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Connectivity and edge-connectivity of strong product graphs

YANG Chao, XU Jun-ming

(*Department of Mathematics, University of Science and Technology of China, Hefei* 230026, *China*)

Abstract: The strong product graph $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 was considered, and it was proved that $\lambda(G_1 \boxtimes G_2) = \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \delta_1 + \delta_2 + \delta_1\delta_2\}$ if G_1 and G_2 were connected, and $\kappa(G_1 \boxtimes G_2) = \min\{\delta_1 n_2, \delta_2 n_1, \delta_1 + \delta_2 + \delta_1\delta_2\}$ if G_1 and G_2 were maximally connected, where n_i, m_i, λ_i and δ_i were the order, the number of edges, the edge-connectivity and the minimum degree of $G_i (i=1,2)$, respectively.

Key words: connectivity; edge-connectivity; strong product graphs

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强乘积图的连通度和边连通度

杨 超，徐俊明

（中国科学技术大学数学系，安徽合肥 230026）

摘要：研究了两个图 G_1 和 G_2 的强乘积图 $G_1 \boxtimes G_2$ 的连通度和边连通度. 这里证明了

$$\lambda(G_1 \boxtimes G_2) = \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \delta_1 + \delta_2 + \delta_1\delta_2\},$$

如果 G_1 和 G_2 都是连通的；还证明了

$$\kappa(G_1 \boxtimes G_2) = \min\{\delta_1 n_2, \delta_2 n_1, \delta_1 + \delta_2 + \delta_1\delta_2\},$$

如果 G_1 和 G_2 都是极大连通的. 其中，n_i，m_i，λ_i 和 δ_i 分别表示 $G_i(i=1,2)$ 的阶数、边数、边连通度和最小度.

关键词：连通度；边连通度；强乘积图

0 Introduction

The concept of several kinds of product graphs including the Cartesian product and the strong product was firstly introduced by Sabidussi[1], who also gave a lower bound for the connectivity of the Cartesian product of two graphs[2]. The product of graphs is an important operation on graphs and also an important method for designing a large-scale interconnection network[3]. The connectivity and the edge-connectivity of product graphs have been studied by several authors recently (see Refs. [4,5]).

We follow Ref. [6] for graph-theoretical

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Biography: YANG Chao, male, born in 1980, PhD. Research field: graph theory.

Corresponding author: XU Jun-ming, Prof. E-mail: xujm@ustc.edu.cn

terminology and notation not defined here. Let $G=(V,E)$ be a connected graph. A subset $B\subset E(G)$ is said to be an edge-cut if there is a subset $S\subset V(G)$ such that $B=[S,\bar{S}]$, which denotes the set of edges with one end in S and the other in $\bar{S}$. The edge-connectivity $\lambda(G)$ of G is the minimum cardinality among all edge-cuts in G. A subset $S\subset V(G)$ is said to be a vertex-cut if $G-S$ is disconnected. The connectivity $\kappa(G)$ of G is the minimum cardinality among all vertex-cuts if G is not a complete graph and $n-1$ if G is a complete graph K_n. A graph G is said to be maximal if its connectivity is equal to its minimum degree $\delta(G)$.

The strong product of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, denoted by $G=G_1\boxtimes G_2$, is a graph with the vertex-set $V_1\times V_2$. Two vertices (x_1,x_2) and (y_1,y_2) are adjacent, denoted by $(x_1x_2,y_1y_2)\in E(G)$, if and only if $x_1=y_1$ and $x_2y_2\in E_2$, or $x_2=y_2$ and $x_1y_1\in E_1$, or $x_1y_1\in E_1$ and $x_2y_2\in E_2$.

By the definition above, the Cartesian product $G_1\times G_2$ is a subgraph of $G_1\boxtimes G_2$. In Ref. [5], we proved $\lambda(G_1\times G_2)=\min\{\delta_1+\delta_2,\lambda_1n_2,\lambda_2n_1\}$. Motivated by the technique in Ref. [7], in this paper, we will completely determine the edge-connectivity of the strong product of two graphs G_1 and G_2 as follows.

$$\lambda(G_1\boxtimes G_2)=\min\{\lambda_1(n_2+2m_2),\ \lambda_2(n_1+2m_1),\ \delta_1+\delta_2+\delta_1\delta_2\},$$

where n_i, m_i, λ_i and δ_i are the order, the number of edges, the edge-connectivity and the minimum degree of $G_i(i=1,2)$, respectively. We also give a lower bound of $\kappa(G_1\boxtimes G_2)$,

$$\kappa(G_1\boxtimes G_2)\geqslant\min\{\kappa_1n_2,\kappa_2n_1,\kappa_2+\kappa_1(\delta_2+1)\}.$$

The equality holds if both G_1 and G_2 are maximally connected.

The proofs of these results are in Section 1 and Section 2, respectively.

1 Edge-connectivity

By the definition, the following lemma holds clearly.

Lemma 1.1 $\delta(G_1\boxtimes G_2)=\delta_1+\delta_2+\delta_1\delta_2$.

For a graph G, let $m_0(G)=|E(G)|-\max\{|E(G')|: G'$ is a spanning bipartite of $G\}$. Clearly, if G is a bipartite, then $m_0(G)=0$.

Lemma 1.2 If G if a graph of order $n\geqslant 2$, then there exist two vertices y_1 and y_2 such that $d_G(y_1)+d_G(y_2)\leqslant n+2m_0$.

Proof The conclusion holds for $n=2$ clearly. We assume $n\geqslant 3$ below. Let G' be a spanning bipartite subgraph of G such that $m_0=|E(G)|-|E(G')|$ and $\{X,Y\}$ be a partition of $V(G')$ such that $|X|\leqslant|Y|$. Thus $|Y|\geqslant 2$. For any vertex $y\in Y$, we have $d_G(y)\leqslant|X|+d_{G[Y]}(y)\leqslant|X|+m_0$. Therefore, for any two distinct vertices y_1 and y_2 in Y, we have $d_G(y_1)+d_G(y_2)\leqslant n+2m_0$. □

In order to determine the edge-connectivity of strong product graphs, we first investigate a spanning subgraph, denoted by $K_2\odot H$, induced from $K_2\boxtimes H$ for a graph H and a complete graph K_2 with the vertex set $\{a,b\}$, defined as $K_2\odot H=K_2\boxtimes H-E(aH)-E(bH)$, where $aH=\{a\}\boxtimes H$ and $bH=\{b\}\boxtimes H$. It is clear that $K_2\odot H$ is bipartite and, moreover, connected if and only if H is connected.

Lemma 1.3 Let H be a nontrivial connected graph with edge-connectivity λ and B an edge-cut of $K_2\odot H$ that separates aH and bH. Then

$$|B|\geqslant 2\lambda.$$

Proof Since B is an edge-cut of $K_2\odot H$, there is a subset $S\subset V(K_2\odot H)$ such that $B=[S,\bar{S}]$. Partition the vertex set $V(H)$ into four parts:

$$P=\{x\in V(H): ax\in\bar{S},\ bx\in\bar{S}\},$$
$$Q=\{x\in V(H): ax\in S,\ bx\in S\},$$
$$R=\{x\in V(H): ax\in S,\ bx\in\bar{S}\}$$

and

$$T=\{x\in V(H): ax\in\bar{S},\ bx\in S\}$$

(see Fig. 1, vertices in S are marked in gray).

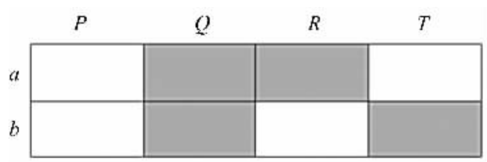

Fig. 1 $K_2\odot H$

Case 1 $R\cup T=\emptyset$. Then $[P,Q]$ is an edge-cut of H. By the definition of $K_2 \odot H$, $|[aP,bQ]| = |[bP,aQ]| = |[P,Q]|$. Thus,

$$|B| = |[aP,bQ]| + |[bP,aQ]| = 2|[P,Q]| \geqslant 2\lambda.$$

Case 2 $|R \cup T| = 1$. Without loss of generality, we assume $R=\{x_0\}$ and $T=\emptyset$. By the hypothesis that B separates both aH and bH, both P and Q are non-empty. Assume that $|[x_0,P]| \leqslant |[x_0,Q]|$. Note that $[P,Q\cup x_0]$ is an edge-cut of H. It is easy to see that $B=[aP,bQ]\cup[aQ,bP]\cup[bP,aR]\cup[aQ,bR]\cup[aR,bR]$, thus

$$\begin{aligned}|B| &= |[aP,bQ]| + |[aQ,bP]| + |[bP,aR]| + |[aQ,bR]| + |[aR,bR]| = \\ &|[P,Q]| + |[P,Q]| + |[P,x_0]| + |[Q,x_0]| + 1 > \\ &2(|[P,Q]| + |[P,x_0]|) = \\ &2|[P,Q\cup\{x_0\}]| \geqslant 2\lambda.\end{aligned}$$

Case 3 $|R\cup T| \geqslant 2$. Let $H'=H[R\cup T]$. By Lemma 1.2, there are y_1 and y_2 in H' such that $|V(H')| + 2m_0(H') \geqslant d_{H'}(y_1) + d_{H'}(y_2)$. We partition B into two parts B_1 and B_2, where B_1 is the set of the edges with both ends in $K_2 \odot H'$, and $B_2 = B \backslash B_1$. So

$$\begin{aligned}|B_1| &= |[aR,bR]| + |[aT,bT]| = \\ &(|R| + 2|E(H[R])|) + \\ &(|T| + 2|E(H[T])|) = \\ &|V(H')| + 2(|E(H[R])| + \\ &|E(H[T])|) \geqslant \\ &|V(H')| + 2m_0(H') = \\ &d_{H'}(y_1) + d_{H'}(y_2).\end{aligned}$$

For each edge $y_i z \in [y_i, P\cup Q]$ in H, either $(ay_i, bz) \in B_2$ or $(by_i, az) \in B_2$ for $i=1,2$. So we have

$$\begin{aligned}|B_2| &\geqslant |[y_1,P\cup Q]| + |[y_2,P\cup Q]| = \\ &(d_H(y_1) - d_{H'}(y_1)) + (d_H(y_2) - d_{H'}(y_2)).\end{aligned}$$

Hence, we have

$$\begin{aligned}|B| &= |B_1| + |B_2| \geqslant \\ &(d_{H'}(y_1) + d_{H'}(y_2)) + \\ &(d_H(y_1) - d_{H'}(y_1) + d_H(y_2) - d_{H'}(y_2)) = \\ &d_H(y_1) + d_H(y_2) \geqslant \\ &2\delta_H \geqslant 2\lambda.\end{aligned}$$

The lemma follows. □

Lemma 1.4 Let H be a nontrivial connected graph with edge-connectivity λ and let B be an edge-cut of $K_2 \odot H$ such that the vertices ax and bx in distinct component of $H \backslash B$ for some $x \in V(H)$. Then

$$|B| \geqslant \delta + 1.$$

Proof Let $B=[S,\overline{S}]$, and assume $ax \in S$ and $bx \in \overline{S}$. Similar to the proof of Lemma 1.3, we partition $N_H(x)$ into four parts: P, Q, R and T (see Fig. 2, black dots denote vertices in S). It is easy to see that for each vertex y in $N_H(x)$, y contributes at least one edge to B: $(ax,by) \in B$ if $y \in P$; $(bx,ay) \in B$ if $y \in Q$; both $(ax,by) \in B$ and $(bx,ay) \in B$ if $y \in R$; $(ay,by) \in B$ if $y \in T$. Noting that (ax, bx) is also an edge in B, we have $|B| \geqslant |N_H(x)| + 1 \geqslant \delta + 1$. □

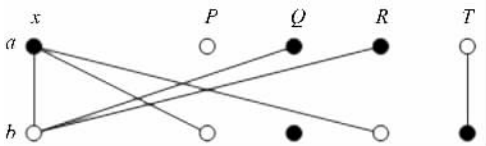

Fig. 2 ***x*** **and its four type neighbors**

Lemma 1.5 Let G be a connected graph, and $\emptyset \neq A \subset V(G)$. Then $|A| + |[A,\overline{A}]| \geqslant \delta + 1$, and the equality holds if and only if $A=\{a\}$ and $d_G(a)=\delta$.

Proof Let $a \in A$, then a has at most $|A|-1$ neighbors inside A, and at most $|[A,\overline{A}]|$ neighbors outside A. So $d_G(a) \leqslant |A| - 1 + |[A,\overline{A}]|$, namely $|A| + |[A,\overline{A}]| \geqslant d_G(a) + 1 \geqslant \delta + 1$. If the equality holds, then a must be a vertex of the minimum degree, and all the edges in $[A,\overline{A}]$ are incident with a. Suppose to the contrary that $|A| \geqslant 2$. Let b be another vertex in A rather than a. Then b has at most $|A|-1$ neighbors inside A and no neighbors outside A, hence $d_G(b) < d_G(a)$, a contradiction. On the other hand, if $A=\{a\}$ and $d_G(a)=\delta$, the equality holds obviously. □

Theorem 1.6 Let G_i be a nontrivial connected graph with n_i vertices, m_i edges and the minimum degree δ_i for $i=1,2$. Then

$$\lambda(G_1 \boxtimes G_2) = \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \delta_1 + \delta_2 + \delta_1\delta_2\}.$$

Proof By Lemma 1.1,

$$\lambda(G_1 \boxtimes G_2) \leqslant \delta(G_1 \boxtimes G_2) = \delta_1 + \delta_2 + \delta_1\delta_2.$$

Let $[X, \overline{X}]$ be a minimum edge-cut G_1. Then $[X \times V(G_2), \overline{X} \times V(G_2)]$ is an edge-cut of $G_1 \boxtimes G_2$ with $\lambda_1(n_2 + 2m_2)$ edges, hence

$$\lambda(G_1 \boxtimes G_2) \leqslant \lambda_1(n_2 + 2m_2).$$

By symmetry, we have

$$\lambda(G_1 \boxtimes G_2) \leqslant \lambda_2(n_1 + 2m_1).$$

So it remains to prove

$$\lambda(G_1 \boxtimes G_2) \geqslant \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \delta_1 + \delta_2 + \delta_1\delta_2\}.$$

Let $B = [S, \overline{S}]$ be an edge-cut of $G = G_1 \boxtimes G_2$. Without loss of generality, assume $\delta_1 \leqslant \delta_2$. For each $x \in V_2$, let $G_2^x = \{x\} \boxtimes G_2$, which is a subgraph of G isomorphic to G_2. We say that G_2^x is separated by B if it has vertices in both S and $\overline{S}$. Let r be the number of vertices $x \in V_1$ such that G_2^x is separated by B. Let $V_1 = \{x_1, x_2, \cdots, x_{n_1}\}$ and $B_i = B \cap E(G_2^{x_i})$. For an edge $e \in E(G_1)$, let $B_e = B \cap (\{e\} \odot G_2)$.

Case 1 $r = 0$

Let $X = \{x \in V_1: V(G_2^x) \subset S\}$. Then $B^* = [X, V_1 \backslash X]$ is an edge-cut of G_1. Therefore,

$$|B| = \sum_{e \in B^*} |B_e| = |B^*|(n_2 + 2m_2) \geqslant \lambda_1(n_2 + 2m_2).$$

Case 2 $r = n_1$

Then $|B_i| \geqslant \lambda_2$. And by Lemma 1.3, $|B_e| \geqslant 2\lambda_2$ for every edge $e = xy \in E(G_1)$ since B_e is an edge-cut of $\{e\} \odot G_2$ that separates both G_2^x and G_2^y.

$$|B| = \sum_{i=1}^{n_1} |B_i| + \sum_{e \in E(G_1)} |B_e| \geqslant n_1\lambda_2 + m_1 \cdot 2\lambda_2 = \lambda_2(n_1 + 2m_1).$$

Case 3 $0 < r < n_1$

Let x_1 and x_k be two adjacent vertices of G_1 such that $G_2^{x_1}$ is separated by B and $G_2^{x_k}$ is not. Assume $V(G_2^{x_k}) \subseteq \overline{S}$, and let $Y = \{y \in V(G_2): x_1 y \in S\}$. Let H be a maximally connected subgraph of G_1 containing x_1 and the set $\{y \in V(G_2): x_i y \in S\} = Y$ for each $x_i \in V(H)$. Let $E^* = E_{G_1}(V(H), \overline{V(H)}) \backslash \{x_1 x_k\}$ (See Fig. 3, vertices in S are marked in gray). Let $a = |V(H)|$, $b = |E(H)|$, $c = |E^*| + 1$, $p = |Y|$ and $q = |E_{G_2}(Y, \overline{Y})|$. For each edge $e = ab \in E^*$, it is obvious that B_e is an edge-cut of $\{e\} \odot G_2$. And by the maximality of H, there exists some $x_0 \in V(H)$ such that ax_0 and bx_0 in distinct component of $\{e\} \odot G_2 - B_e$, thus $|B_e| \geqslant \delta_2 + 1$ by Lemma 1.4. And it is easy to see that

$$|B_{x_1 x_k}| = \sum_{x \in Y} (d_H(x) + 1) \geqslant p(\delta_2 + 1).$$

Therefore,

$$|B| \geqslant \sum_{x_i \in V(H)} |B_i| + \sum_{e \in E(H)} |B_e| + |B_{x_1 x_k}| + \sum_{e \in E^*} |B_e| \geqslant aq + 2bq + p(\delta_2 + 1) + (c - 1)(\delta_2 + 1). \quad (1)$$

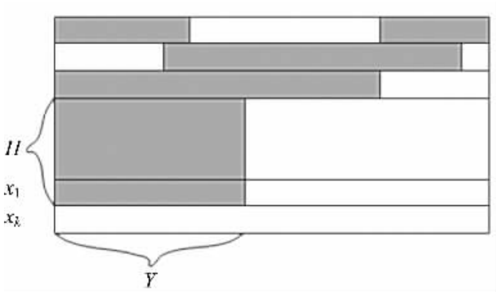

Fig. 3 Illustration for Case 3

If $p = 1$, then $q \geqslant \delta_2$. And $b \geqslant a - 1$ because H is connected; $a + c \geqslant \delta_1 + 1$ by Lemma 1.5. Hence by Eq. (1),

$$\begin{aligned}|B| &\geqslant aq + 2bq + p(\delta_2 + 1) + (c - 1)(\delta_2 + 1) \geqslant \\ &a\delta_2 + 2(a - 1)\delta_2 + c(\delta_2 + 1) = \\ &(a + c)(\delta_2 + 1) + 2(a - 1)\delta_2 - a \geqslant \\ &(\delta_1 + 1)(\delta_2 + 1) + (a - 1) - a = \\ &\delta_1 + \delta_2 + \delta_1\delta_2.\end{aligned}$$

In the rest of the proof, assume $p \geqslant 2$. Then by Lemma 1.5, $p + q \geqslant \delta_2 + 2$. If $a \geqslant \delta_2 + 1$, by Eq. (1), we have

$$\begin{aligned}|B| &\geqslant aq + 2bq + p(\delta_2 + 1) + (c - 1)(\delta_2 + 1) \geqslant \\ &q(\delta_2 + 1) + 0 + p(\delta_2 + 1) + \\ &(c - 1)(\delta_2 + 1) = \\ &(p + q + c - 1)(\delta_2 + 1) > \\ &(\delta_2 + 1)(\delta_2 + 1) \overset{(*)}{\geqslant} \\ &(\delta_1 + 1)(\delta_2 + 1) > \delta_1 + \delta_2 + \delta_1\delta_2.\end{aligned}$$

Here, the inequality (*) holds since $\delta_1 \leqslant \delta_2$ by our assumption. The only case remaining is $a < \delta_2 + 1$.

In this case,

$$|B| \geqslant aq + 2bq + p(\delta_2 + 1) + (c-1)(\delta_2 + 1) = aq + 2bq + (p-1)(\delta_2 + 1) + c(\delta_2 + 1) > aq + 0 + (p-1)a + c(\delta_2 + 1) = a(p+q-1) + c(\delta_2 + 1) \geqslant a(\delta_2 + 1) + c(\delta_2 + 1) \geqslant (\delta_1 + 1)(\delta_2 + 1) > \delta_1 + \delta_2 + \delta_1\delta_2.$$

Thus, in all cases, we have

$$|B| \geqslant \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \delta_1 + \delta_2 + \delta_1\delta_2\},$$

this completes the proof. □

Corollary 1.7 Let H be a nontrivial connected graph with edge-connectivity λ and the minimum degree δ. Then

$$\lambda(K_n \boxtimes H) = \min\{n(\delta + 1) - 1, n^2\lambda\}.$$

Specially, $\lambda(K_n \boxtimes K_m) = nm - 1$.

The corollary follows directly from Theorem 1.6 by simple calculation and comparison. Note that $K_n \boxtimes K_m = K_{nm}$, so its edge-connectivity can be seen easily without using Theorem 1.6.

2 Connectivity

First, we consider the case that at least one of the factor graphs is complete.

Proposition 2.1 $\kappa(K_n \boxtimes K_m) = nm - 1$.

Proof Note that $K_n \boxtimes K_m = K_{nm}$. □

Proposition 2.2 Let H be a non-complete connected graph with connectivity κ. Then

$$\kappa(K_n \boxtimes H) = n\kappa.$$

Proof The conclusion holds clearly for $n=1$. Assume $n \geqslant 2$ below. It is sufficient to prove that $\kappa(K_n \boxtimes H) \geqslant n\kappa$ because the reversed inequality holds clearly. Let $\{x_1, x_2, \cdots, x_n\}$ be the vertex set of K_n, S any cut set (not necessary minimum) of $K_n \boxtimes H$ and $S_i = S \cap (\{x_i\} \times V(H))$. We claim that $|S_i| \geqslant \kappa$. Suppose to the contrary, for example, that $|S_1| < \kappa$. Then $\{x_1\} \boxtimes H - S_1$ is connected. For each vertex $x_k (k \neq i)$ in $\{x_k\} \boxtimes H - S_k$, the number of neighbors of u in $\{x_1\} \boxtimes H$ is at least $\delta(H) + 1 \geqslant \kappa + 1$, therefore at least one of them remains after the removal of S_1. This implies that $K_n \boxtimes H - S$ is connected, which contradicts our hypothesis that S is a cut set. Thus, $|S| = \sum_{i=1}^{n} |S_i| \geqslant n\kappa$. □

In the rest of this section, we assume that both G_1 and G_2 are non-complete graphs. In this case, it is easy to see that

$$\kappa(G_1 \boxtimes G_2) \leqslant \min\{\kappa_1 n_2, \kappa_2 n_1, \delta_1 + \delta_2 + \delta_1\delta_2\}. \quad (2)$$

Unlike the case of edge-connectivity, the reversed inequality in Eq. (2) does not always hold. The simplest counter example may be the strong product of the path P_4 and the graph H, where H is shown in Fig. 4. The strong product graph $P_4 \boxtimes H$ is shown in Fig. 5, and the four black vertices form a vertex separating set of $P_4 \boxtimes H$. It is easy to check

$$\kappa(P_4 \boxtimes H) = 4 < 5 = \min\{\kappa(P_4)|V(H)|, \kappa(H)|V(P_4)|, \delta(P_4 \boxtimes H)\}.$$

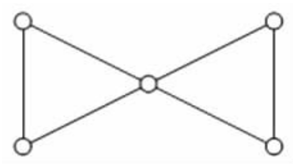

Fig. 4 A graph H

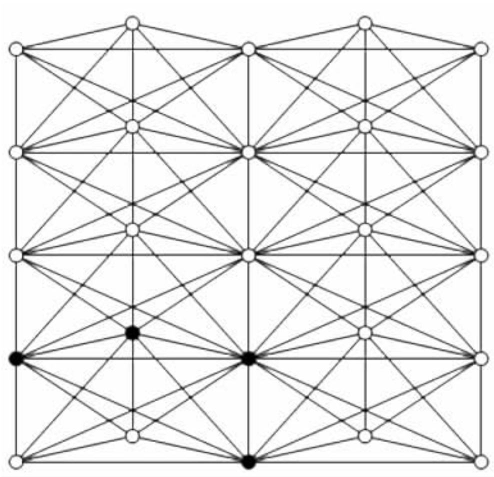

Fig. 5 $P_4 \boxtimes H$

Theorem 2.3 Let G_i be a nontrivial non-complete connected graph with order n_i, connectivity κ_i, and the minimum degree δ_i, for $i=1,2$. Then

$$\kappa(G_1\boxtimes G_2)\geqslant\min\{\kappa_1 n_2,\kappa_2 n_1,\kappa_1+\kappa_2+\kappa_1\delta_2\}.$$

Proof Let S be a minimum vertex separating set. For each $x\in V(G_1)$, G_2^x is said to be separated by S if it contains vertex of at least two components of $G_1\boxtimes G_2-S$. Let $S_x=S\cap V(G_2^x)$ and r the number of vertices x's in G_1 for which G_2^x is separated by S. Note that $|S_x|\geqslant\kappa_2$ if G_2^x is separated by S.

Case 1 $r=0$. Then for each x, G_2^x contains vertices of at most one component of $G_1\boxtimes G_2-S$. If there is no edge $e=xy\in E(G_1)$ such that G_2^x and G_2^y contain vertices in distinct components of $G_1\boxtimes G_2-S$, then let

$$T=\{x\in V(G_1):V(G_2^x)\subset S\}.$$

Then $|T|\geqslant\kappa_1$ since T is a separating set of G_1. Thus

$$|S|\geqslant\sum_{x\in T}|S_x|\geqslant\kappa_1 n_2.$$

Now, let x_1 and x_2 be two adjacent vertices of $V(G_1)$ such that $V(G_2^{x_1}-S_{x_1})$ and $V(G_2^{x_2}-S_{x_2})$ belong to distinct components C and C' of $G_1\boxtimes G_2-S$. Thus there are κ_1 internally disjoint x_1x_2-paths $P_1,P_2,\cdots,P_{\kappa_1}$ in G_1, where $P_1=x_1x_2$ and all other paths have lengths at least 2. For each x_1x_2-path $P=x_{t_1=1}x_{t_2}\cdots x_{t_k=2}$ of length at least 2, namely $k\geqslant 3$, we will find an internal vertex $y(P)$ with $|S_{y(P)}|$ as large as possible. Let x_{t_i} be such a vertex that $G_2^{x_{t_i}}$ contains vertices in C and nearest to x_2 along the path P. Obviously, $x_{t_i}\neq x_2$. If $x_{t_i}=x_1$, let $y(P)=x_{t_2}$, then $|S_{x_{t_2}}|\geqslant\delta_2+1$, since $G_2^{x_1}-S_{x_1}$ has at least δ_2+1 neighbors in $G_2^{x_{t_2}}$. If $x_{t_i}\neq x_1$, consider $G_2^{x_{t_{i+1}}}$. If $V(G_2^{x_{t_{i+1}}})\subset S$ (which implies $x_{t_{i+1}}\neq x_2$), let $y(P)=x_{t_{i+1}}$ then

$$|S_{x_{t_{i+1}}}|=|V(G_2)|\geqslant\delta_2+2;$$

if $G_2^{x_{t_{i+1}}}$ has some vertices not in S, say u, then u lies in a component other than C by our choice of x_{t_i}, and u has at least δ_2+1 neighbors in $G_2^{x_{t_i}}$, let $y(P)=x_{t_i}$, thus $|S_{x_{t_i}}|\geqslant\delta_2+1$. So, in all cases, we have found an internal vertex of the path P, denoted by $y(P)$, such that $|S_{y(P)}|\geqslant\delta_2+1$. Note that also $|S_{x_1}|\geqslant\delta_2+1$ and $|S_{x_2}|\geqslant\delta_2+1$. So

$$\begin{aligned}|B|&\geqslant\sum_{j=2}^{\kappa_1}|S_{y(P_j)}|+|S_{x_1}|+|S_{x_2}|\geqslant\\&(\kappa_1-1)(\delta_2+1)+2(\delta_2+1)=\\&(\kappa_1+1)(\delta_2+1)>\\&\kappa_2+\kappa_1(\delta_2+1).\end{aligned}$$

Case 2 $r>0$. Let

$$X=\{x\in V(G_1):G_2^x\text{ is separated by }S\}$$

and let $Y=N_{G_1}(X)$. It is easy to see that $|S_y|\geqslant\delta_2+1$ for each $y\in Y$. Suppose to the contrary that $|S_y|\leqslant\delta_2$. Let $x^*\in X$ be a neighbor of y. Then each vertex in $G_2^x-S_x$ has a neighbor in $G_2^y-S_y$. But $G_2^y-S_y$ is connected since $y\notin X$. Therefore $G_2^x-S_x$ must lie in one component, a contradiction.

If $X\cup Y=V(G_1)$, then

$$\begin{aligned}|B|&=\sum_{x\in X}|S_x|+\sum_{y\in Y}|S_y|\geqslant\\&|X|\kappa_2+|Y|(\delta_2+1)\geqslant\\&(|X|+|Y|)\kappa_2=n_1\kappa_2.\end{aligned}$$

Now assume that $X\cup Y$ is a proper subset of $V(G_1)$. Then $|Y|\geqslant\kappa_1$ since Y separates X and $V(G_1)-X-Y$. Therefore,

$$|B|=\sum_{x\in X}|S_x|+\sum_{y\in Y}|S_y|\geqslant\kappa_2+\kappa_1(\delta_2+1).$$

The proof of the theorem is complete. □

Ref. [4] showed that

$$\kappa(G_1\boxtimes G_2)\geqslant\min\{\kappa_1(\delta_2+1),\kappa_2(\delta_1+1)\}.$$

Theorem 2.3 improves this result, since

$$\begin{aligned}&\min\{\kappa_1 n_2,\kappa_2 n_1,\kappa_2+\kappa_1(\delta_2+1)\}\geqslant\\&\min\{\kappa_1(\delta_2+1),\kappa_2(\delta_1+1)\}+1.\end{aligned}$$

The graph shown in Fig. 5 shows that the lower bound in Theorem 2.3 is the best possible.

As we have already pointed out, the reversed inequality in Eq. (2) does not hold generally, but it holds if both of the factor graphs are maximally connected, which is a direct corollary of Theorem 2.3.

Corollary 2.4 Let G_i be a nontrivial non-complete maximally connected graph with order n_i, connectivity κ_i, and the minimum degree δ_i, for $i=1,2$. Then

$$\kappa(G_1\boxtimes G_2)=\min\{\kappa_1 n_2,\kappa_2 n_1,\ \delta_1+\delta_2+\delta_1\delta_2\}.$$

The condition that G_i is non-complete for each

$i=1,2$ in Corollary 2.4 can not be omitted in view of $K_n \boxtimes W_{2n+1}$, where W_{2n+1} is constructed by a complete graph K_{2n-1} and two extra vertices each of which is joined by an edge to all vertices in K_{2n-1}. Obviously, $\kappa(W_{2n+1}) = 2n-1$. By Proposition 2.2,

$$\kappa(K_n \boxtimes W_{2n+1}) = n\kappa(W_{2n+1}) = n(2n-1) = 2n^2 - n,$$

but

$$\min\{\kappa(K_n)\,|V(W_{2n+1})|, \kappa(W_{2n+1})\,|V(K_n)|, \delta(K_n \boxtimes W_{2n+1})\} = (n-1)(2n+1) = 2n^2 - n - 1.$$

We conclude this paper with a conjecture about the connectivity of strong product graphs. Let G be a graph with connectivity κ and the minimum degree δ. For each j ($\kappa \leqslant j \leqslant \delta$), let

$$c^j = \min\{|C| : C \text{ is a component of } G-S, S \text{ is a separating set with } |S| = j\}.$$

It follows immediately that $c^\delta = 1$ and $c^{j+1} \leqslant c^j$.

Conjecture 2.5 Let G_i be a nontrivial non-complete connected graph with order n_i, connectivity κ_i, and the minimum degree δ_i, for $i=1,2$. And c_i^j are defined as the above for $\kappa_i \leqslant j \leqslant \delta_i$, $i=1,2$. Then

$$\kappa(G_1 \boxtimes G_2) = \min\{\kappa_1 n_2, \kappa_2 n_1, \min_{\kappa_i \leqslant j_i \leqslant \delta_i} \{j_1 j_2 + j_1 c_2^{j_2} + j_2 c_1^{j_1}\}\}.$$

If both G_1 and G_2 are maximally connected, namely $\kappa_i = \delta_i$ ($i=1,2$), Conjecture 2.5 can be referred to Corollary 2.4.

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