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至多一个斜率变点模型的收敛速度

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摘要:对线性模型中至多只有一个斜率变点 τ_0 的模型,利用局部比较方法给出了变点 τ_0 估计 $\hat{\tau}$ 的强、弱相合性和强、弱收敛速度,同时在局部对立假设下研究了变点估计的 O_p 收敛速度.

关键词:变点;强相合性;收敛速度

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Strong convergence rate for slope change point estimator

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Abstract: The problem of slope-change point was discussed. The consistency and convergence rate of the change point estimator were presented. At the same time, the convergence rate of the change point estimator was obtained when the magnitude of the slope change was small.

Key words: change point; strong consistency; convergence rate

0 引言

变点问题自上世纪70年代以来一直是统计中的一个热门话题,它广泛应用于工业质量控制、生物统计、经济、金融等领域[1~4].许多学者对独立随机变量序列中的变点问题都进行了研究,如 Chernoff et al[6]提出了检验正态分布均值变点的统计量;Haccou et al[11], Ramanayake et al[15]对独立指数随机变量序列中的变点提出了似然比检验方法;Daniel et al[10]利用 Bayes 方法讨论了变点检验的问题;Csörgö et al[8,9], Krishnasah et al[12]利用非参数方法讨论了变点的检测和估计,Parka et al[14]用小波的方法讨论了突变点的问题并给出了 O_p 的速度.缪柏其等[17]研究了变点个数及位置的检测和估计问题;史晓平等[18]利用密度函数的核估计研究了对称的稳定分布参数变点估计的相合性;谭常春等[19,20]则利用累积和方法对 Γ 分布参数两个参数同时发生变化时的变点进行了统计推断.

关于斜率变点的研究,Miao[13]利用局部比较法研究了变点的检测和估计,Chu et al[7]利用似然比方法研究了不同误差条件下的斜率变点问题.但是这种估计是否是相合的?如果是相合的,则随着样本量 n 的增大,估计量与真实值的误差有多大?这些都是在实际应用中所关心的问题.因此本文在Miao[13]给出变点估计的基础上,进一步研究了变点估计的强、弱相合性,并给出了强、弱收敛速度以及在局部对立假设下变点的 O_p 收敛速度.考虑Miao[13]中模型

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$$X_k=\begin{cases}\mu+\beta_1\left(\tau_0-\frac{k}{n}\right)+\varepsilon_k,\ 1\leqslant k\leqslant[n\tau_0];\\ \mu+\beta_2\left(\frac{k}{n}-\tau_0\right)+\varepsilon_k,\ [n\tau_0]<k\leqslant n.\end{cases}\tag{1}$$

式中，μ, β_1, β_2, τ_0 为未知参数，$0<\tau_0<1$，称 τ_0 为变点；ε_k 为随机误差，其分布 F 与 k 无关，并假设 $E(\varepsilon_k)=0$，$\mathrm{Var}(\varepsilon_k)=\sigma^2$（$\sigma^2$ 存在且有限）；符号[·]表示向下取整.

为方便起见，文中使用下面的记号：

$$\left.\begin{array}{l}Y_k=\frac{1}{\sqrt{4m}}\Big(\sum_{i=k+m+1}^{k+2m}X_i-\sum_{i=k+1}^{k+m}X_i-\sum_{i=k-m+1}^{k}X_i+\\ \quad\sum_{i=k-2m+1}^{k-m}X_i\Big),\ k=2m,\cdots,n-2m,\\ A_n(x)=\left[2\log\left(\frac{5n}{4m}-5\right)\right]^{-\frac{1}{2}}\Big(x+2\log\left(\frac{5n}{4m}-5\right)+\\ \quad\frac{1}{2}\log\log\left(\frac{5n}{4m}-5\right)-\frac{1}{2}\log\pi\Big),\\ X_i^*=X_i-EX_i,\ \xi_i^*=X_i^*-X_{i-m}^*.\end{array}\right\}\tag{2}$$

同时记 $c, c_1, c_2, \cdots$ 为与 n 无关的常数，每次出现可以不同.

1 变点估计的相合性和收敛速度

假定本节中出现的 $g_i(x)$, $i=1,2$ 均为 x 的非负单调增函数，且 $\lim\limits_{x\to\infty}g_i(x)=\infty$；为了证明下面的定理，先给出两个引理.

引理 1.1 令 $\{X_i, i\geqslant1\}$ 是 i. i. d. r. v. 序列，$X_1\sim N(0,\sigma^2)$，其矩母函数为 $M(t)=\mathrm{e}^{\frac{\sigma^2t^2}{2}}$. 记 $m(x)=\inf\limits_t \mathrm{e}^{-xt}M(t)$. 则对任意的 $x>0$，有

$$P(S_n\geqslant nx)\leqslant m^n(x)=\mathrm{e}^{-\frac{nx^2}{2\sigma^2}}.\tag{3}$$

证明 参见文献[16]. □

引理 1.2 设 $X_1, X_2, \cdots, X_n$ 独立，且

$$X_k=\mu+\frac{k}{n}\beta+\varepsilon_k,\ k=1,2,\cdots,n.\tag{4}$$

$\varepsilon_1,\cdots,\varepsilon_n$ i. i. d. $\sim N(0,\sigma^2)$. $m=m_n$ 为正整数，满足条件

$$n^{\frac{2}{3}}\log^{\frac{2}{3}}n\ll m\ll n,\tag{5}$$

$m\ll n$ 表示 $\lim\limits_{n\to\infty}m/n=0$. 则当 σ^2 已知时，有

$$\left.\begin{array}{l}\lim\limits_{n\to\infty}P\Big(\max\limits_{2m\leqslant k\leqslant n-2m}\frac{1}{\sigma}\mid Y_k\mid\leqslant A_n(x)\Big)=\exp\{-2\mathrm{e}^{-x}\},\\ \qquad -\infty<x<+\infty,\end{array}\right\}\tag{6}$$

当 σ^2 未知时，有

$$\left.\begin{array}{l}\lim\limits_{n\to\infty}P\Big(\max\limits_{2m\leqslant k\leqslant n-2m}\frac{1}{\hat\sigma_n}\mid Y_k\mid\leqslant A_n(x)\Big)=\exp\{-2\mathrm{e}^{-x}\},\\ \qquad -\infty<x<+\infty.\end{array}\right\}\tag{7}$$

式中，$\hat\sigma_n^2$ 为 σ^2 的估计，满足 $\lim\limits_{n\to\infty}|\hat\sigma_n^2-\sigma^2|\log n=0$, in P, $A_n(x)$ 见式(2).

证明 参见文献[13]. □

引理 1.2 可以检测变点的存在性. 在检验出存在变点或事先已知变点存在的情况下，由文献[13]，定义变点 τ_0 的估计为

$$\hat\tau=\frac{\hat k}{n}=\frac{1}{n}\min\{k:\mid Y_k\mid=\max_{2m\leqslant j\leqslant n-2m}\mid Y_j\mid\}.\tag{8}$$

定理 1.3 假设式(1)，(5)，(8)成立，ε_1, ε_2, $\cdots$, ε_n i. i. d. $\sim N(0,\sigma^2)$. 并假定

$$k_0=[n\tau_0],\ k=[n\tau],\ 0<\tau<1,\tag{9}$$

当坡变度 $\beta_0\hat{=}\beta_2-\beta_1$ 为常数时，则 $\hat\tau$ 为 τ_0 的相合估计，且 $|\hat\tau-\tau_0|=o_p(g_1^{-1}(n))$，其中 $g_1(n)=o(n^{\frac{1}{2}}m^{-\frac{1}{4}})$.

证明 不妨假定 $\beta_0>0$. 欲证 $\hat\tau$ 的相合性及其收敛速度，对任何的 $\varepsilon>0$，先考虑 $P(g_1(n)|\hat\tau-\tau_0|>\varepsilon)$. 因为

$$X_k=\begin{cases}\mu+\frac{k-k_0}{n}\beta_1+\varepsilon_k,\ k=1,2,\cdots,k_0;\\ \mu+\frac{k-k_0}{n}\beta_2+\varepsilon_k,\ k=k_0+1,k_0+2,\cdots,n.\end{cases}$$

计算可得，

$$EY_{k_0}=\frac{m^{\frac{3}{2}}\beta_0}{2n},\tag{10}$$

$$EY_k=\frac{\beta_0}{n\sqrt{4m}}\cdot\begin{cases}0,\quad k\leqslant k_0-2m;\\ \frac{(k-k_0+2m)(k-k_0+2m+1)}{2},\\ \qquad k_0-2m<k\leqslant k_0-m;\\ m^2-\frac{(k-k_0)(k-k_0+1)}{2},\ k_0-m<k\leqslant k_0;\\ m^2-\frac{(k_0-k)(k_0-k-1)}{2},\ k_0<k\leqslant k_0+m;\\ \frac{(k_0-k+2m)(k_0-k+2m-1)}{2},\\ \qquad k_0+m<k\leqslant k_0+2m;\\ 0,\quad k>k_0+2m.\end{cases}\tag{11}$$

由式(10),(11)得

$$|EY_{k_0}|-|EY_k|=\frac{\beta_0}{n\sqrt{4m}}\cdot\begin{cases}m^2, & k\leqslant k_0-2m;\\ m^2-\dfrac{(k-k_0+2m)(k-k_0+2m+1)}{2}, & k_0-2m<k\leqslant k_0-m;\\ \dfrac{(k-k_0)(k-k_0+1)}{2}, & k_0-m<k\leqslant k_0;\\ \dfrac{(k_0-k)(k_0-k-1)}{2}, & k_0<k\leqslant k_0+m;\\ m^2-\dfrac{(k_0-k+2m)(k_0-k+2m-1)}{2}, & k_0+m<k\leqslant k_0+2m;\\ m^2, & k>k_0+2m.\end{cases}\tag{12}$$

由三角不等式有

$$|EY_{k_0}|-|EY_k|\leqslant |Y_k-EY_k|+|Y_{k_0}-EY_{k_0}|+|Y_{k_0}|-|Y_k|\leqslant 2\max_{2m\leqslant k\leqslant n-2m}|Y_k-EY_k|+|Y_{k_0}|-|Y_k|.\tag{13}$$

(ⅰ) 当 $k_0-2m<k\leqslant k_0-m$ 时,记 $k_0-k=sm$,则 $1\leqslant s<2$. 因为

$$m^2-\frac{(k-k_0+2m)(k-k_0+2m+1)}{2}=\frac{m^2}{2}\left(2-(2-s)\left(2-s+\frac{1}{m}\right)\right)=\frac{m^2}{2}\left(-s^2+\left(4+\frac{1}{m}\right)s-2-\frac{2}{m}\right)\geqslant\frac{m^2}{2}\times\frac{1}{4}s^2=\frac{m^2s^2}{8},\tag{14}$$

所以有

$$|EY_{k_0}|-|EY_k|=\frac{\beta_0}{n\sqrt{4m}}\left(m^2-\frac{(k-k_0+2m)(k-k_0+2m+1)}{2}\right)\geqslant\frac{\beta_0}{n\sqrt{4m}}\times\frac{1}{8}s^2m^2=\frac{\beta_0}{16n\sqrt{m}}(k-k_0)^2=\frac{n\beta_0}{16\sqrt{m}}(\tau-\tau_0)^2.\tag{15}$$

(ⅱ) 当 $k_0-m<k\leqslant k_0$ 时, 因为只要 $k_0-k\neq 1$, 即有 $(k-k_0)(k-k_0+1)\geqslant\frac{(k_0-k)^2}{2}$, 所以

$$|EY_{k_0}|-|EY_k|=\frac{\beta_0}{4n\sqrt{m}}(k-k_0)(k-k_0+1)\geqslant\frac{\beta_0}{8n\sqrt{m}}(k-k_0)^2=\frac{n\beta_0}{8\sqrt{m}}(\tau-\tau_0)^2.\tag{16}$$

同理, 对 $k_0<k\leqslant k_0+2m$ 有相同结果. 综合式(13)～(16)知,当 $k\in[k_0-2m,k_0+2m]$时, 有

$$n\beta_0(\hat{\tau}-\tau_0)^2\leqslant 32\sqrt{m}\max_{2m\leqslant k\leqslant n-2m}|Y_k-EY_k|.\tag{17}$$

为了证明 $\hat{\tau}$ 的相合性,我们先证明

$$\left.\begin{aligned}&\hat{k}\in(k_0-2m,k_0+2m)\ \text{in}\ P,\\ &\text{即}\ P\{\hat{k}\notin(k_0-2m,k_0+2m)\}\to 0.\end{aligned}\right\}\tag{18}$$

首先考虑 $P(\hat{k}\leqslant k_0-2m)$. 取 $a_m=\frac{1}{2}EY_{k_0}=\frac{m^{\frac{3}{2}}\beta_0}{4n}$. 注意到$|Y_{\hat{k}}|\geqslant|Y_{k_0}|$,则

$$\begin{aligned}&P(\hat{k}\leqslant k_0-2m)=\\ &P(|Y_{\hat{k}}|\geqslant|Y_{k_0}|,\hat{k}\leqslant k_0-2m)\leqslant\\ &P(\max_{k\leqslant k_0-2m}|Y_k|\geqslant|Y_{k_0}|)\leqslant\\ &P(\max_{k\leqslant k_0-2m}|Y_k|\geqslant|Y_{k_0}|,\bigcap_{k\leqslant k_0-2m}(|Y_k|<a_m))+\\ &P(\bigcup_{k\leqslant k_0-2m}(|Y_k|\geqslant a_m))\hat{=}A_1+A_2,\end{aligned}\tag{19}$$

式中,

$$\begin{aligned}A_1&\leqslant P\left(|Y_{k_0}|<\frac{1}{2}EY_{k_0}\right)\leqslant\\ &P\left(Y_{k_0}<\frac{1}{2}EY_{k_0}\right)=\\ &P\left(Y_{k_0}-EY_{k_0}<-\frac{1}{2}EY_{k_0}\right)\leqslant\\ &P\left(|Y_{k_0}-EY_{k_0}|>\frac{1}{2}EY_{k_0}\right)\leqslant\\ &P\left(|\sum_{i=k_0+m+1}^{k_0+2m}(X_i-EX_i)|>\frac{m^{\frac{1}{2}}}{4}EY_{k_0}\right)+\\ &P\left(|\sum_{i=k_0+1}^{k_0+m}(X_i-EX_i)|>\frac{m^{\frac{1}{2}}}{4}EY_{k_0}\right)+\\ &P\left(|\sum_{i=k_0-m+1}^{k_0}(X_i-EX_i)|>\frac{m^{\frac{1}{2}}}{4}EY_{k_0}\right)+\\ &P\left(|\sum_{i=k_0-2m+1}^{k_0-m}(X_i-EX_i)|>\frac{m^{\frac{1}{2}}}{4}EY_{k_0}\right)=\\ &4P\left(|\sum_{i=k_0+m+1}^{k_0+2m}X_i^*|>\frac{m^{\frac{1}{2}}}{4}EY_{k_0}\right)=\\ &4P\left(|\sum_{i=k_0+m+1}^{k_0+2m}X_i^*|>\frac{m^2\beta_0}{8n}\right).\end{aligned}$$

由引理 1.1 得

$$A_1\leqslant 4P\left(|\sum_{i=k_0+m+1}^{k_0+2m}X_i^*|>\frac{m^2\beta_0}{8n}\right)=$$

$$8P\Big(\sum_{i=k_0+m+1}^{k_0+2m} X_i^* > \frac{m^2\beta_0}{8n}\Big) \leqslant$$

$$8\exp\Big\{-\frac{m^3\beta_0^2}{128\sigma^2 n^2}\Big\} = 8\exp\Big\{-c_1\frac{m^3}{n^2}\Big\}. \quad (20)$$

注意到当 $k\leqslant k_0-2m$ 时,$EY_k=0$,故

$$A_2 = P\Big(\bigcup_{k\leqslant k_0-2m}(|Y_k|\geqslant a_m)\Big)\leqslant$$

$$\sum_{k\leqslant k_0-2m} P\Big(|Y_k|\geqslant \frac{1}{2}EY_{k_0}\Big)\leqslant$$

$$4nP\Big(\Big|\sum_{i=k+m+1}^{k+2m}(X_i-EX_i)\Big| > \frac{m^{\frac{1}{2}}}{4}EY_{k_0}\Big) =$$

$$4nP\Big(\Big|\sum_{i=k+m+1}^{k+2m} X_i^*\Big| > \frac{m^2\beta}{8n}\Big)\leqslant$$

$$8n\exp\Big\{-\frac{m^3\beta_0{}^2}{128\sigma^2 n^2}\Big\} = 8n\exp\Big\{-c_2\frac{m^3}{n^2}\Big\}. \quad (21)$$

综合式(20),(21)得

$$P(\hat{k}\leqslant k_0-2m)\leqslant c_3 n\exp\Big\{-c_4\frac{m^3}{n^2}\Big\}. \quad (22)$$

由式(5)有$\frac{m^3}{n^2}\geqslant \log^2 n$,即得 $P(\hat{k}\leqslant k_0-2m)<\frac{1}{n^4}$,同理有 $P(\hat{k}\geqslant k_0-2m)<\frac{1}{n^4}$. 所以有

$$\sum_{n=1}^{\infty} P(\hat{k}\leqslant k_0-2m)<\infty,$$

$$\sum_{n=1}^{\infty} P(\hat{k}\geqslant k_0-2m)<\infty,$$

故 $\hat{k}\in(k_0-2m,k_0+2m)$,a. s..

记 $S_k^* = \sum_{i=k_0-2m+1}^{k} X_i^*$, $k>k_0-2m$,$S_k=\sum_{i=1}^{k} X_i^*$. 综合式(17),(18) 知,

$$(\hat{\tau}-\tau_0)^2 \leqslant \frac{16}{n\beta_0}\max_{2m\leqslant k\leqslant n-2m}\Big|\sum_{i=k+m+1}^{k+2m}X_i^* - \sum_{i=k+1}^{k+m}X_i^* - \sum_{i=k-m+1}^{k}X_i^* + \sum_{i=k-2m+1}^{k-m}X_i^*\Big|\leqslant$$

$$\frac{16}{n\beta_0}\max_{k_0-2m\leqslant k\leqslant k_0+2m}\Big|\sum_{i=k+m+1}^{k+2m}X_i^* - \sum_{i=k+1}^{k+m}X_i^* - \sum_{i=k-m+1}^{k}X_i^* + \sum_{i=k-2m+1}^{k-m}X_i^*\Big|, \text{ in } P =$$

$$\frac{16}{n\beta_0}\max_{k_0-2m\leqslant k\leqslant k_0+2m}|S_{k+2m}^*-2S_{k+m}^*+2S_{k-m}^*-S_{k-2m}^*|\leqslant$$

$$\frac{96}{n\beta_0}\max_{k_0-2m\leqslant k\leqslant k_0+2m}|S_k^*| =$$

$$\frac{96}{n\beta_0}\max_{k_0-2m\leqslant k\leqslant k_0+2m}\Big|\sum_{i=1}^{k-k_0+2m}X_{i+k_0-2m}^*\Big| \overset{d}{=}$$

$$\frac{96}{n\beta_0}\max_{1\leqslant k\leqslant 4m}\Big|\sum_{i=1}^{k}X_i^*\Big|. \quad (23)$$

式中,$X\overset{d}{=}Y$ 表示 X 与 Y 的分布相同. 由式(23),Levy 不等式及引理 1.1 知

$$P(g_1(n)|\hat{\tau}-\tau_0|>\varepsilon) =$$

$$P(g_1{}^2(n)(\hat{\tau}-\tau_0)^2>\varepsilon^2)\leqslant$$

$$P\Big(\max_{1\leqslant k\leqslant 4m}\Big|\sum_{i=1}^{k}X_i^*\Big| > \frac{n\beta_0\varepsilon^2}{96g_1^2(n)}\Big)\leqslant$$

$$2P\Big(|S_{4m}| > \frac{n\beta_0\varepsilon^2}{96g_1^2(n)}\Big)\leqslant$$

$$4\exp\Big\{-\frac{n^2\beta_0{}^2\varepsilon^4\cdot 4m}{384^2\cdot 2\sigma^2 m^2 g_1^4(n)}\Big\} =$$

$$4\exp\Big\{-c_5\frac{n^2}{mg_1^4(n)}\Big\}. \quad (24)$$

由式(24)知,若取 $g_1(n)=o(n^{\frac{1}{2}}m^{-\frac{1}{4}})$,且 m 满足式(5)时,则有 $P(g_1(n)|\hat{\tau}-\tau_0|>\varepsilon)\to 0$,命题得证.

注 1.4 ①若取 $m=n^{\frac{2}{3}+\alpha}$, $0<\alpha<\frac{1}{3}$时, 则可取 $g_1(n)=o(n^{\frac{1}{3}-\frac{\alpha}{4}})$;

②若取 $m=n^{\frac{2}{3}}\log^{\gamma}n$,$\frac{2}{3}<\gamma$ 时,则可取 $g_1(n)=o(n^{\frac{1}{3}}\log^{-\frac{1}{4}\gamma}n)$.

定理 1.5 假设式(1),(5),(8),(9)成立, ε_1, ε_2,…, ε_n i. i. d. $\sim N(0,\sigma^2)$. 当坡变度 β_0 为常数时,则 $\hat{\tau}$ 为 τ_0 的强相合估计. 若窗宽 $m=n^{\frac{2}{3}+\delta}$, $0<\delta<\frac{1}{3}$, 则$|\hat{\tau}-\tau_0|=o(g_2^{-1}(n))$, a. s. ,其中 $g_2(n)=o(n^{\frac{1}{3}-\frac{\delta}{4}}\log^{-\frac{1}{4}}n)$.

证明 由定理 1.3 的证明, 我们有

$$\hat{k}\in(k_0-2m,k_0+2m) \text{ a. s.}. \quad (25)$$

由式(25), 类似式(23)的证明有

$$(\hat{\tau}-\tau_0)^2\leqslant\frac{96}{n\beta_0}\max_{1\leqslant k\leqslant 4m}\Big|\sum_{i=1}^{k}X_i^*\Big|,\text{a. s.}. \quad (26)$$

综合式(5)和式(26)知, 对 $\forall\varepsilon>0$,

$$\sum_{n=1}^{\infty}P(g_2(n)|\hat{\tau}-\tau_0|>\varepsilon) =$$

$$\sum_{n=1}^{\infty}P(g_2^2(n)(\hat{\tau}-\tau_0)^2>\varepsilon^2)\leqslant$$

$$\sum_{n=1}^{\infty}P\Big(\max_{1\leqslant k\leqslant 4m}\Big|\sum_{i=1}^{k}X_i^*\Big| > \frac{n\beta_0\varepsilon^2}{96g_2^2(n)}\Big)\leqslant$$

$$\sum_{n=1}^{\infty}2P\Big(|S_{4m}| > \frac{n\beta_0\varepsilon^2}{96g_2^2(n)}\Big)\leqslant$$

$$\sum_{n=1}^{\infty}4\exp\Big\{-\frac{n^2\beta_0^2\varepsilon^4\cdot 4m}{384^2\cdot 2\sigma^2 m^2 g_2^4(n)}\Big\} =$$

$$\sum_{n=1}^{\infty} 4\exp\left\{-c_6 \frac{n^2}{mg_2^4(n)}\right\}. \tag{27}$$

由式(27)知，当窗宽 $m=n^{\frac{2}{3}+\delta}$，$0<\delta<\frac{1}{3}$ 时，取 $g_2(n)=o(n^{\frac{1}{3}-\frac{\delta}{4}}\log^{-\frac{1}{4}} n)$，有

$$\sum_{n=1}^{\infty} P(g_2(n)\mid \hat{\tau}-\tau_0\mid>\varepsilon)<\infty,$$

命题得证. □

注 1.6　由注 1.4 及定理 1.5 知，当 $m=n^{\frac{2}{3}+\theta}$，$0<\theta<\frac{1}{3}$ 时，

$$g_1(n)=o(n^{\frac{1}{3}-\frac{\theta}{4}}),g_2(n)=o(n^{\frac{1}{3}-\frac{\theta}{4}}\log^{-\frac{1}{4}} n);$$

估计 $\hat{\tau}$ 强、弱收敛速度之间仅相差一个慢变函数.

定理 1.3，1.5 给出了 $\hat{\tau}$ 的强、弱相合性和强、弱收敛速度. 下面的定理则考虑在局部对立假设条件下 $\hat{\tau}$ 的收敛速度. 局部对立条件即 β_0 与样本量 n 有关，记 β_0 为 β_n，且当 $n\to\infty$ 时，$\beta_n\to 0$.

2　局部对立条件下的 O_P 收敛速度

定理 2.1　假定定理 1.3 的条件满足，设存在 $0<\theta_0<\frac{1}{2}$，使得 $\tau_0\in(\theta_0,1-\theta_0)$. 若 β_n 满足条件

$$\beta_n>0,\text{且 } \beta_n\to 0,\ \beta_n^2\log n\to\infty(\text{或常数 } c), \tag{28}$$

则有

$$\mid\hat{\tau}-\tau_0\mid=O_p\left(\frac{m}{n}\right).$$

证明　由定理 1.3 证明知，$\hat{\tau}$ 为 τ_0 的相合估计，则 $\forall\varepsilon>0$，有 $P(\hat{\tau}\notin(\theta_0,1-\theta_0))<\varepsilon$. 这是因为 $\forall\delta>0$，

$$P(\hat{\tau}\notin(\theta_0,1-\theta_0))\leqslant P(\mid\hat{\tau}-\tau_0\mid>\delta).$$

而由定理 1.5 中的式(4)，令 $g_1(n)=1$，则有

$$P(\hat{\tau}\notin(\theta_0,1-\theta_0))\leqslant P(\mid\hat{\tau}-\tau_0\mid>\delta)\to 0.$$

欲证 $|\hat{\tau}-\tau_0|=O_p\left(\frac{m}{n}\right)$，即需证对 $T=T(n)\to\infty$，有

$$P\left(\mid\hat{\tau}-\tau_0\mid>\frac{m}{n}T\right)\to 0. \tag{29}$$

记 $D_{n,T}=\{k: |k-k_0|>Tm,[n\theta_0]\leqslant k\leqslant[n(1-\theta_0)]\}$. 因为

$$P\left(\mid\hat{\tau}-\tau_0\mid>T\frac{m}{n}\right)\leqslant$$
$$P(\hat{\tau}\notin(\theta_0,1-\theta_0))+$$
$$P\left(\mid\hat{\tau}-\tau_0\mid>T\frac{m}{n},\hat{\tau}\in(\theta_0,1-\theta_0)\right)\leqslant$$
$$\varepsilon+P(\mid\hat{k}-k_0\mid>Tm,\hat{k}\in([n\theta_0],[n(1-\theta_0)]))\leqslant$$
$$\varepsilon+P(\max_{k\in D_{n,T}}\mid Y_k\mid>\mid Y_{k_0}\mid). \tag{30}$$

因为

$$P(\max_{k\in D_{n,T}}\mid Y_k\mid>\mid Y_{k_0}\mid)=$$
$$P(\max_{k\in D_{n,T}}\mid Y_k\mid>\mid Y_{k_0}\mid,Y_{k_0}<0)+$$
$$P(\max_{k\in D_{n,T}}\mid Y_k\mid>\mid Y_{k_0}\mid,Y_{k_0}\geqslant 0)=$$
$$P(\max_{k\in D_{n,T}}\mid Y_k\mid>\mid Y_{k_0}\mid,Y_{k_0}<0)+$$
$$P(\max_{k\in D_{n,T}}\mid Y_k\mid>\mid Y_{k_0}\mid,Y_{k_0}\geqslant 0,\bigcap_{k\in D_{n,T}}\{Y_k>0\})+$$
$$P(\max_{k\in D_{n,T}}\mid Y_k\mid>\mid Y_{k_0}\mid,Y_{k_0}\geqslant 0,\bigcup_{k\in D_{n,T}}\{Y_k\leqslant 0\})\leqslant \tag{31}$$

$$P(Y_{k_0}<0)+$$
$$P(\max_{k\in D_{n,T}}(Y_k-Y_{k_0})>0,Y_{k_0}\geqslant 0,\bigcap_{k\in D_{n,T}}\{Y_k>0\})+$$
$$P(\inf_{k\in D_{n,T}}(Y_k+Y_{k_0})\leqslant 0,Y_{k_0}\geqslant 0,\bigcup_{k\in D_{n,T}}\{Y_k\leqslant 0\})+$$
$$P(\max_{k\in D_{n,T}}(Y_k-Y_{k_0})\geqslant 0,Y_{k_0}\geqslant 0,\bigcup_{k\in D_{n,T}}\{Y_k\leqslant 0\})\leqslant$$
$$P(Y_{k_0}<0)+2P(\max_{k\in D_{n,T}}(Y_k-Y_{k_0})>0)+$$
$$P(\inf_{k\in D_{n,T}}(Y_k+Y_{k_0})\leqslant 0)\hat{=}$$
$$B_1+B_2+B_3. \tag{32}$$

由引理 1.1 即有

$$B_1=P(Y_{k_0}<0)=P(Y_{k_0}-EY_{k_0}<-EY_{k_0})=$$
$$P\left(\frac{1}{\sqrt{4m}}(\sum_{i=k_0+m+1}^{k_0+2m}\xi_i^*-\sum_{i=k_0-m+1}^{k_0}\xi_i^*)<-\frac{\beta_n m^{\frac{3}{2}}}{2n}\right)\leqslant$$
$$P\left(\mid\sum_{i=k_0+m+1}^{k_0+2m}\xi_i^*-\sum_{i=k_0-m+1}^{k_0}\xi_i^*\mid>\frac{\beta_n m^2}{n}\right)\leqslant$$
$$2P\left(\mid\sum_{i=k_0+m+1}^{k_0+2m}\xi_i^*\mid>\frac{\beta_n m^2}{2n}\right)=$$
$$4P\left(S_m>\frac{\beta_n m^2}{2n}\right)\leqslant$$
$$4\exp\left\{-\frac{\beta_n^2 m^3}{16\sigma^2 n^2}\right\}=4\exp\left\{-c_1\frac{\beta_n^2 m^3}{n^2}\right\}. \tag{33}$$

当 $k\in D_{n,T}$ 时，有 $EY_k=0$，故有

$$B_3=P(\inf_{k\in D_{n,T}}(Y_k+Y_{k_0})\leqslant 0)\leqslant$$
$$P(\inf_{k\in D_{n,T}}(Y_k-EY_k+Y_{k_0}-EY_{k_0})\leqslant -EY_{k_0})\leqslant$$
$$P(\sup_{k\in D_{n,T}}\mid Y_k-EY_k+Y_{k_0}-EY_{k_0}\mid>EY_{k_0})\leqslant$$
$$P\left(\sup_{k\in D_{n,T}}\mid Y_k-EY_k\mid>\frac{1}{2}EY_{k_0}\right)+$$
$$P\left(\mid Y_{k_0}-EY_{k_0}\mid>\frac{1}{2}EY_{k_0}\right)\hat{=}D_1+D_2. \tag{34}$$

利用式(6)知，

$$P(\max_{[n\theta_0]\leqslant k\leqslant k_0-Tm} |Y_k - EY_k| \geqslant \sigma A_n(x)) \to 1-\exp\{-2e^{-x}\},$$

这里，

$$A_n(x) = \left[2\log\left(\frac{5(k_0-n\theta_0)}{4m}-5T\right)\right]^{-\frac{1}{2}} \cdot \left(x+2\log\left(\frac{5(k_0-n\theta_0)}{4m}-5T\right)+\frac{1}{2}\log\log\left(\frac{5(k_0-n\theta_0)}{4m}-5T\right)-\frac{1}{2}\log\pi\right),$$

对任意 $-\infty<x<+\infty$，$A_n(x)=O(\log^{\frac{1}{2}} n)$，令 $x\to+\infty$，

$$\lim_{n\to\infty} P(\max_{[n\theta_0]\leqslant k\leqslant k_0-Tm} |Y_k - EY_k| \geqslant c\log^{\frac{1}{2}} n) = 0, \tag{35}$$

其中 $c>0$ 的常数. 同理有

$$\lim_{n\to\infty} P(\max_{k_0+Tm\leqslant k\leqslant n-[n\theta_0]} |Y_k - EY_k| \geqslant c\log^{\frac{1}{2}} n) = 0, \tag{36}$$

综合式(34),(35)得

$$D_1 = P\left(\sup_{k\in D_{n,T}} |Y_k - EY_k| > \frac{\beta_n m^{\frac{3}{2}}}{4n}\right) \leqslant P\left[\sup_{[n\theta_0]\leqslant k\leqslant k_0-Tm} |Y_k - EY_k| > \log^{\frac{1}{2}} n \frac{\beta_n m^{\frac{3}{2}}}{4n\log^{\frac{1}{2}} n}\right] + P\left[\sup_{k_0+Tm\leqslant k\leqslant n-[n\theta_0]} |Y_k - EY_k| > \log^{\frac{1}{2}} n \frac{\beta_n m^{\frac{3}{2}}}{4n\log^{\frac{1}{2}} n}\right] \leqslant P(\sup_{[n\theta_0]\leqslant k\leqslant k_0-Tm} |Y_k - EY_k| > c\log^{\frac{1}{2}} n) + P(\sup_{k_0+Tm\leqslant k\leqslant n-[n\theta_0]} |Y_k - EY_k| > c\log^{\frac{1}{2}} n) \to 0. \tag{37}$$

且类似由式(32)得，

$$D_2 = P\left(\left|\sum_{i=k_0+m+1}^{k_0+2m} \xi_i^* - \sum_{i=k_0-m+1}^{k_0} \xi_i^*\right| > \frac{\beta_n m^2}{2n}\right) \leqslant 2P\left(\left|\sum_{i=k_0+m+1}^{k_0+2m} \xi_i^*\right| > \frac{\beta_n m^2}{4n}\right) = 4P\left(S_m > \frac{\beta_n m^2}{4n}\right) \leqslant 4\exp\left\{-\frac{\beta_n^2 m^3}{96\sigma^2 n^2}\right\} = 4\exp\left\{-c_2\frac{\beta_n^2 m^3}{n^2}\right\}. \tag{38}$$

综合条件(33)～(37)知 $B_3\to 0$. 同理有

$$B_2 \leqslant 2P(\sup_{k\in D_{n,T}} |Y_k - EY_k - Y_{k_0} + EY_{k_0}| > EY_{k_0}) \leqslant 2P\left(\sup_{k\in D_{n,T}} |Y_k - EY_k| > \frac{1}{2}EY_{k_0}\right) + 2P\left(|Y_{k_0} - EY_{k_0}| > \frac{1}{2}EY_{k_0}\right) \to 0. \tag{39}$$

综上所述知，当 β_n 满足条件(28)时，式(29)均成立，即 $|\hat{\tau}-\tau_0| = O_p\left(\frac{m}{n}\right)$. □

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