

常用矢量波函数的坐标变换关系*

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摘要 本文导出标准直角、圆柱和圆球矢量波函数的坐标变换关系,为实际使用提供了方便

关键词 矢量波函数;标量波函数;加法定理;坐标变换

1. 引言

为了自动、快速、准确地获取更多的电磁信息,广泛地使用了微波扫描技术.微波遥感^[1]、成像^[2]、多体多次散射^[3,4]、逆散射^[5]和天线近区场测量^[6]中,经常人为地实现源点或场点以某一确定形式有规律运动.处理这一类问题,进行坐标变换并使用矢量波函数的变换关系,将会给问题求解带来很多方便.

2. 标准直角矢量波函数的变换关系

如图 1 所示,直角坐标系 $o'-x'y'z'$ 是由 $o-xyz$ 经过平移(图 2(a))和旋转(图 2(b))构成的.坐标平移后,两个系统的关系是

$$\left. \begin{aligned} x &= x'' \\ y &= y'' \\ z &= z'' \\ \vec{R} &= \vec{R}_0 + \vec{R}'' \end{aligned} \right\} \quad (1)$$

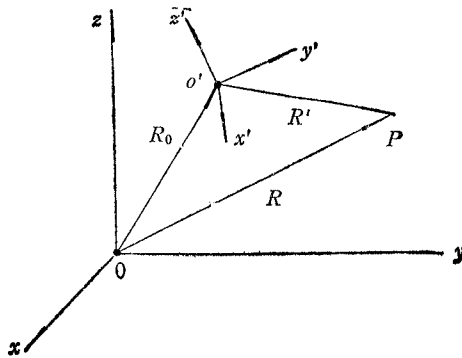


图 1

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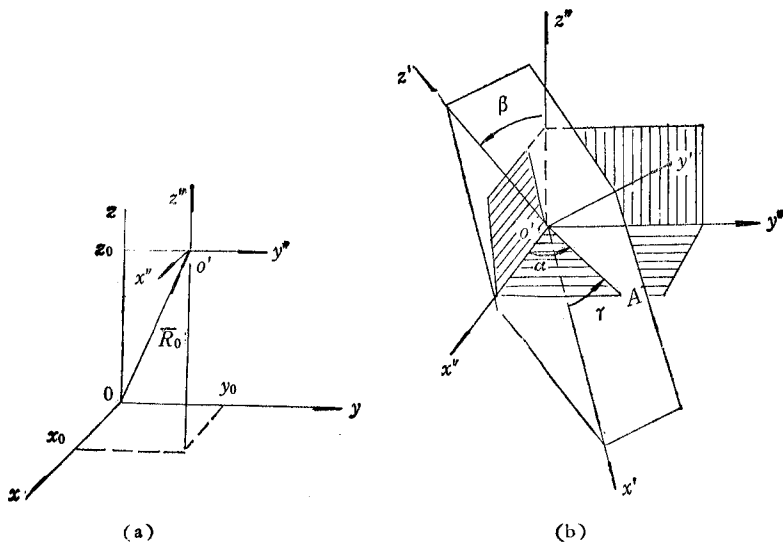


图 2

任取 \hat{z}'' (或 \hat{x}'' , \hat{y}'') 为领示矢量, $\psi(\vec{k}, \vec{R}'') = e^{i\vec{k} \cdot \vec{R}''}$ 为生成函数, 可以得到坐标系统平移后标准直角矢量波函数^[7]的变换关系是

$$\begin{bmatrix} \bar{L}^{(z'')}(\vec{k}, \vec{R}'') \\ \bar{M}^{(z'')}(\vec{k}, \vec{R}'') \\ \bar{N}^{(z'')}(\vec{k}, \vec{R}'') \end{bmatrix} = e^{-i\vec{k} \cdot \vec{R}_0} \begin{bmatrix} \bar{L}^{(z)}(\vec{k}, \vec{R}) \\ \bar{M}^{(z)}(\vec{k}, \vec{R}) \\ \bar{N}^{(z)}(\vec{k}, \vec{R}) \end{bmatrix} \quad (2)$$

直角坐标系旋转后, 假设 oA 是面 $o'x''y''$ 与面 $o'x'y'$ 的交线, 旋转 Euler 角定义为自转角 α , 章动角 β 和进动角 γ , 则两个系统的坐标关系是

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = [c_{ij}] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \gamma - \cos \beta \sin \alpha \sin \gamma & -\cos \alpha \sin \gamma - \cos \beta \sin \alpha \cos \gamma & \sin \beta \sin \alpha \\ \sin \alpha \cos \gamma + \cos \beta \cos \alpha \sin \gamma & -\sin \alpha \sin \gamma + \cos \beta \cos \alpha \cos \gamma & -\sin \beta \cos \alpha \\ \sin \beta \sin \gamma & \sin \beta \cos \gamma & \cos \beta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad (3a)$$

$$\begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = [c_{ij}] \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}$$

和

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [d_{ij}] \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \quad (3b)$$

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = [d_{ij}] \begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix}$$

其中 $[d_{ij}] = [c_{ij}]^{-1}$, 选 z' 为领示矢量, $\phi(\bar{k}, \bar{R}') = e^{i\bar{k} \cdot \bar{R}'}$ 为生成函数, 可得坐标系统旋转后标准直角矢量波函数的变换关系是

$$\begin{aligned} \bar{L}^{(z')}(\bar{k}, \bar{R}') &= \bar{L}^{(z'')}(\bar{k}, \bar{R}'') \\ \bar{M}^{(z')}(\bar{k}, \bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2 + k_y^2} - \frac{d_{23}k_y h}{k_x^2 + k_y^2} + d_{33} \right) \bar{M}^{(z'')}(\bar{k}, \bar{R}'') \\ &\quad + i \left(\frac{-d_{31}k_y k}{k_x^2 + k_y^2} + \frac{d_{32}k_x k}{k_x^2 + k_y^2} \right) \bar{N}^{(z'')}(\bar{k}, \bar{R}'') \\ \bar{N}^{(z')}(\bar{k}, \bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2 + k_y^2} - \frac{d_{32}k_y h}{k_x^2 + k_y^2} + d_{33} \right) \bar{N}^{(z'')}(\bar{k}, \bar{R}'') \\ &\quad + i \left(-\frac{d_{31}k_y k}{k_x^2 + k_y^2} + \frac{d_{32}k_x k}{k_x^2 + k_y^2} \right) \bar{M}^{(z'')}(\bar{k}, \bar{R}'') \end{aligned} \quad (4)$$

式中 $k^2 = k_x^2 + k_y^2 + h^2$. 综合(2)和(4)式最后即导出直角坐标系统经过平移和旋转以后, 标准直角矢量波函数的变换关系是

$$\begin{aligned} \bar{L}^{(z')}(\bar{k}, \bar{R}') &= e^{-i\bar{k} \cdot \bar{R}_0} \bar{L}^{(z)}(\bar{k}, \bar{R}) \\ \bar{M}^{(z')}(\bar{k}, \bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2 + k_y^2} - \frac{d_{32}k_y h}{k_x^2 + k_y^2} + d_{33} \right) e^{-i\bar{k} \cdot \bar{R}_0} \bar{M}^{(z)}(\bar{k}, \bar{R}) \\ &\quad + i \left(-\frac{d_{31}k_y k}{k_x^2 + k_y^2} + \frac{d_{32}k_x k}{k_x^2 + k_y^2} \right) e^{-i\bar{k} \cdot \bar{R}_0} \bar{N}^{(z)}(\bar{k}, \bar{R}) \\ \bar{N}^{(z')}(\bar{k}, \bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2 + k_y^2} - \frac{d_{32}k_y h}{k_x^2 + k_y^2} + d_{33} \right) e^{-i\bar{k} \cdot \bar{R}_0} \bar{N}^{(z)}(\bar{k}, \bar{R}) \\ &\quad + i \left(-\frac{d_{31}k_y k}{k_x^2 + k_y^2} + \frac{d_{32}k_x k}{k_x^2 + k_y^2} \right) e^{-i\bar{k} \cdot \bar{R}_0} \bar{M}^{(z)}(\bar{k}, \bar{R}) \end{aligned} \quad (5)$$

3. 标准圆柱矢量波函数的变换关系

如图3所示, 圆柱坐标系统 $o-r\varphi z$ 变换到 $o'-r'\varphi'z'$ 有下列关系, 即

$$\begin{aligned} r &= r_0 \cos(\varphi - \varphi_0) + r' \cos(\varphi' - \varphi) \\ \varphi &= \varphi' - \eta \\ z &= z_0 + z' \end{aligned} \quad (6)$$

由此得到柱谱函数 $J_n(\lambda r') e^{in\varphi'}$ 的表示式是^[8]

$$\begin{aligned} J_n(\lambda r') e^{in\varphi'} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda r' \cos\theta + im(\theta + \varphi' - \frac{\pi}{2})} d\theta \\ &= \sum_{m=-\infty}^{\infty} J_m(\lambda r_0) J_{n+m}(\lambda r) e^{in\varphi + im(\varphi - \varphi_0)} \end{aligned} \quad (7a)$$

或

$$J_n(\lambda r') e^{in\eta} = \sum_{m=-\infty}^{\infty} J_m(\lambda r_0) J_{n+m}(\lambda r) e^{im(\varphi - \varphi_0)} \quad (7b)$$

同理, 当 $|r| > |r_0 \cos(\varphi - \varphi_0)|$ 时, 有

$$H_n^{(1)}(\lambda r') e^{in\eta} = \sum_{m=-\infty}^{\infty} J_m(\lambda r_0) H_{n+m}^{(1)}(\lambda r) e^{im(\varphi - \varphi_0)} \quad (8a)$$

当 $|r| < |r_0 \cos(\varphi - \varphi_0)|$ 时, 有

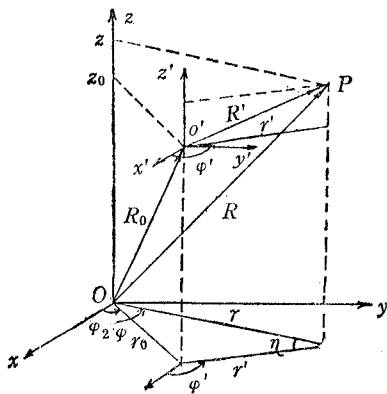


图 3

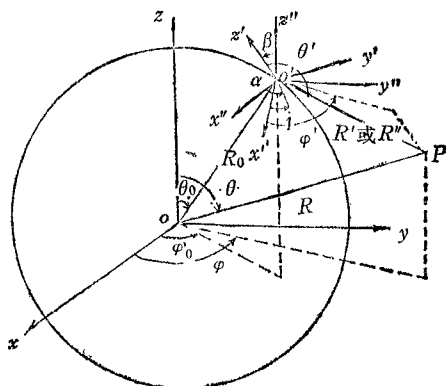


图 4

$$H_n^{(1)}(\lambda r') e^{in\eta} = \sum_{m=-\infty}^{\infty} H_m^{(1)}(\lambda r_0) J_{n+m}(\lambda r) e^{im(\varphi-\varphi_0)} \quad (8b)$$

利用标量柱谐函数加法定理(7)和(8)式,可以导得标准圆柱矢量波函数的坐标关系是

$$\begin{bmatrix} \bar{L}_n^{(1)}(\bar{k}, \bar{R}') \\ \bar{M}_n^{(1)}(\bar{k}, \bar{R}') \\ \bar{N}_n^{(1)}(\bar{k}, \bar{R}') \end{bmatrix} = \sum_{m=-\infty}^{\infty} J_m(\lambda r_0) e^{-i(m\varphi_0+h z_0)} \begin{bmatrix} \bar{L}_{n+m}^{(1)}(\bar{k}, \bar{R}) \\ \bar{M}_{n+m}^{(1)}(\bar{k}, \bar{R}) \\ \bar{N}_{n+m}^{(1)}(\bar{k}, \bar{R}) \end{bmatrix} \quad (9a)$$

当 $|r| > |r_0 \cos(\varphi - \varphi_0)|$ 时

$$\begin{bmatrix} \bar{L}_n^{(3)}(\bar{k}, \bar{R}') \\ \bar{M}_n^{(3)}(\bar{k}, \bar{R}') \\ \bar{N}_n^{(3)}(\bar{k}, \bar{R}') \end{bmatrix} = \sum_{m=-\infty}^{\infty} J_m(\lambda r_0) e^{-i(m\varphi_0+h z_0)} \begin{bmatrix} \bar{L}_{n+m}^{(3)}(\bar{k}, \bar{R}) \\ \bar{M}_{n+m}^{(3)}(\bar{k}, \bar{R}) \\ \bar{N}_{n+m}^{(3)}(\bar{k}, \bar{R}) \end{bmatrix} \quad (9b)$$

当 $|r| < |r_0 \cos(\varphi - \varphi_0)|$ 时

$$\begin{bmatrix} \bar{L}_n^{(3)}(\bar{k}, \bar{R}') \\ \bar{M}_n^{(3)}(\bar{k}, \bar{R}') \\ \bar{N}_n^{(3)}(\bar{k}, \bar{R}') \end{bmatrix} = \sum_{m=-\infty}^{\infty} H_m^{(1)}(\lambda r_0) e^{-i(m\varphi_0+h z_0)} \begin{bmatrix} \bar{L}_{n+m}^{(1)}(\bar{k}, \bar{R}) \\ \bar{M}_{n+m}^{(1)}(\bar{k}, \bar{R}) \\ \bar{N}_{n+m}^{(1)}(\bar{k}, \bar{R}) \end{bmatrix} \quad (9c)$$

4. 标准圆球矢量波函数的变换关系

如图 4 所示,圆球坐标系 $o-R\theta\varphi$ 经过平移 \bar{R}_0 后成 $o'-R''\theta''\varphi''$,再旋转 Euler 角,成系统 $o'-R'\theta'\varphi'$. S. Stein^[9] 曾导出平移后圆球标量波函数的加法定理是

$$z_n(kR) P_n^m(\cos\theta) e^{im\varphi} = \begin{cases} \sum_{p=0}^{\infty} \sum_{\mu=-p}^p \sum_{\nu} (-1)^\mu i^{\nu+p-n} (2\nu+1) a(m, n | -\mu\varphi | p) \\ \quad \times j_\nu(kR'') z_p(kR_0) P_\nu^\mu(\cos\theta'') P_p^{m-\mu}(\cos\theta_0) \\ \quad \times e^{i(m-\mu)\varphi_0} e^{i\mu\varphi''}, & R'' \leq R_0 \\ \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{p} (-1)^\mu i^{\nu+p-n} (2\nu+1) a(m, n | -\mu, \nu | p) \\ \quad \times j_\nu(kR_0) z_p(kR'') P_\nu^\mu(\cos\theta_0) P_p^{m-\mu}(\cos\theta'') \\ \quad \times e^{i(m-\mu)\varphi_0} e^{i\mu\varphi''}, & R'' \geq R_0 \end{cases} \quad (10)$$

其中下标 $p = \nu + n, \nu + n - 2, \nu + n - 4, \dots$, 但不低于 $|\nu - n|$, 即 $\nu + n \geq p \geq |\nu - n|$. 系数 $a(m, n | -\mu, \nu | p)$ 由展开式

$$P_n^m(\cos\theta)P_\nu^\mu(\cos\theta) = \sum_p a(m, n | \mu, \nu | p) P_p^{m+\mu}(\cos\theta) \quad (11)$$

定义,可以写成

$$a(m, n | \mu, \nu | p) = (-1)^{m+\mu} (2p+1) \left[\frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m+\mu)!} \right]^{1/2} \\ \times \begin{bmatrix} n & \nu & p \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n & \nu & p \\ m & \mu & -m-\mu \end{bmatrix} \quad (12)$$

式中 $\begin{bmatrix} i_1 & i_2 & i_3 \\ m_1 & m_2 & m_3 \end{bmatrix}$ 是 Wigner 3-j 符号^[10]. 经过复杂地运算^[11], 无论 $R'' \leq R_0$ 或 $R'' \geq R_0$, 式(10)可以改写成

$$z_n(kR)P_n^m(\cos\theta)e^{im\varphi} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A(\mu, \nu) z_\nu(kR'')P_\nu^\mu(\cos\theta'')e^{i\mu\varphi''} \quad (13)$$

其中

$$A(\mu, \nu) = (-1)^\mu i^{\nu-n} (2\nu+1) \sum_p i^p a(m, n | -\mu, \nu | p) z_p(kR_0)P_p^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0} \quad (14)$$

由此, 坐标系统平移后, 标准圆球矢量波函数的变换关系是

$$\bar{K}_{mn}(\bar{k}, \bar{R}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A(\mu, \nu) \bar{L}_{\mu\nu}(\bar{k}, \bar{R}'') \\ \bar{M}_{mn}(\bar{k}, \bar{R}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} [A_{\mu\nu}^{mn} \bar{M}_{\mu\nu}(\bar{k}, \bar{R}'') + B_{\mu\nu}^{mn} \bar{N}_{\mu\nu}(\bar{k}, \bar{R}'')] \\ \bar{N}_{mn}(\bar{k}, \bar{R}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} [A_{\mu\nu}^{mn} \bar{N}_{\mu\nu}(\bar{k}, \bar{R}'') + B_{\mu\nu}^{mn} \bar{M}_{\mu\nu}(\bar{k}, \bar{R}'')] \quad (15)$$

其中^[12]

$$A_{\mu\nu}^{mn} = (-1)^\mu \sum_p a(m, n | -\mu, \nu | p) a(n, \nu, p) z_p(kR_0)P_p^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0} \\ B_{\mu\nu}^{mn} = (-1)^\mu \sum_p a(m, n | -\mu, \nu | p, p-1) b(n, \nu, p) z_p(kR_0)P_p^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0} \\ a(n, \nu, p) = i^{\nu+p-n} [2\nu(\nu+1)(2\nu+1) + (\nu+1)(n-\nu+p+1)(n+\nu-p)] \\ b(n, \nu, p) = i^{\nu+p-n} [(n+\nu+p+1)(\nu-n+p)(n-\nu+p)(n+\nu-p+1)]^{1/2} \frac{2\nu+1}{2\nu(\nu+1)} \quad (16)$$

A. R. Edmonds^[10] 用转动群表示证明了标量圆球波函数的加法定理是

$$z_n(kR'')P_n^m(\cos\theta'')e^{im\varphi''} = \sum_{\mu=-n}^n \beta(m, \mu, n) z_n(kR')P_n^\mu(\cos\theta')e^{i\mu\varphi'} \quad (17)$$

式中

$$\beta(m, \mu, n) = (-1)^{\mu+m} \left[\frac{(n+m)!(n-\mu)!}{(n-m)!(n+\mu)!} \right]^{1/2} D_{m\mu}^{(n)}(\alpha, \beta, \gamma) \\ D_{m\mu}^{(n)}(\alpha, \beta, \gamma) = e^{i\mu\alpha} \alpha_{m\mu}^{(n)}(\beta) e^{im\gamma} \\ d_{m\mu}^{(n)}(\beta) = \left[\frac{(n+\mu)!(n-\mu)!}{(n+m)!(n-m)!} \right]^{1/2} \sum_{\sigma} \begin{pmatrix} n+m \\ n-\mu-\sigma \end{pmatrix} \begin{pmatrix} n-m \\ \sigma \end{pmatrix} (-1)^{n-\mu-\sigma} \\ \times \left(\cos \frac{\beta}{2} \right)^{2\sigma+\mu+m} \left(\sin \frac{\beta}{2} \right)^{2n-2\sigma-\mu-m} \quad (18)$$

由此得到坐标旋转后标准圆球矢量波函数的变换关系是

$$\begin{aligned}\bar{L}_{mn}(\bar{k}, \bar{R}') &= \sum_{\mu=-n}^n \beta(m, \mu, n) \bar{L}_{\mu n}(\bar{k}, \bar{R}') \\ \bar{M}_{mn}(\bar{k}, \bar{R}') &= \sum_{\mu=-n}^n \beta(m, \mu, n) \bar{M}_{\mu n}(\bar{k}, \bar{R}') \\ \bar{N}_{mn}(\bar{k}, \bar{R}') &= \sum_{\mu=-n}^n \beta(m, \mu, n) \bar{N}_{\mu n}(\bar{k}, \bar{R}')\end{aligned}\quad (19)$$

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TRANSFORMATION RELATIONS OF NORMAL VECTOR WAVE FUNCTIONS

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Abstract The transformation relations of normal rectangular, cylindrical and spherical vector wave functions are derived. With these relations, it will be convenient for engineering.

Key words Vector wave function; Scalar wave function; Addition theorem; Coordinate transformation