus U_2$ have non-empty interiors); and has uniformly positive sequence entropy (u. p. s. e.) if for each standard open cover $\mathscr{U}$ of X there exists a sequence $\mathscr{A}\subseteq\mathbb{Z}_+$ such that $h^{\mathscr{A}}(T, \mathscr{U})>0$. Moreover, we say that (x_1, x_2) is an entropy pair of (X, T) along $\mathscr{A}$ if $x_1\neq x_2$ and $h^{\mathscr{A}}(T, \mathscr{U})>0$ when $\mathscr{U}=\{U_1, U_2\}$ is a standard open cover of X with x_2 (resp. x_1) in the interior of $X\setminus U_1$ (resp. $X\setminus U_2$). Then we have

Theorem 2.6 Let (X, T) be a TDS and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence.

(Ⅰ) Assume that (x_1, x_2) is an entropy pair of (X, T) along $\mathscr{A}$. Then $h^{\mathscr{A}}(T, f)>0$ if $f\in C(X)$ satisfies $f(x_1)\neq f(x_2)$.

(Ⅱ) Assume that (X, T) has u. p. e. along $\mathscr{A}$. Then $h^{\mathscr{A}}(T, f)>0$ if $f\in C(X)$ is not a constant function.

(Ⅲ) Assume that (X, T) has u. p. s. e. Then for each non-constant function $f\in C(X)$ there exists a sequence $\mathscr{A}\subseteq\mathbb{Z}_+$ such that $h^{\mathscr{A}}(T, f)>0$.

Proof Note that (X, T) has u. p. e. along $\mathscr{A}$ iff (x_1, x_2) is an entropy pair of (X, T) along $\mathscr{A}$ for any $x_1\neq x_2$, Part (Ⅱ) follows from Part (Ⅰ). Since the proof of Part (Ⅲ) is the same as Part (Ⅰ), we only present the proof of Part (Ⅰ).

Let (x_1, x_2) be an entropy pair of (X, T) along $\mathscr{A}$ and $f\in C(X)$ with $f(x_1)\neq f(x_2)$. Without loss of generality we assume $f(x_1)=0$ (using Proposition 2.3 (Ⅱ)). Moreover, by Lemma 2.1 it makes no difference to assume $f\in C^+(X)$ with $f(x_2)=1$. Set

$$U_1=\left\{x\in X: f(x)<\frac{3}{4}\right\}$$

and

$$U_2=\left\{x\in X: f(x)>\frac{1}{4}\right\}.$$

Then $\mathscr{U}=\{U_1, U_2\}$ is a standard open cover of X with x_2 (resp. x_1) in the interior of $X\setminus U_1$ (resp. $X\setminus U_2$), and so $h^{\mathscr{A}}(T, \mathscr{U})>0$, as (x_1, x_2) is an entropy pair of (X, T) along $\mathscr{A}$. Obviously, for each $x\in X$, $\{z\in X: d_f(x, z)\leqslant\frac{1}{6}\}$ is contained in either U_1 or U_2. Then conducting a similar discussion as in Lemma 2.4 we have $N(\bigvee_{j=1}^{n}T^{-t_j}\mathscr{U})\leqslant \mathrm{span}_n(d_f, T, \frac{1}{6}, \mathscr{A})$, which implies $h^{\mathscr{A}}(T, f)\geqslant h^{\mathscr{A}}(T, \mathscr{U})>0$. □

Remark 2.7 In general, the converse of the above statements need not hold. There exists a zero-dimensional TDS (X, T) such that each function $f\in C(X)$ satisfying $h^{\mathbb{Z}_+}(T,f)=0$ must be a constant function, however it is not transitive, and so not u. p. e. along $\mathbb{Z}_+$ (each TDS having u. p. e. along $\mathbb{Z}_+$ must be weakly mixing[10], and so transitive). For example, let (X_1,T) be any zero-dimensional TDS having u. p. e. along $\mathbb{Z}_+$ and $x_1\in X_1$ a fixed point. Set X to be the space $X_1\times\{0,1\}$ identifying $(x_1,0)$ and $(x_1,1)$. Clearly, it is not transitive. Now assume that $f\in C(X)$ satisfies $h^{\mathbb{Z}_+}(T,f)=0$. Let $f_i\in C(X_1\times\{i\})$ be the restriction of f on $X_1\times\{i\}$, we have $h^{\mathbb{Z}_+}(T,f_i)=0$, and so f_i is a constant function (applying Theorem 2.6 (Ⅰ) to $X_1\times\{i\}$), $i=0,1$. Thus f is a constant function.

3 Topo-null TDSs

As an application of previous sections, we prove that, if X is zero-dimensional then for any given sequence $\mathscr{A}\subseteq\mathbb{Z}_+$, (X,T) has zero topological entropy along $\mathscr{A}$ iff $(\mathscr{M}(X),T)$ has zero topological entropy along $\mathscr{A}$, thus if X is zero-dimensional then (X,T) is topo-null iff $(\mathscr{M}(X),T)$ is topo-null.

First we need Ref. [3, Proposition 2.1].

Lemma 3.1 For any $\varepsilon>0$ and $b>0$ there exist $N\in\mathbb{N}$ and $c>0$ such that when $n\geqslant N$, if $\phi: l_1^{L_n}\to l_\infty^n$ is a linear map with $\|\phi\|\leqslant 1$, and if $\phi(B_1(l_1^{L_n}))$ contains at least 2^{bn} points $x_1,\cdots,x_l$ with $\min\limits_{1\leqslant i<j\leqslant l} d(x_i,x_j)>\varepsilon$, then $L_n\geqslant 2^{cn}$. Here $\|\phi\|$ (resp. $B_1(l_1^{L_n})$, d) denotes the norm of the linear operator ϕ (resp. the unit ball of $l_1^{L_n}$, the metric on l_∞^n).

Remark 3.2 A compatible metric on $l_1^{L_n}$ (resp. l_∞^n) is given by $\sum\limits_{i\geqslant 1}\sum\limits_{1\leqslant j\leqslant L_n}|a(i,j)-b(i,j)|$ (resp. $\sup\limits_{i\geqslant 1}\max\limits_{1\leqslant j\leqslant n}|a(i,j)-b(i,j)|$).

Let (X,T) be a TDS. Then the space $C(X)$ is separable, as X is a compact metric space. Let $\{f_i\}_{i\in\mathbb{N}}\subseteq C(X)$ be a dense subset. Note that each $h\in C(X)$ determines on $\mathscr{M}(X)$ a pseudo-metric $\rho_h^*(\mu_1,\mu_2)=\dfrac{\left|\int h d\mu_1-\int h d\mu_2\right|}{\|h\|+1}$ and a compatible metric on $\mathscr{M}(X)$ is given by $\rho=\sum\limits_{i\in\mathbb{N}}\dfrac{\rho_i}{2^i}$ with $\rho_i=\rho_{f_i}^*$. Then by Remark 1.6, for any given sequence $\mathscr{A}\subseteq\mathbb{Z}_+$, $h_\rho^{\mathscr{A}}(\mathscr{M}(X),T)=0$ iff $h_{\rho_i}^{\mathscr{A}}(\mathscr{M}(X),T)=0$ for all $i\in\mathbb{N}$. Now let $\{g_i\}_{i\in\mathbb{N}}\subseteq C(X)$ with span$(\{g_i: i\in\mathbb{N}\})$ dense in $C(X)$. Note that for each $f=\sum\limits_{1\leqslant i\leqslant N}\lambda_i g_i$, $\lambda_1,\cdots,\lambda_N\in\mathbb{R}$, the pseudo-metric $\sum\limits_{1\leqslant i\leqslant N}|\lambda_i|\rho_{g_i}^*(\|g_i\|+1)$ dominates the pseudo-metric ρ_f^*. Then using Lemma 1.1 and Corollary 1.3 it is not hard to obtain that

$$h_\rho^{\mathscr{A}}(\mathscr{M}(X),T)=0 \quad\text{iff}\quad h_{\rho_i'}^{\mathscr{A}}(\mathscr{M}(X),T)=0$$
$$\text{for each}\quad i\in\mathbb{N}\quad\text{with}\quad \rho_i'=\rho_{g_i}^*. \tag{7}$$

Now let (X,T) be a zero-dimensional TDS. For each clopen (closed and open) subset $A\subseteq X$ we denote by $\chi_A\in C(X)$ the characteristic function of A and write $\rho_A=\rho_{\chi_A}^*$. It is not hard to check that span$(\{\chi_A: A\subseteq X$ is clopen$\})$ is dense in $C(X)$. Note that in any compact zero-dimensional metric space, there are at most countably many clopen subsets in the space, thus using Eq. (7) we have

$$h_\rho^{\mathscr{A}}(\mathscr{M}(X),T)=0 \quad\text{iff}\quad h_{\rho_A}^{\mathscr{A}}(\mathscr{M}(X),T)=0$$
$$\text{for each clopen subset } A\subseteq X. \tag{8}$$

Then following the ideas of Ref. [3, Section 2] we have

Theorem 3.3 Let (X,T) be a zero-dimensional TDS and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. Then (X,T) has zero topological entropy along $\mathscr{A}$ iff $(\mathscr{M}(X),T)$ has zero topological entropy along $\mathscr{A}$.

Proof First assume that $(\mathscr{M}(X),T)$ has zero topological entropy along $\mathscr{A}$. (X,T), as a subsystem of $(\mathscr{M}(X),T)$, obviously has zero topological entropy along $\mathscr{A}$ as well.

Now assume that (X,T) has zero topological entropy along $\mathscr{A}$. Using Eq. (8) it suffices to prove $h^{\mathscr{A}}(\mathscr{M}(X),T)=0$ by showing $h_{\rho_A}^{\mathscr{A}}(\mathscr{M}(X),T)=0$ for each clopen $A\subseteq X$.

Set $\mathscr{U}=\{A, X\setminus A\}$, a clopen partition of X.

And for each $n\in\mathbb{N}$ we set

$$\left.\begin{array}{l}\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}=\{A_1^n,\cdots,A_{L_n}^n\},\\ \text{where}\quad L_n=N(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U})\quad\text{and}\\ \lim_{n\to\infty}\frac{1}{n}\log L_n=0.\end{array}\right\}\qquad(9)$$

We define the matrix $\phi_n=(\phi_n(j,i))_{1\leqslant j\leqslant L_n,1\leqslant i\leqslant n}$ by setting $\phi_n(j,i)=1$ if $T^{t_i}A_j^n\subseteq A$ and $\phi_n(j,i)=0$ if $T^{t_i}A_j^n\subseteq X\setminus A$. Then ϕ_n induces naturally a linear map from $l_1^{L_n}$ to l_∞^n by setting

$$\phi_n((a(i,j))_{i\geqslant 1,1\leqslant j\leqslant L_n})(k,l)=\sum_{1\leqslant j\leqslant L_n}a(k,j)\phi_n(j,l).$$

It is easy to check that $\|\phi_n\|\leqslant 1$.

Note that the space $\mathcal{M}(X)$ can be mapped into $l_1^{L_n}$ as follows

$$\mu\in\mathcal{M}(X)\mapsto(\mu(A_1^n),\cdots,\mu(A_{L_n}^n);\ 0,\cdots,0;0,\cdots)\in l_1^{L_n}.\qquad(10)$$

In fact, with the above mapping it may be viewed as $\mathcal{M}(X)\subseteq B_1(l_1^{L_n})$ (here, for our proof it makes no difference though the mapping may be not injective), so $\phi_n(B_1(l_1^{L_n}))\supseteq\phi_n(\mathcal{M}(X))$. Moreover, as $\sum_{1\leqslant j\leqslant L_n}\mu(A_j^n)\phi_n(j,l)=\mu(T^{-t_l}A)=T^{t_l}\mu(A)$ for each $\mu\in\mathcal{M}(X)$ and $1\leqslant l\leqslant n$, we have

$$\phi_n(\mu)=\Big(\sum_{1\leqslant j\leqslant L_n}\mu(A_j^n)\,\phi_n(j,1),\cdots,\sum_{1\leqslant j\leqslant L_n}\mu(A_j^n)\phi_n(j,n);0,\cdots,0;0,\cdots\Big)=(T^{t_1}\mu(A),\cdots,T^{t_n}\mu(A);0,\cdots,0;0,\cdots),$$

which implies that for all $\mu,\nu\in\mathcal{M}(X)$ we have (here, d denotes the metric on l_∞^n)

$$d(\phi_n(\mu),\phi_n(\nu))=\max_{1\leqslant i\leqslant n}|T^{t_i}\mu(A)-T^{t_i}\nu(A)|.\qquad(11)$$

If we assume the contrary that $h_{\rho_A}^{\mathcal{A}}(\mathcal{M}(X),T)>0$, then there exist $\varepsilon>0$ and $b>0$ such that for infinitely many n we have $\mathrm{sep}_n(\rho_A,T,\varepsilon,\mathcal{A})\geqslant 2^{bn}$, thus there exists $\{\mu_1,\cdots,\mu_{s_n}\}\subseteq\mathcal{M}(X)$ such that $\{\mu_1,\cdots,\mu_{s_n}\}$ is $(\rho_A,n,\varepsilon,\mathcal{A})$-separated w. r. t. T and $s_n=\mathrm{sep}_n(\rho_A,T,\varepsilon,\mathcal{A})$. That is,

$$\max_{1\leqslant i\leqslant n}|T^{t_i}\mu_{j_1}(A)-T^{t_i}\mu_{j_2}(A)|>\varepsilon\quad\text{if}\quad 1\leqslant j_1<j_2\leqslant s_n,$$

i. e. $d(\phi_n(\mu_{j_1}),\phi_n(\mu_{j_2}))>\varepsilon$ if $1\leqslant j_1<j_2\leqslant s_n$ (using Eq. (11)). Now applying Lemma 3. 1 we have that there exists $c>0$ such that $L_n\geqslant 2^{cn}$ if n is large enough, which contradicts Eq. (9). Thus $h_{\rho_A}^{\mathcal{A}}(\mathcal{M}(X),T)=0$. This completes the proof. □

Let (Y,S) be a TDS. A TDS (X,T) is a factor of (Y,S) if there exists a continuous surjective map $\pi: Y\to X$ such that $\pi S=T\pi$. We also say that (Y,S) is an extension of (X,T) and $\pi:(Y,S)\to(X,T)$ is a factor map between TDSs. Then, as a direct corollary of Theorem 3. 3 we have

Theorem 3. 4 Let (X,T) be a TDS and $\mathcal{A}\subseteq\mathbb{Z}_+$ a given sequence.

(Ⅰ) If $(\mathcal{M}(X),T)$ has zero topological entropy along $\mathcal{A}$, then (X,T) also has zero topological entropy along $\mathcal{A}$.

(Ⅱ) Suppose that TDS (X,T) admits a zero-dimensional extension (Y,S) with $h^{\mathcal{A}}(X,T)=h^{\mathcal{A}}(Y,S)$. Then $(\mathcal{M}(X),T)$ has zero topological entropy along $\mathcal{A}$ iff (X,T) has zero topological entropy along $\mathcal{A}$.

Proof Part (Ⅰ) is obvious. Now we turn to the proof of Part (Ⅱ). We assume that TDS (X,T) admits a zero-dimensional extension (Y,S) with $h^{\mathcal{A}}(X,T)=h^{\mathcal{A}}(Y,S)$. Using Part (Ⅰ), it suffices to prove that if (X,T) has zero topological entropy along $\mathcal{A}$ then $(\mathcal{M}(X),T)$ has zero also topological entropy along $\mathcal{A}$. Note that there exists a natural homomorphism which maps $\mathcal{M}(Y)$ onto $\mathcal{M}(X)$. In particular, the topological entropy along $\mathcal{A}$ of $(\mathcal{M}(X),T)$ is not more than that of $(\mathcal{M}(Y),S)$. If (X,T) has zero topological entropy along $\mathcal{A}$ and so does (Y,S) (by assumption), then using Theorem 3. 3 we have that TDS $(\mathcal{M}(Y),S)$ has zero topological entropy along $\mathcal{A}$, and so does $(\mathcal{M}(X),T)$. □

Due to the contribution of Boyle, any TDS with zero topological entropy admits a zero-dimensional extension with zero topological entropy[3, Proposition 2. 4]. Moreover, using the classical variational principle, for each TDS with finite topological entropy, Ref. [11, Fact 4. 0. 5] constructs a zero-dimensional extension with the same topological entropy. It is natural to conjecture similar results in view of topological sequence entropy. However, it seems difficult to construct a zero-dimensional topo-null extension for any topo-null TDS. Thus we ask

Question 3.5 Let (X,T) be a TDS and $\mathscr{A}\subseteq\mathbb{Z}_+$ a given sequence. Does it admit a zero-dimensional extension with the same topological entropy along $\mathscr{A}$? Does it hold if in addition we assume $h^{\mathscr{A}}(T)=0$?

Assume that Question 3.5 has an affirmative answer for a TDS (X,T) and a given sequence $\mathscr{A}\subseteq\mathbb{Z}_+$ satisfying $h^{\mathscr{A}}(T)=0$. Then using Theorem 3.4 (Ⅱ) (X,T) has zero topological entropy along $\mathscr{A}$ iff $(\mathscr{M}(X),T)$ has zero topological entropy along $\mathscr{A}$. Moreover, if Question 3.5 has an affirmative answer for a TDS (X,T) and any given sequence $\mathscr{A}\subseteq\mathbb{Z}_+$ satisfying $h^{\mathscr{A}}(T)=0$, then (X,T) is topo-null iff $(\mathscr{M}(X),T)$ is topo-null. It seems more possible that for any given sequence $\mathscr{A}\subseteq\mathbb{Z}_+$ we can construct a zero-dimensional extension for each topo-null TDS such that the extension has zero topological entropy along $\mathscr{A}$. Of course, if it is true, this also implies that (X,T) is topo-null iff $(\mathscr{M}(X),T)$ is topo-null.

Acknowledgement We thank Huang W for many useful comments. We also thank Prof Ye X for the valuable discussions.

References

[1] Kolmogorov A N. A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces[J]. Dokl Akad Nauk SSSR, 1958, 119: 861-864.

[2] Adler R L, Konheim A G, McAndrew M H. Topological entropy[J]. Trans Amer Math Soc, 1965, 114: 309-319.

[3] Glasner E, Weiss B. Quasi-factors of zero-entropy systems[J]. J Amer Math Soc, 1995, 8: 665-686.

[4] Bauer W, Sigmund K. Topological dynamics of transformations induced on the space of probability measures[J]. Monatsh Math, 1975, 79: 81-92.

[5] Glasner E. Distal and semisimple affine flows[J]. Amer J Math, 1987, 109: 115-131.

[6] Kushnirenko A G. On metric invariants of entropy type [J]. Russian Math Surveys, 1967, 22: 53-61.

[7] Goodman T N T. Topological sequence entropy[J]. Proc London Math Soc, 1974, 29: 331-350.

[8] Kerr D, Li H F. Dynamical entropy in Banach spaces [J]. Invent Math, 2005, 162: 649-686.

[9] Kerr D. Entropy and induced dynamics on state spaces [J]. Geom Funct Anal, 2004, 14: 575-594.

[10] Blanchard F. Fully positive topological entropy and topological mixing [J]. Contemporary Mathematics, 1992, 135: 95-105.

[11] Downarowicz T. Entropy structure[J]. J Anal Math, 2005, 96: 57-116.