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On bondage number of toroidal graphs

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Abstract: The bondage number $b(G)$ of a nonempty graph G is the cardinality of a smallest edge set whose removal from G results in a graph with the domination number greater than the domination number $\gamma(G)$ of G. [Fischermann M, Rautenbach D, Volkmann L. Remarks on the bondage number of planar graphs. Discrete Math, 2003, 260: 57-67] showed that for a connected planar graph G with girth $g(G)$, $b(G)\leqslant 6$ if $g(G)\geqslant 4$, $b(G)\leqslant 5$ if $g(G)\geqslant 5$, $b(G)\leqslant 4$ if $g(G)\geqslant 6$ and $b(G)\leqslant 3$ if $g(G)\geqslant 8$. This result was generalized to a connected toroidal graph that was embeddable on the torus.

Key words: bondage number; domination number; crossing number; planar graph

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超环面图上的约束数

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摘要:非空图 G 的约束数 $b(G)$ 是指使得图 G 的控制数 $\gamma(G)$ 增大而删除的最少的边数. [Fischermann M, Rautenbach D, Volkmann L. Remarks on the bondage number of planar graphs. Discrete Math, 2003, 260: 57-67]已经证明,对于一个围长为 $g(G)$ 的平面图 G,如果 $g(G)\geqslant 4$ 则 $b(G)\leqslant 6$,如果 $g(G)\geqslant 5$ 则 $b(G)\leqslant 5$,如果 $g(G)\geqslant 6$ 则 $b(G)\leqslant 4$,如果 $g(G)\geqslant 8$ 则 $b(G)\leqslant 3$. 我们把这个结果推广到连通的超环面图中.

关键词:约束数;控制数;交叉数;平面图

0 Introduction

For terminology and notation on graph theory not given here, the reader is referred to Ref. [9]. Let $G=(V,E)$ be a finite, undirected and simple graph. For $u\in V(G)$ let $N_G(u)$ be the neighborhood of u, that is, $N_G(u)=\{v\in V(G): uv\in E(G)\}$, and $N_G(X)=\bigcup_{u\in X}N_G(u)$ for a set $X\subseteq V(G)$. We denote the degree of u by $d_G(u)=|N_G(u)|$, the minimum and the maximum degree of G by $\delta(G)$ and $\Delta(G)$. For a subset $A\subseteq V(G)$, let $G[A]$ be the subgraph induced by A. We denote the distance between the vertices x and y in the graph G by $d_G(x,y)$. The girth $g(G)$ of G is

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the length of a shortest cycle in G. If G has no cycles we define $g(G)=\infty$. A set $D\subseteq V(G)$ is called a dominating set if $D\cup N(D)=V(G)$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets in G. The bondage number of a nonempty graph G, denoted by $b(G)$, is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$.

The first result on bondage number was obtained in Ref. [1]. Ref. [3] conjectured that $b(G)\leqslant\Delta(G)+1$ for any nontrivial planar graph G. Ref. [7] confirmed this conjecture for $\Delta(G)\geqslant 7$ by proving that $b(G)\leqslant\min\{8,\Delta(G)+2\}$, and proved that $b(G)\leqslant 7$ for any connected planar graph without vertices of degree five. Ref. [4] generalized the latter result, and showed that the conjecture is valid for all connected planar graphs with $g(G)\geqslant 4$ and $\Delta(G)\geqslant 5$ as well as all planar graphs with $g(G)\geqslant 5$ unless they are 3-regular. In particular, they proved that for a connected planar graph G,

$$b(G)\leqslant\begin{cases}6, \text{ if } g(G)\geqslant 4;\\ 5, \text{ if } g(G)\geqslant 5;\\ 4, \text{ if } g(G)\geqslant 6;\\ 3, \text{ if } g(G)\geqslant 8.\end{cases} \tag{1}$$

Recently, Ref. [6] has generalized the result in Eq. (1) to a connected graph with small crossing number. In this paper, we generalize the result in Eq. (1) to a connected toroidal graph which can be embedded on the torus, that is,

$$b(G)\leqslant\begin{cases}6, \text{ if } g(G)\geqslant 4 \text{ and } G \text{ is not 4-regular};\\ 5, \text{ if } g(G)\geqslant 5;\\ 4, \text{ if } g(G)\geqslant 6 \text{ and } G \text{ is not 3-regular};\\ 3, \text{ if } g(G)\geqslant 8.\end{cases}$$

In the next section, we recall some results to be used in our discussions. The proofs of our main results are given in Section 2.

1 Preliminary results

In the first place, let us recall the concept of embedding a graph into a surface. Let S be a given surface. We say a graph G to be embeddable on S if G can be drawn on S such that its edges intersect only at their end-vertices. It is well known that a graph G is embeddable on the sphere if and only if it is embeddable on the plane. A graph G is called a planar graph if G is embeddable on the plane or the sphere. There exist many graphs, such as K_5 and $K_{3,3}$, which are not embeddable on the plane or the sphere. To avoid crossings of edges when we draw a graph G in the sphere, we could change the surface by adding overpasses, called handles, to the sphere. The torus is the surface obtained by adding one handle to a sphere. A graph is called a toroidal graph if it can be embedded on the torus. For example, K_5 and $K_{3,3}$ both are toroidal graphs. A more complicated example, the cartesian product $C_m\times C_n$ of two cycles C_m and C_n is a toroidal graph.

We use $\omega(G)$ to denote the number of components in a graph G. An edge e of G is a cut-edge if $\omega(G-e)>\omega(G)$. Use $c(G)$ to denote the number of cut-edges in G.

As we know, see, for example, Theorem 4.22 in Ref. [2], if G is a toroidal graph with $n(G)$ vertices, $m(G)$ edges, $\omega(G)$ components and $\phi(G)$ regions, then

$$\phi(G)=m(G)-n(G)+\omega(G)-1. \tag{2}$$

The following lemma is an analogy of Ref. [4] for a planar graph.

Lemma 1.1 If G is a toroidal graph with $3\leqslant g(G)<\infty$, then

$$m(G)\leqslant\frac{g(G)n(G)-c(G)}{g(G)-2}.$$

Proof Every noncut-edge is on a common boundary of two regions and every cut-edge is on a boundary of exactly one region, so we have that $g(G)\phi(G)\leqslant 2m(G)-c(G)$. Then, by Eq. (2) we have

$$g(G)(m(G)-n(G)+\omega(G)-1)\leqslant 2m(G)-c(G).$$

Thus,

$$(g(G)-2)m(G)\leqslant g(G)(n(G)-\omega(G)+1)-c(G).$$

It follows that

$$m(G)\leqslant\frac{g(G)(n(G)-\omega(G)+1)-c(G)}{g(G)-2}\leqslant$$

$$\frac{g(G)n(G)-c(G)}{g(G)-2}$$

as required. □

Let G be a toroidal graph and let $n_i(G)$ denote the number of vertices of degree i in G. We use g, Δ, m, n and n_i to denote $g(G)$, $\Delta(G)$, $m(G)$, $n(G)$ and $n_i(G)$, respectively. Then

$$\left.\begin{aligned} n &= n_1+n_2+\cdots+n_\Delta, \\ 2m &= n_1+2n_2+\cdots+\Delta n_\Delta. \end{aligned}\right\} \quad (3)$$

Noting that $c(G)\geqslant n_1$, from Lemma 1.1 we have

$$m\leqslant\frac{gn-n_1}{g-2}. \quad (4)$$

And the function $f(g)=\frac{gn-n_1}{g-2}$ is descending on $[4,+\infty)$. Substituting Eq. (3) into Eq. (4) yields

$$2g\sum_{i=1}^{\Delta}n_i-2n_1\geqslant(g-2)\sum_{i=1}^{\Delta}in_i.$$

Thus,

$$gn_1+4n_2+(6-g)n_3\geqslant\sum_{i=4}^{\Delta}(g(i-2)-2i)n_i. \quad (5)$$

Let $\tau_i=n_i+n_{i+1}+\cdots+n_\Delta$ for $i=1,2,\cdots,\Delta$.

Lemma 1.2[1,8] If G is a nontrivial graph, then $b(G)\leqslant d_G(u)+d_G(v)-1$ for any two distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G.

2 Bondage number of toroidal graphs

In this section, we present our main results.

Theorem 2.1 Let G be a connected toroidal graph. If G is not 4-regular and $g(G)\geqslant 4$, then $b(G)\leqslant 6$.

Proof By Lemma 1.2, we only need to show that $d_G(u)+d_G(v)\leqslant 7$ for some pair of distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G. Suppose to the contrary that $d_G(u)+d_G(v)\geqslant 8$ for any two distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G with $g(G)\geqslant 4$. Then $d_G(v)\geqslant 7$ if $d_G(u)=1$, $d_G(v)\geqslant 6$ if $d_G(u)=2$ and $d_G(v)\geqslant 5$ if $d_G(u)=3$. Thus,

$$\left.\begin{aligned} \tau_5 &\geqslant n_1+2n_2+3n_3, \\ \tau_6 &\geqslant n_1+2n_2, \\ \tau_7 &\geqslant n_1. \end{aligned}\right\} \quad (6)$$

Substituting $g=4$ and Eq. (6) into Eq. (5) yields

$$2n_1+2n_2+n_3\geqslant n_5+2n_6+3n_7+\sum_{8}^{\Delta}(i-4)n_i=$$

$$\tau_5+\tau_6+\tau_7+\sum_{8}^{\Delta}(i-7)n_i\geqslant$$

$$3n_1+4n_2+3n_3+\sum_{8}^{\Delta}(i-7)n_i,$$

that is,

$$0\geqslant n_1+2n_2+2n_3+\sum_{8}^{\Delta}(i-7)n_i.$$

This inequality holds if and only if $n_i=0$ $(i\neq 4)$, a contradiction, and so the theorem follows. □

Theorem 2.2 Let G be a connected toroidal graph. If $g(G)\geqslant 5$, then $b(G)\leqslant 5$.

Proof By Lemma 1.2, we only need to show that $d_G(u)+d_G(v)\leqslant 6$ for some pair of distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G. Suppose to the contrary that $d_G(u)+d_G(v)\geqslant 7$ for any two distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G with $g(G)\geqslant 5$. Then $d_G(v)\geqslant 6$ if $d_G(u)=1$, $d_G(v)\geqslant 5$ if $d_G(u)=2$ and $d_G(v)\geqslant 4$ if $d_G(u)=3$. Thus,

$$\left.\begin{aligned} \tau_4 &\geqslant n_1+2n_2+3n_3, \\ \tau_5 &\geqslant n_1+2n_2, \\ \tau_6 &\geqslant n_1. \end{aligned}\right\} \quad (7)$$

Substituting $g=5$ and Eq. (7) into Eq. (5) yields

$$5n_1+4n_2+3n_3\geqslant$$

$$2n_4+5n_5+8n_6+\sum_{7}^{\Delta}(3i-10)n_i=$$

$$2\tau_4+3\tau_5+3\tau_6+\sum_{7}^{\Delta}(3i-18)n_i\geqslant$$

$$8n_1+10n_2+6n_3+\sum_{7}^{\Delta}(3i-18)n_i,$$

that is,

$$0\geqslant 3n_1+6n_2+5n_3+\sum_{7}^{\Delta}(3i-18)n_i,$$

which is impossible, a contradiction, and so the theorem follows. □

Theorem 2.3 Let G be a connected toroidal graph. If $g(G)\geqslant 6$ and G is not 3-regular, then $b(G)\leqslant 4$.

Proof By Lemma 1.2, we only need to show that $d_G(u)+d_G(v)\leqslant 5$ for some pair of distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G. Suppose to the contrary that $d_G(u)+d_G(v)\geqslant 6$ for any two distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G with $g(G)\geqslant 6$. Then $d_G(v)\geqslant 5$ if $d_G(u)=1$ and $d_G(v)\geqslant 4$ if $d_G(u)=2$. Thus,

$$\tau_4 \geqslant n_1 + 2n_2,\ \tau_5 \geqslant n_1. \tag{8}$$

Substituting $g=6$ and Eq. (8) into Eq. (5) yields

$$3n_1 + 2n_2 \geqslant 2n_4 + 4n_5 + \sum_6^{\Delta}(2i-6)n_i =$$

$$2\tau_4 + 2\tau_5 + \sum_6^{\Delta}(2i-10)n_i \geqslant$$

$$4n_1 + 4n_2 + \sum_6^{\Delta}(2i-10)n_i,$$

that is,

$$0 \geqslant n_1 + 2n_2 + \sum_6^{\Delta}(2i-10)n_i.$$

This inequality holds if and only if $n_i=0(i\neq 3)$, a contradiction to the hypothesis that G is not a 3-regular graph. The theorem follows. □

Theorem 2.4 Let G be a connected toroidal graph. If $g(G)\geqslant 8$, then $b(G)\leqslant 3$.

Proof By Lemma 1.2, we only need to show that $d_G(u)+d_G(v)\leqslant 4$ for some pair of distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G. Suppose to the contrary that $d_G(u)+d_G(v)\geqslant 5$ for any two distinct vertices u and v with $d_G(u,v)\leqslant 2$ in G with $g(G)\geqslant 8$. Then $d_G(v)\geqslant 4$ if $d_G(u)=1$ and $d_G(v)\geqslant 3$ if $d_G(u)=2$. Thus,

$$\tau_3 \geqslant n_1 + 2n_2,\ \tau_4 \geqslant n_1. \tag{9}$$

Substituting $g=8$ and Eq. (9) into Eq. (5) yields

$$4n_1 + 2n_2 \geqslant n_3 + 4n_4 + \sum_5^{\Delta}(3i-8)n_i =$$

$$\tau_3 + 3\tau_4 + \sum_5^{\Delta}(3i-12)n_i \geqslant$$

$$4n_1 + 2n_2 + \sum_5^{\Delta}(3i-12)n_i,$$

that is,

$$0 \geqslant \sum_5^{\Delta}(3i-12)n_i, \tag{10}$$

which is impossible, a contradiction, and so the theorem follows. □

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