INTEGRABLE GL(2) GEOMETRY AND HYDRODYNAMIC PARTIAL DIFFERENTIAL EQUATIONS

ABRAHAM D. SMITH

ABSTRACT. This article is a local analysis of integrable GL(2)-structures of degree 4. A GL(2)-structure of degree n corresponds to a distribution of rational normal cones over a manifold of dimension $n+1$. Integrability corresponds to the existence of many submanifolds that are spanned by lines in the cones. These GL(2)-structures are important because they naturally arise from a certain family of second-order hyperbolic PDEs in three variables that are integrable via hydrodynamic reduction. Familiar examples include the wave equation, the first flow of the dKP equation, and the Boyer–Finley equation.

The main results are a structure theorem for integrable GL(2)-structures, a classification for connected integrable GL(2)-structures, and an equivalence between local integrable GL(2)-structures and Hessian hydrodynamic hyperbolic PDEs in three variables.

This yields natural geometric characterizations of the wave equation, the first flow of the dKP equation, and several others. It also provides an intrinsic, coordinate-free infrastructure to describe a large class of hydrodynamic integrable systems in three variables.

CONTENTS

Introduction

A fundamental problem in analysis is to understand why some differential equations (particularly hyperbolic equations arising from wave-like equations or from differential geometry) are *integrable*. This problem is compounded by the many competing notions of integrability [32]. Informally, a PDE is called integrable if it can be solved thanks to the existence of an infinite hierarchy of conservation laws or of a family of invariant foliations along characteristics. Integrable PDEs of all types are well-studied in two $(1 + 1)$ independent variables; however, examples are increasingly rare and poorly understood in higher dimensions.

A particularly interesting class of integrable PDEs are those that can be integrated by decomposing the equation to a set of coupled first-order equations that represent traveling waves [28]. This technique is called "integration by hydrodynamic reduction," and it has been extensively studied for various special classes of second-order PDEs in three or more independent variables [29, 30, 13, 6, 14, 12, 11].

Consider a second-order hyperbolic equation of the form $F(\xi^i, u, u_i, u_{ij}) = 0$ on scalar functions $u(\xi^1, \xi^2, \xi^3)$. In the method of hydrodynamic reduction, one hopes to integrate the PDE $F = 0$ by stipulating that the solution function u (and its derivatives) may be written as $u(R^1, \dots, R^k)$ for an *a priori* unknown number k of functions $R^1(\xi), \dots, R^k(\xi)$ whose derivatives admit the "commuting" relations

$$(1) \qquad \frac{\partial}{\partial \xi^2} R^i = \rho_2^i(R) \frac{\partial}{\partial \xi^1} R^i, \qquad \frac{\partial}{\partial \xi^3} R^i = \rho_3^i(R) \frac{\partial}{\partial \xi^1} R^i$$

which also imply

$$(2) \qquad \frac{1}{\rho_2^i - \rho_2^j} \frac{\partial \rho_2^i}{\partial R^j} = \frac{1}{\rho_3^i - \rho_3^j} \frac{\partial \rho_3^i}{\partial R^j}, \quad \forall i \neq j.$$

Equations (1) and (2) can be solved by other methods [28]. The reduction of $F = 0$ to Equation (2) is called a k-parameter hydrodynamic reduction. The equation $F = 0$ is called integrable via k-parameter hydrodynamic reduction if it admits an infinite family of k-parameter hydrodynamic reductions and this family is itself parametrized by k functions of one variable.. The functions $R^i(\xi^1, \xi^2, \xi^3)$ are called Riemann invariants, and their level sets define the foliations along characteristics in the informal definition above. See the works cited above, particularly [14], [12] and [11], for examples and detailed exposition of this technique.

Many examples of these so-called hydrodynamic equations are known, such as the wave equation $u_{22} = u_{13}$, the dispersionless Kadomtsev–Petviashvili (dKP) equation $u_{22} = (u_1 - uu_3)_3$, the first flow of the dKP equation $u_{22} = u_{13} - \frac{1}{2}(u_{33})^2$, the Boyer–Finley equation $u_{22} + u_{33} = e^{u_{11}}$, and some well-known integrable hierarchies. Tests for hydrodynamic integrability also exist, but so far there is no intrinsic, coordinate-free theory that classifies and generates all of these equations.

For the case of second-order hydrodynamic equations in three variables that involve only the Hessian of the solution function, recent work by Ferapontov, *et al.*, reveals a connection between the integrability of the PDEs, the symplectic contact symmetries of the equations, and the equations' underlying degree-4 GL(2)-structures [11]. Integrable GL(2)-structures are not well-known in the literature; they arise in the search for exotic affine holonomies in differential geometry [4] and a similar conformal geometry appears useful in the equivalence problem for ODEs [22], but the application to PDEs is very recent [11, 24].

The purpose of this article is to fully analyze the local geometry of integrable GL(2)-structures of degree 4, with an eye towards understanding PDEs that are integrable via hydrodynamic reduction. The main result is a natural classification of local integrable GL(2)-structures and the associated PDEs. This is achieved using the techniques of exterior differential systems and geometry *à la Cartan*.

Here is a summary of the contents of this article, including abbreviated and *non-technical* versions of the main theorems:

Section 1 defines GL(2)-structures and their integrability. It also introduces the motivating observation by Ferapontov *et al.* that hyperbolic PDEs in three variables that are integrable by means of hydrodynamic reductions lead to integrable GL(2)-structures.

Section 2 contains a description of the necessary $SL(2)$ representation theory and an explicit solution to the equivalence problem for GL(2)-structures. Corollary 2.2 defines a global canonical connection for GL(2)-structures that has essential torsion in $\mathbb{R}^3$, $\mathbb{R}^7$, $\mathbb{R}^9$, and $\mathbb{R}^{11}$.

In Section 3, the exterior differential system establishing 2-integrability of a GL(2)-structure is analyzed. The most important object in this article, the $\mathbb{R}^9$-valued torsion, is first emphasized in Theorem 3.2.

Theorem 3.2. *A GL(2)-structure is 2-integrable if and only if its torsion takes values only in $\mathbb{R}^9$.*

Section 4 provides Theorem 4.3, a complete local description of GL(2)-structures that are both 2-integrable and 3-integrable. The immediate consequence of this theorem is that the value of the torsion at a single point completely determines the local structure of a 2,3-integrable GL(2)-structure.

Theorem 4.3. *A GL(2)-structure is 2,3-integrable if and only if its structure equations are of the form*

$$
\begin{aligned}
\mathrm{d}\omega &= -\theta \wedge \omega + T(\omega \wedge \omega) \\
\mathrm{d}\theta &= -\theta \wedge \theta + T^2(\omega \wedge \omega) \\
\mathrm{d}T &= J(T)(\omega, \theta)
\end{aligned}
$$

(3)

for an essential torsion T in $\mathbb{R}^9$ and a 9×9 matrix $J(T)$.

Section 5 presents a classification of connected 2,3-integrable GL(2)-structures. This classification is provided by a natural stratification of $\mathbb{R}^9$ into equivalence classes that are defined by the matrix $J(T)$. These equivalence classes are explicitly identified in Theorem 5.3.

Theorem 5.3. *Every connected 2,3-integrable GL(2)-structure belongs to one of 55 equivalence classes, which are given by a stratification of $\mathbb{R}^9$ into the root factorization types of real binary octic polynomials.*

In Section 6, the structure equations from Section 4 are applied to reproduce the integrable PDEs, yielding Theorem 6.3.

Theorem 6.3. *Every local 2,3-integrable GL(2)-structure arises from a Hessian hydrodynamic hyperbolic PDE.*

Therefore, the classification in Theorem 5.3 is also a classification of Hessian hydrodynamic hyperbolic PDEs. Some previously known examples of such PDEs

are identified in the new classification, and several new integrable PDEs are constructed.

This work would have been impossible to complete without the uncanny speed and accuracy of computer algebra software, and there is no sense in retyping dozens of pages of formulas and identities. Thus, a reader looking for gory details may be disappointed by this article (in which case I redirect the reader to [24]). However, most proofs rely only on basic concepts from differential geometry, linear algebra, and representation theory, and a reader comfortable with the standard methods of exterior differential systems (and their various software implementations) can readily supply the computational details. A MAPLE file containing the structure equations appearing in Theorem 4.3 and the Maurer–Cartan form appearing in Theorem 6.3 will remain available at [23] as long as possible. This file should also be bundled with this article on arXiv.org. The content of this file allows rapid reproduction of all results following Theorem 4.3.

Please note that the terms "hyperbolic" and "integrable" are heavily used in slightly different contexts throughout this article. "Hyperbolic" refers to the maximal intersection of projective hyperplanes with projective varieties, to PDEs in both two and three independent variables with appropriate leading symbol, and to exterior differential systems with appropriate tableaux. The term "integrable" refers both to PDEs that admit infinitely many exact solutions and to GL(2)-structures that admit many secant submanifolds (which one might also call "hyperbolic"). Of course, all of these concepts are intimately related, so the standard overloading of these definitions is reasonable.

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1. BACKGROUND

Let $\mathcal{V}_n$ denote the vector space of degree n homogeneous polynomials in x and y with real coefficients. Identify $\mathcal{V}_n$ with $\mathbb{R}^{n+1}$ using the terms from the binomial theorem to produce a basis; for example, $\mathcal{V}_4 \to \mathbb{R}^5$ by

$$(4) \quad v_{-4}\, x^4 + v_{-2}\, 4x^3y + v_0\, 6x^2y^2 + v_2\, 4xy^3 + v_4\, y^4 \mapsto (v_{-4}, v_{-2}, v_0, v_2, v_4) \in \mathbb{R}^5.$$

Let M denote a 5-dimensional smooth manifold, and let $\mathcal{F}$ denote the $\mathcal{V}_4$-valued co-frame bundle over M, so the fiber $\mathcal{F}_p$ is the set of all isomorphisms $\mathbf{T}_pM \to \mathcal{V}_4$. The co-frame bundle $\mathcal{F}$ is a principal right $GL(\mathcal{V}_4)$ bundle.

Recall that $\mathcal{V}_n$ is the unique irreducible representation of $\mathfrak{sl}(2)$ of dimension $n+1$, and its action is generated by

$$(5) \qquad X = y\frac{\partial}{\partial x}, \ Y = -x\frac{\partial}{\partial y}, \ \text{and} \ H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Definition 1.1. *A **GL(2)-structure** $\mathcal{B} \to M$ is a reduction of $\mathcal{F}$ with fiber group $GL(2) \subset GL(\mathcal{V}_n)$ infinitesimally generated by X, Y, H, and the scaling action I.*

Generally, a GL(2)-structure is said to have "degree n" if the base space M has dimension $n+1$. The degree 3 case is thoroughly studied in [4], and various observations are made for all n in [24]. This article considers only the degree 4 case. The $PGL(2)$ action generated by X, Y, and H is the symmetry group of a rational

normal curve in $\mathbb{P}(\mathcal{V}_4)$. A rational normal curve is a curve of degree n in $\mathbb{P}(\mathbb{R}^{n+1})$, all of which are $PGL(n+1)$-equivalent to $\{[g^n : g^{n-1}h : \cdots : gh^{n-1} : g^n]\}$ [18]. The de-projectivization of the rational normal curve is the rational normal cone, which has a $GL(2)$ symmetry group and is usually described as $\{(g^n, g^{n-1}h, \ldots, gh^{n-1}, h^n)\} \subset \mathbb{R}^{n+1}$. Thus, the geometric content of Definition 1.1 is contained in the following lemma.

Lemma 1.1. *A GL(2)-structure $\mathcal{B} \to M^5$ is equivalent to a distribution of rational normal cones $\mathbf{C} \subset \mathbf{T}M$. For any $b \in \mathcal{B}_p$, $b(\mathbf{C}_p) = \{(gx + hy)^4 : g, h \in \mathbb{R}\} \subset \mathcal{V}_4$.*

Definition 1.2 (Integrability)**.** *Let $\mathcal{B} \to M$ be a GL(2)-structure.*

*A k-dimensional linear subspace $E \subset \mathbf{T}_pM$ is **k-secant** if $E \cap \mathbf{C}_p$ is k distinct lines. Equivalently, E is k-secant if E is spanned by vectors in $\mathbf{C}_p$.*

*A k-dimensional submanifold $N \subset M$ is **k-secant** if $\mathbf{T}_pN$ is a k-secant subspace of $\mathbf{T}_pM$ for every $p \in N$.*

*A GL(2)-structure is **k-integrable** if, for every k-secant linear subspace $E \in Gr_k(\mathbf{T}M)$, there exists a k-secant submanifold N with $E = \mathbf{T}_pN$.*

Locally, 1-integrability is uninteresting, as it is simply the existence of a local flow for a vector field. Since M has dimension 5, being 4-secant is an open condition on $E \in Gr_4(\mathbf{T}M)$; however, the condition is closed for $k=2$ (bi-secant) and $k=3$ (tri-secant). A GL(2)-structure that is both 2-integrable and 3-integrable is called "2,3-integrable," or simply "integrable."

Notice that all the definitions are projectively invariant. Indeed, one could equivalently study the principal right $PGL(2)$ bundle over M defined by the symmetries of rational normal curves in $\mathbb{P}\mathbf{T}M$. By considering the de-projectivized $GL(2)$ geometry, one must occasionally observe an additional $\mathbb{Z}/2\mathbb{Z}$ symmetry by $\pm I$. For the purposes of this article, it is easier to proceed with the affine geometry and deal with this symmetry when it arises. In fact, GL(2)-structures of degree 4 induce conformal geometries with signature $(2, 3)$. The essentials of this relationship are mentioned in [4] and [11]; it is examined in great detail in [17]. The simplicity of $SL(2)$ representation theory as compared to the Weyl tensor justifies the omission of conformal geometry from this article.

The geometric content of this article is a complete local description and classification of GL(2)-structures of degree 4 that are 2,3-integrable. Such GL(2)-structures are particularly interesting because of their deep connection to integrable PDEs. The remainder of this section outlines how [11] obtains 2,3-integrable GL(2)-structures from certain hydrodynamic PDEs.

1.1. **Hessian Hydrodynamic Hyperbolic PDEs.**
The space of 1-jets over $\mathbb{R}^3 \to \mathbb{R}$ is $\mathbb{J}^1 = \{(\xi^i, z, p_i)\}$ where $1 \leq i, j \leq 3$. The space of 2-jets is $\mathbb{J}^2 = \{(\xi^j, z, p_i, U_{ij})\}$ where $1 \leq i, j \leq 3$ and $U_{ij} = U_{ji}$. The contact system on $\mathbb{J}^2$ is

$$(6) \qquad \mathcal{J}^2 = \{\mathrm{d}z - p_i\mathrm{d}\xi^i, \ \mathrm{d}p_i - U_{ij}\mathrm{d}\xi^j\}.$$

A second-order PDE on $u : \mathbb{R}^3 \to \mathbb{R}$ is merely the level set $F^{-1}(0)$ of some smooth function $F : \mathbb{J}^2 \to \mathbb{R}$ on which the projection $\mathbb{J}^2 \to \mathbb{J}^1$ is a submersion, and a solution is a function $u : \mathbb{R}^3 \to \mathbb{R}$ whose jet-graph is a subset of $F^{-1}(0)$. Recall that a section of $\mathbb{J}^2$ is the jet-graph of a function if and only if it is integral to the contact system.

For this article, consider only PDEs on $u : \mathbb{R}^3 \to \mathbb{R}$ of the form

$$(7) \qquad F(u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0.$$

Instead of considering the orbit of this PDE under all contact transformations, consider only those contact transformations that preserve the Hessian-only form of Equation (7). To understand these transformations and to fix notation, an alternative system $(\mathbb{J}^\nabla, \sigma)$ is more convenient than the traditional contact system on $\mathbb{J}^2$.

Let $\mathbb{J}^\nabla$ denote gradient space, which is $\mathbb{R}^6$ with coordinates $(\xi^1, \xi^2, \xi^3, p_1, p_2, p_3)$. Gradient space admits the symplectic 2-form,

$$(8) \qquad \sigma = \mathrm{d}p_1 \wedge \mathrm{d}\xi^1 + \mathrm{d}p_2 \wedge \mathrm{d}\xi^2 + \mathrm{d}p_3 \wedge \mathrm{d}\xi^3.$$

In each tangent space $\mathbf{T}_{(\xi,p)}\mathbb{J}^\nabla$, the symplectic 2-form defines a Lagrangian Grassmannian,

$$(9) \qquad \Lambda = \{U \in Gr_3(\mathbb{R}^6) \ : \ \sigma|_U = 0\},$$

and its non-vertical open subset

$$(10) \qquad \Lambda^o = \{U \in \Lambda \ : \ \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3|_U \neq 0\} = \mathrm{Sym}^2(\mathbb{R}^3).$$

The alternative version of the contact condition can now be expressed.

Lemma 1.2. *Let $N \subset \mathbb{J}^\nabla$ be a submanifold of dimension three. Then N is locally the gradient-graph of a function $u : \mathbb{R}^3 \to \mathbb{R}$ such that $u_i = N^*(p_i)$ if and only if $\mathbf{T}_{(\xi,p)}N \in \Lambda^o$ for all $(\xi, p) \in N$.*

A solution to Equation (7) is a function $u : \mathbb{R}^3 \to \mathbb{R}$ whose gradient-graph is everywhere tangent to $F^{-1}(0) \subset \Lambda^o$. Thus, the contact transformations that preserve the Hessian-only form of Equation (7) are elements of the conformal symplectic group,

(11)
$$CSp(3) = \{g \in GL(6, \mathbb{R}) \ : \ \sigma(gv, gw) = \lambda\sigma(v, w) \ \forall v, w\} \text{ (any } \lambda \in \mathbb{R}^*)$$

$$= \left\{ \begin{pmatrix} B & C \\ A & D \end{pmatrix} \ : \ 0 = A^t B - B^t A = D^t C - C^t D, \lambda I_3 = D^t B - C^t A \right\},$$

which naturally forms a bundle, $\Pi : CSp(3) \to \Lambda$. In fact, only the transformations in $CSp(3)^o = \Pi^{-1}(\Lambda^o)$ are permissible, as the condition $\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3 \neq 0$ must be preserved. As above, one could eliminate the conformal factor and consider only $Sp(3)^o$ actions by projectivizing, but there is little utility in doing so.

To be explicit, suppose $U \in \Lambda^o$, so $\mathrm{d}p_i = U_{ij}\mathrm{d}x^j$ and $U_{ij} = U_{ji}$. Then

$$(12) \qquad \begin{pmatrix} \mathrm{d}\hat{x} \\ \mathrm{d}\hat{p} \end{pmatrix} = \begin{pmatrix} B & C \\ A & D \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}p \end{pmatrix}.$$

The assumption that $\mathrm{d}x$ and $\mathrm{d}\hat{x}$ both have maximum rank when pulled-back to N implies that $(B+CU)$ is nonsingular. Moreover, Equation (12) shows that $CSp(3)^o$ acts on Λ^o by

$$(13) \qquad g : U \mapsto g(U) = (A + DU)(B + CU)^{-1}.$$

Because I_6 is in the fiber over $0 \in \Lambda^o$, the projection $\Pi : CSp(3)^o \to \Lambda^o$ can be computed from Equation (13) as

$$(14) \qquad \Pi : g \mapsto g(0) = AB^{-1}.$$

The most important fact about Equation (7) is that $M = F^{-1}(0)$ admits a natural GL(2)-structure whenever the PDE is hyperbolic. To be precise, one must note some real algebraic geometry.

Lemma 1.3. *Let $F : \Lambda^o \to \mathbb{R}$ be a smooth function, and suppose $U \in \Lambda^o$ such that $F(U) = 0$, $\mathrm{d}F_U \neq 0$, and $\ker(\mathrm{d}F_U)$ is hyperbolic as a hyperplane in $\mathbf{T}_U\Lambda^o = \mathrm{Sym}^2(\mathbb{R}^3)$. Then there is an open 5-dimensional submanifold $M \subset \Lambda^o$ defined by $F|_M = 0$ in a neighborhood of U, and M admits a distribution $\mathbf{C}$ of rational normal cones. That is, M admits a GL(2)-structure.*

Proof. Aside from the condition on hyperbolicity of the tangent space, the lemma is simply a statement of the implicit function theorem. Here is an explanation of hyperbolicity and its relation to GL(2)-structures.

The 6-dimensional manifold Λ^o admits an $Sp(3)$-invariant distribution of 3-dimensional 'Veronese' cones defined by $\{P \in \mathbf{T}\Lambda^o : \mathrm{rank}\, P \leq 1\}$. In the complex case, it is easy to check that the intersection of a generic hyperplane with the 3-dimensional Veronese cone produces a 2-dimensional rational normal cone in $\mathbb{C}^6 = \mathrm{Sym}^2(\mathbb{C}^3)$. The real case needed here requires a little detail to describe accurately. Consider the intersection of a generic hyperplane with with the Veronese cone. This intersection is given by the polynomial equation in Z_1, Z_2, Z_3 such as

$$(15) \qquad a_{11}(Z_1)^2 + a_{12}Z_1Z_2 + a_{13}Z_1Z_3 + a_{22}(Z_2)^2 + a_{23}Z_2Z_3 + a_{33}(Z_3)^2 = 0.$$

Depending on the (real) coefficients a_{ij}, Equation (15) may or may not have real solutions. If Equation (15) has real solutions, then the solution in $\mathbb{P}(\mathbb{R}^3)$ is a real quadric surface. This quadric may or may not be degenerate. The existence of real non-degenerate solutions is an open condition on the hyperplane in the topology of $Gr_3(\mathbb{R}^6)$. If this condition is satisfied, the hyperplane defining $\{a_{ij}\}$ is called hyperbolic. The precise algebraic condition for hyperbolicity is that the real symmetric matrix (a_{ij}) is nonsingular and has split signature [33]. A hyperbolic hyperplane in $\mathrm{Sym}^2(\mathbb{R}^3)$ intersects the Veronese cone in a rational normal cone, and every rational normal cone in $\mathrm{Sym}^2(\mathbb{R}^3)$ can be written this way [18].

In the context of the lemma, Equation (15) describes $\ker(\mathrm{d}F_U)$, so the symmetric matrix (a_{ij}) is precisely the leading symbol of F at U. Therefore, if U is a regular point for the regular value 0 of $F : \Lambda^o \to \mathbb{R}$ and if the PDE $F = 0$ is hyperbolic at U, then $F^{-1}(0)$ admits a field of rational normal cones near U. $\qquad\square$

The fact that every hyperbolic PDE of the form in Equation (7) admits a GL(2)-structure would only be an algebraic curiosity, except that the integrability of the PDE is intimately related to the 2,3-integrability of the GL(2)-structure.

Theorem 1.4 (Theorem 3 in [11]). *Fix a hyperbolic PDE of the form in Equation (7) and its corresponding GL(2)-structure $\pi : \mathcal{B} \to M$. Then $\mathcal{B}$ is 2-integrable. Moreover, the PDE is integrable via 3-parameter hydrodynamic reductions if and only if $\mathcal{B}$ is also 3-integrable.*

Theorem 1.4 as presented in [11] apparently requires that, for each 3-secant $N^3 \subset M^5$, the characteristic net defined by the intersection $\mathbf{T}N \cap \mathbf{C}$ is a coordinate net. According to Corollary 4.2, this requirement is redundant.

As discussed in [11], for PDEs in three independent variables, the existence of 3-parameter hydrodynamic reductions implies the existence of k-parameter hydrodynamic reductions for all $k \geq 3$.

Corollary 1.5. *$F(u_{ij}) = 0$ is integrable via k-parameter hydrodynamic reductions for all $k \geq 2$ if and only if the induced GL(2)-structure over $F^{-1}(0)$ is 2,3-integrable.*

For the remainder of this article, the term "Hessian hydrodynamic" describes hyperbolic PDEs in three independent variables of the form in Equation (7) that are integrable by 3-parameter hydrodynamic reduction.

2. A Canonical Connection

In this section, Cartan's method of equivalence is applied to GL(2)-structures. Cartan's method of equivalence is a standard tool in the field of exterior differential systems; it is an algorithm for canonically fixing a preferred connection among all the possible pseudo-connections of a geometric structure [16, 20, 3]. (The distinction between a connection and a pseudo-connection is simply whether it was obtained canonically via such an algorithm.) The result of the algorithm is a canonical global co-frame for the GL(2)-structure that splits into the semi-basic "tautological" form, ω, and the vertical $\mathfrak{gl}(2)$-valued "connection" form, θ.

For a GL(2)-structure $\pi : \mathcal{B} \to M$, let ω denote the tautological 1-form defined by $\omega_b = b \circ \pi_* : \mathbf{T}_b\mathcal{B} \to \mathcal{V}_4$. As always, ω is a globally-defined 1-form of maximum rank 5 on $\mathcal{B}$, and it is semi-basic, meaning $\omega|_{\ker \pi_*} = 0$. A pseudo-connection on $\mathcal{B}$ is a 1-form θ taking values in the non-trivial representation of $\mathfrak{gl}(2)$ in $\mathfrak{gl}(\mathcal{V}_4)$ such that $d\omega = -\theta \wedge \omega + T(\omega \wedge \omega)$ for some torsion $T(b) : \mathcal{V}_4 \wedge \mathcal{V}_4 \to \mathcal{V}_4$.

The goal is to fix a particular θ that canonically minimizes the torsion T. To carry out this process, one needs to understand the $SL(2)$ representation theory on $\mathcal{V}_n$. The decomposition of the tensor product $\mathcal{V}_m \otimes \mathcal{V}_n$ into irreducible components is

$$(16) \qquad \mathcal{V}_m \otimes \mathcal{V}_n = \mathcal{V}_{|m-n|} \oplus \mathcal{V}_{|m-n|+2} \oplus \cdots \oplus \mathcal{V}_{m+n-2} \oplus \mathcal{V}_{m+n}.$$

The projections onto the various components are given by the Clebsch–Gordan [19, 4] pairings $\langle \cdot, \cdot \rangle_p : \mathcal{V}_m \otimes \mathcal{V}_n \to \mathcal{V}_{m+n-2p}$, which are provided by the formula

$$(17) \qquad \langle u, v \rangle_p = \frac{1}{p!} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \frac{\partial^p u}{\partial x^{p-k} \partial y^k} \cdot \frac{\partial^p v}{\partial x^k \partial y^{p-k}}.$$

This pairing has some important properties. Notice that $\langle u, v \rangle_p = (-1)^p \langle v, u \rangle_p$ and that the pairing is nontrivial for $0 \le p \le \min\{m, n\}$. Hence, the tensor decomposition can be further refined in terms of the symmetric and alternating tensors:

$$(18) \qquad \begin{aligned} \mathcal{V}_n \circ \mathcal{V}_n &= \mathcal{V}_{2n} \oplus \mathcal{V}_{2n-4} \oplus \cdots \oplus \mathcal{V}_{0 \text{ or } 2}, \\ \mathcal{V}_n \wedge \mathcal{V}_n &= \mathcal{V}_{2n-2} \oplus \mathcal{V}_{2n-6} \oplus \cdots \oplus \mathcal{V}_{2 \text{ or } 0}. \end{aligned}$$

Notice too that $\langle \cdot, \cdot \rangle_n : \mathcal{V}_n \otimes \mathcal{V}_n \to \mathcal{V}_0 = \mathbb{R}$ is a non-degenerate symmetric- or skew-bilinear form. Hence, for fixed $u \in \mathcal{V}_n$ the map $\langle u, \cdot \rangle_n : \mathcal{V}_n \to \mathcal{V}_0 = \mathbb{R}^1$ provides a natural identification, $\mathcal{V}_n = \mathcal{V}_n^*$, and one need never distinguish between dual spaces when considering representations.

For any derivation over $\mathbb{R}[x, y]$, a Leibniz rule over the pairing holds. Because $SL(2)$ is infinitesimally generated by X, Y, and H, this means that the pairings are $SL(2)$-equivariant. That is, $\alpha(\langle u, v \rangle_p) = \langle \alpha(u), v \rangle_p + \langle u, \alpha(v) \rangle_p$ for any $\alpha \in \mathfrak{sl}(2)$ implies $a \cdot \langle u, v \rangle_p = \langle a \cdot u, a \cdot v \rangle_p$ for any $a \in SL(2)$. The pairing is not $GL(2)$-equivariant, but the scaling action is easily computed easily where required. The geometric objects encountered here are projectively defined, so this variance in scaling is of little concern.

Most importantly, the pairing can be generalized to binary-polynomial-valued alternating forms over a manifold. If $u \in \Gamma(\wedge^r \mathbf{T}^* \mathcal{B} \otimes \mathcal{V}_m)$ and $v \in \Gamma(\wedge^s \mathbf{T}^* \mathcal{B} \otimes \mathcal{V}_n)$, then extend the definition by using the wedge-product:

$$(19) \qquad \langle u, v \rangle_p = \frac{1}{p!} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \frac{\partial^p u}{\partial x^{p-k} \partial y^k} \wedge \frac{\partial^p v}{\partial x^k \partial y^{p-k}}.$$

In this generalization, the symmetry of the pairing is further altered by the degree of the forms: $\langle u, v \rangle_p = (-1)^{rs+p} \langle v, u \rangle_p$. If λ is a $\mathbb{R}I_m$-valued 1-form, then $\lambda \wedge u$ may be written as the trivial pairing $\langle \lambda, u \rangle_0 = (-1)^r \langle u, \lambda \rangle_0 \in \Gamma(\wedge^{r+1} \mathbf{T}^* \mathcal{B} \otimes \mathcal{V}_m)$.

As $SL(2)$ representations, $\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathbb{R} = \mathcal{V}_2 \oplus \mathcal{V}_0$. Thus, a $\mathfrak{gl}(2)$-valued pseudo-connection θ decomposes as (φ, λ) with $\varphi \in \Gamma(\mathbf{T}^* \mathcal{B} \otimes \mathcal{V}_2)$ and $\lambda \in \Gamma(\mathbf{T}^* \mathcal{B} \otimes \mathcal{V}_0)$. The torsion of a generic θ is

$$(20) \qquad \begin{aligned} T : \mathcal{B} &\to \mathcal{V}_4 \otimes (\mathcal{V}_4^* \wedge \mathcal{V}_4^*) = (\mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6) \oplus (\mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}), \text{ so} \\ T &= (T_2^2 + T_4^2 + T_6^2) + (T_2^6 + T_4^6 + T_6^6 + T_8^6 + T_{10}^6). \end{aligned}$$

In this notation, each T_n^r is a distinct irreducible component of T. The lower index n indicates the weight of the representation in which T_n^r takes values, and the upper index r indicates the factor from which it was obtained.

Thus, Cartan's first structure equation for a GL(2)-structure may be written in either vector form or polynomial form:

$$(21) \qquad \begin{aligned} \mathrm{d}\omega &= -\theta \wedge \omega + T(\omega \wedge \omega) \\ &= - \langle \varphi, \omega \rangle_1 - \langle \lambda, \omega \rangle_0 + \langle T_2^2, \langle \omega, \omega \rangle_3 \rangle_0 + \langle T_4^2, \langle \omega, \omega \rangle_3 \rangle_1 + \langle T_6^2, \langle \omega, \omega \rangle_3 \rangle_2 \\ &\quad + \langle T_2^6, \langle \omega, \omega \rangle_1 \rangle_2 + \langle T_4^6, \langle \omega, \omega \rangle_1 \rangle_3 + \langle T_6^6, \langle \omega, \omega \rangle_1 \rangle_4 \\ &\quad + \langle T_8^6, \langle \omega, \omega \rangle_1 \rangle_5 + \langle T_{10}^6, \langle \omega, \omega \rangle_1 \rangle_6. \end{aligned}$$

Explicitly, the connection term is

$$(22) \qquad \theta \wedge \omega = \begin{pmatrix} 8\varphi_0 - \lambda & -8\varphi_{-2} & 0 & 0 & 0 \\ 2\varphi_2 & 4\varphi_0 - \lambda & -6\varphi_{-2} & 0 & 0 \\ 0 & 4\varphi_2 & -\lambda & -4\varphi_{-2} & 0 \\ 0 & 0 & 6\varphi_2 & -4\varphi_0 - \lambda & -2\varphi_{-2} \\ 0 & 0 & 0 & 8\varphi_2 & -8\varphi_0 - \lambda \end{pmatrix} \wedge \begin{pmatrix} \omega^{-4} \\ \omega^{-2} \\ \omega^0 \\ \omega^2 \\ \omega^4 \end{pmatrix},$$

so the matrix representation of θ is $(2\varphi_{-2}X + 2\varphi_0 H + 2\varphi_2 Y + \lambda I_5)$, which takes values in $\mathfrak{gl}(2) \subset \mathfrak{gl}(\mathcal{V}_4)$.

Theorem 2.1. *A generic GL(2)-structure $\pi : \mathcal{B} \to M$ admits a two-dimensional family of connections such that the essential torsion T decomposes irreducibly as*

$$(23) \qquad T = T_2 + T_6 + T_8 + T_{10} \in \mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10} \subset \mathcal{V}_4 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6^*).$$

Proof. Changes of pseudo-connection are of the form $\hat{\varphi} = \varphi + P(\omega)$ and $\hat{\lambda} = \lambda + Q(\omega)$ where $P \in \mathcal{V}_2 \otimes \mathcal{V}_4^* = \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6$ and $Q \in \mathcal{V}_0 \otimes \mathcal{V}_4^* = \mathcal{V}_4$. Canonical connections are obtained by analyzing the exact sequence

$$(24) \qquad 0 \to \mathfrak{gl}(2)^{(1)} \longrightarrow (\mathcal{V}_2 \oplus \mathcal{V}_0) \otimes \mathcal{V}_4^* \overset{\delta}{\longrightarrow} \mathcal{V}_4 \otimes (\wedge^2 \mathcal{V}_4^*) \longrightarrow H^{0,2}(\mathfrak{gl}(2)) \to 0.$$

For current purposes, $\mathfrak{gl}(2)^{(1)}$ and $H^{0,2}(\mathfrak{gl}(2))$ are defined by the exactness of the sequence [16]. To find the space of essential torsion, $H^{0,2}(\mathfrak{gl}(2))$, one must compute δP and δQ.

Fix $P \in \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6$, where $\mathcal{V}_2 \ni P(\omega) = \langle P_2, \omega \rangle_2 + \langle P_4, \omega \rangle_3 + \langle P_6, \omega \rangle_4$. Let $\delta P \in \mathcal{V}_4 \otimes \wedge^2(\mathcal{V}_4)$ have components $\delta P = \delta P_2^2 + \delta P_4^2 + \delta P_6^2 + \delta P_2^6 + \delta P_4^6 + \delta P_6^6 + \delta P_8^6 + \delta P_{10}^6$, similar to the decomposition of T. Since these are irreducible components and the action of δ must be $SL(2)$-equivariant, Schur's lemma implies δ must preserve the weights of the representations. In particular there must exist constants a_2, a_4, a_6, b_2, b_4, and b_6 such that $\delta P_2^2 = a_2 P_2$, $\delta P_2^6 = b_2 P_2$, and so on. Thus,

$$
\begin{aligned}
(25) \qquad 0 &= \langle P(\omega), \omega \rangle_1 - \delta P(\omega, \omega) \\
&= \langle \langle P_2, \omega \rangle_2, \omega \rangle_1 + \langle \langle P_4, \omega \rangle_3, \omega \rangle_1 + \langle \langle P_6, \omega \rangle_4, \omega \rangle_1 \\
&\quad - \langle a_2 P_2, \langle w, w \rangle_3 \rangle_0 - \langle a_4 P_4, \langle w, w \rangle_3 \rangle_1 - \langle a_6 P_6, \langle w, w \rangle_3 \rangle_2 \\
&\quad - \langle b_2 P_2, \langle w, w \rangle_1 \rangle_2 - \langle b_4 P_4, \langle w, w \rangle_1 \rangle_3 - \langle b_6 P_6, \langle w, w \rangle_1 \rangle_4 .
\end{aligned}
$$

Carrying out this computation shows that

$$
(26) \qquad a_2 = \frac{3}{10}, \ b_2 = \frac{1}{5}, \ a_4 = \frac{1}{2}, \ b_4 = 0, \ a_6 = -\frac{1}{5}, \ b_6 = -\frac{1}{20}.
$$

Similarly, fix $Q \in \mathcal{V}_4$, where $\mathcal{V}_0 \ni Q(\omega) = \langle Q_4, \omega \rangle_4$. Let $\delta Q \in \mathcal{V}_4 \otimes \wedge^2(\mathcal{V}_4)$ have components $\delta Q = \delta Q_2^2 + \delta Q_4^2 + \delta Q_6^2 + \delta Q_2^6 + \delta Q_4^6 + \delta Q_6^6 + \delta Q_8^6 + \delta Q_{10}^6$, but again the image must have the same weight as the domain. In particular there must exist constants c_4, and d_4 such that

$$
\begin{aligned}
(27) \qquad 0 &= \langle Q(\omega), \omega \rangle_0 - \delta Q(\omega, \omega) \\
&= \langle \langle Q_4, \omega \rangle_4, \omega \rangle_0 - \langle c_4 Q_4, \langle w, w \rangle_3 \rangle_1 - \langle d_4 Q_4, \langle w, w \rangle_1 \rangle_3 .
\end{aligned}
$$

Carrying out this computation shows that

$$
(28) \qquad c_4 = -\frac{1}{40}, \ d_4 = -\frac{1}{160}.
$$

Fix a generic pseudo-connection (φ, λ) with torsion T. Consider another pseudo-connection $\hat{\varphi} = \varphi + P(\omega)$, $\hat{\lambda} = \lambda + Q(\lambda)$ with torsion $\hat{T}$. Then

$$
\begin{aligned}
(29) \qquad \hat{T}(\omega, \omega) &= \mathrm{d}\omega + \langle \hat{\varphi}, \omega \rangle_1 + \left\langle \hat{\lambda}, \omega \right\rangle_0 \\
&= \mathrm{d}\omega + \langle \varphi, \omega \rangle_1 + \langle P(\omega), \omega \rangle_1 - \langle Q(\omega), \omega \rangle_0 \\
&= (T + \delta P + \delta Q)(\omega, \omega).
\end{aligned}
$$

Using Equation (26) and Equation (28), the absorption of torsion is dictated by the solvability of the equations

(30)
$$\hat{T}_2^2 = T_2^2 + \frac{3}{10}P_2,$$
$$\hat{T}_4^2 = T_4^2 + \frac{1}{2}P_4 - \frac{1}{40}Q_4,$$
$$\hat{T}_6^2 = T_6^2 - \frac{1}{5}P_6,$$
$$\hat{T}_2^6 = T_2^6 + \frac{1}{5}P_2,$$
$$\hat{T}_4^6 = T_4^6 - \frac{1}{160}Q_4,$$
$$\hat{T}_6^6 = T_6^6 - \frac{1}{20}P_6,$$
$$\hat{T}_8^6 = T_8^6,$$
$$\hat{T}_{10}^6 = T_{10}^6.$$

Generally, one may choose P_2 to force exactly one linear combination of $\hat{T}_2^6$ and $\hat{T}_2^2$ to vanish. Similarly, one may choose P_6 to force exactly one linear combination of $\hat{T}_6^6$ and $\hat{T}_6^2$ to vanish. Unique Q_4 and P_4 eliminate $\hat{T}_4^6$ and $\hat{T}_4^2$. All other components of $\hat{T}$ are fixed. Thus, $\mathfrak{gl}(2)^{(1)} = \mathfrak{sl}(2)^{(1)} = 0$ and $H^{0,2}(\mathfrak{sl}(2)) = \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}$, while $H^{0,2}(\mathfrak{gl}(2)) = \mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}$. $\square$

Corollary 2.2 (Canonical GL(2) Connection). *A GL(2)-structure $\mathcal{B} \to M$ admits a unique connection φ, λ such that $\mathcal{B}$ has first structure equation*

(31)
$$\begin{aligned} d\omega = &- \langle \varphi, \omega \rangle_1 - \langle \lambda, \omega \rangle_0 \\ &+ \langle T_2, \langle \omega, \omega \rangle_1 \rangle_2 + \langle T_6, \langle \omega, \omega \rangle_1 \rangle_4 \\ &+ \langle T_8, \langle \omega, \omega \rangle_1 \rangle_5 + \langle T_{10}, \langle \omega, \omega \rangle_1 \rangle_6. \end{aligned}$$

Proof. Of the possible connections, choose the one that absorbs T_2^2 and T_6^2. The remaining essential torsion is $T = T_2^6 + T_6^6 + T_8^6 + T_{10}^6$. $\square$

Henceforth, all references to θ, φ, λ, and T assume this connection. The specification of this connection over the others is arbitrary, but it does not affect any subsequent theorems in this article.

3. 2-INTEGRABILITY

Theorem 3.1. *If a GL(2)-structure $\pi : \mathcal{B} \to M$ is 2-integrable, then $T = T_8$ (that is, $T_2 = T_6 = T_{10} = 0$), and the bi-secant surfaces in M are parametrized by two functions of one variable. Conversely, if $\mathcal{B}$ is a smooth GL(2)-structure with $T = T_8$, then $\mathcal{B}$ is 2-integrable.*

Proof. To prove the theorem, one must find the conditions on $\mathcal{B}$ that allow an arbitrary bi-secant plane $E \in Gr_2(\mathbf{T}_p M)$ to be extended to a bi-secant surface $N \subset M$.

In a neighborhood M' of p, let $u : M' \to \mathcal{B}(M')$ be a section of $\pi : \mathcal{B} \to M$; that is, let u be a $GL(2)$ co-frame on M' such that $u(p) = b$. Since E is bi-secant, $b(E) \subset \mathcal{V}_4$ is spanned by $(g_1 x - h_1 y)^4$ and $(g_2 x - h_2 y)^4$ with $g_1 h_2 \neq g_2 h_1$. Through a

$GL(2)$ frame adaptation redefining u, one may assume that $b(E) = \mathrm{span}\{x^4, y^4\} \subset \mathcal{V}_4$. Let $\tilde{E}$ denote the lift of E, so $\tilde{E} = u_*(E) \in Gr_2(\mathbf{T}_b\mathcal{B})$. Then $\omega^{-4} \wedge \omega^4|_{\tilde{E}} \neq 0$.

Consider the linear Pfaffian exterior differential system $\mathcal{I}$ differentially generated by the 1-forms $\{w^{-2}, w^0, w^2\}$ with independence condition the $\omega^{-4} \wedge \omega^4 \neq 0$. It suffices to prove the existence of a surface $\tilde{N} \subset \mathcal{B}$ that is integral to $\mathcal{I}$, because $u(\pi_*(\mathbf{T}\tilde{N})) = \omega_u(\mathbf{T}\tilde{N}) = \mathrm{span}\{x^4, y^4\}$ implies that the surface $N = \pi(\tilde{N}) \subset M'$ is bi-secant. In fact, by adapting the moving co-frame u appropriately, every bi-secant surface through E must arise this way.

The generating 2-forms of $\mathcal{I}$ are

$$(32) \qquad \mathrm{d}\begin{pmatrix} \omega^{-2} \\ \omega^0 \\ \omega^2 \end{pmatrix} \equiv \begin{pmatrix} -2\varphi_2 & 0 \\ 0 & 0 \\ 0 & 2\varphi_{-2} \end{pmatrix} \wedge \begin{pmatrix} \omega^{-4} \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \tau^{-2} \\ \tau^0 \\ \tau^2 \end{pmatrix} \omega^{-4} \wedge \omega^4$$

modulo $\omega^{-2}, \omega^0, \omega^2$, where

$$\tau^{-2} = 48\, T_{2,-2} + 8640\, T_{6,-2} + 322560\, T_{8,-2} - 4838400\, T_{10,-2}$$

$$(33) \qquad \tau^0 = -96\, T_{2,0} + 23040\, T_{6,0} - 4838400\, T_{10,0}$$

$$\tau^2 = 48\, T_{2,2} + 8640\, T_{6,2} - 322560\, T_{8,2} - 4838400\, T_{10,2}$$

Because of the independence condition, integral elements exist only when the torsion can be absorbed. The torsion component τ^0 can never be absorbed, so integral manifolds exist only when $\tau^0 = 0$. The condition of 2-integrability means that every 2-secant plane is tangent to a 2-secant surface, but the $GL(2)$ action is transitive on 2-secant planes in $\mathbf{T}_pM$; therefore, it must be true that $\tau^0 = 0$ for every element in the $GL(2)$ orbit of T. Under the $GL(2)$ action, the coordinates of the irreducible representations of T will change, so each irreducible representation that appears in τ^0 must vanish identically. Hence, 2-integrability of M by integral manifolds implies

$$(34) \qquad\qquad\qquad T_{10} = T_6 = T_2 = 0.$$

The remaining torsion components, τ^{-2} and τ^2, are absorbed by setting $\pi_1 = 2\varphi_2 - 322560 T_{8,-2}\, \omega^4$ and $\pi_2 = -2\varphi_{-2} - 322560 T_{8,2}\, \omega^{-4}$, so

$$(35) \qquad \mathrm{d}\begin{pmatrix} \omega^{-2} \\ \omega^0 \\ \omega^2 \end{pmatrix} \equiv \begin{pmatrix} \pi_1 & 0 \\ 0 & 0 \\ 0 & \pi_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^{-4} \\ \omega^4 \end{pmatrix}, \quad \mathrm{mod}\ \omega^{-2}, \omega^0, \omega^2.$$

This proves the torsion condition in the theorem.

To establish existence of existence and parametrization of bi-secant surfaces, one can apply Cartan's test for involutivity to the tableau in Equation (35) [2, 20]. For a generic flag of $\mathbf{T}_pN$ obtained from generic linear combinations of ω^{-4} and ω^4, the tableau has Cartan characters $s_1 = 2$ and $s_2 = 0$. The space of integral elements for the EDS $(\mathcal{I}, \omega^{-4} \wedge \omega^4)$ is 2-dimensional, as parametrized by the coefficients $p_{1,4}$ and $p_{3,-4}$ that appear in the pulled-back forms $\tilde{N}^*(\pi_1) = p_{1,4}\tilde{N}^*(\omega^4)$ and $\tilde{N}^*(\pi_2) = p_{3,-4}\tilde{N}^*(\omega^{-4})$. Therefore, if $\pi : \mathcal{B} \to M$ is 2-integrable, then bi-secant surfaces in M depend on two functions of one variable. With the Cartan characters computed, Cartan's test for involutivity applies, so integral surfaces for the linear Pfaffian system exist in the real-analytic category.

More can be said by employing modern theorems regarding hyperbolic exterior differential systems [31]. The characteristic variety of the tableau consists of two

real points, and because a generic tableau is involutive with $s_1 = 2$, Yang's generalization of the Cartan–Kähler theorem to smooth hyperbolic systems implies that integral surfaces exist and are parametrized by two functions of one variable in the smooth category [31, Theorem 1.19]. A special case of this observation is revisited in Section 6.1. $\qquad\square$

By restricting the torsion, Theorem 3.1 provides the necessary first-order conditions for 2-integrability. Necessary second-order conditions for 2-integrability arise by via the Bianchi identity.

Corollary 3.2. *Suppose the GL(2)-structure $\pi : \mathcal{B} \to M$ is 2-integrable with torsion T. Let $S = \nabla(T)$ denote the covariant derivative of T, and let $Q = T \circ T$ denote the symmetric product of T. Then $\mathcal{B}$ has structure equations of the form*

$$\mathrm{d}\omega = - \langle \varphi, \omega \rangle_1 - \langle \lambda, \omega \rangle_0 + \langle T, \langle \omega, \omega \rangle_1 \rangle_5$$

(36)
$$\mathrm{d}\varphi = -\frac{1}{2} \langle \varphi, \varphi \rangle_1 + \langle R_0^2, \langle \omega, \omega \rangle_3 \rangle_0 + \langle 48Q_4 + 42S_4, \langle \omega, \omega \rangle_3 \rangle_2$$
$$+ \langle 45S_6, \langle \omega, \omega \rangle_1 \rangle_5 + \langle 33S_8, \langle \omega, \omega \rangle_1 \rangle_6 + \langle -8Q_4 - 12S_4, \langle \omega, \omega \rangle_1 \rangle_4$$
$$\mathrm{d}\lambda = 960 \langle S_6, \langle \omega, \omega \rangle_1 \rangle_6$$

for a scalar curvature function $R_0^2 : \mathcal{B} \to \mathcal{V}_0$.

Proof. Write $\nabla = \mathrm{d} + \theta$ for the covariant derivative on $\mathcal{B}$ defined by the connection 1-form (φ, λ). Second-order conditions for 2-integrability arise by via the Bianchi identity, $\nabla(\theta) \wedge \omega = \nabla(T(\omega \wedge \omega))$. Curvature appears in $\nabla(\theta)$, which splits into $R(\omega \wedge \omega) = \mathrm{d}\varphi + \frac{1}{2} \langle \varphi, \varphi \rangle_1$ and $r(\omega \wedge \omega) = \mathrm{d}\lambda$. The covariant derivative of the torsion two-form, $\nabla(T(\omega \wedge \omega))$, expands as $\nabla(T(\omega \wedge \omega)) = \nabla(T)(\omega \wedge \omega) + 2Q(T,T)(\omega \wedge \omega \wedge \omega)$, so the Bianchi identity for a GL(2)-structure is

(37) $$R(\omega \wedge \omega) + r(\omega \wedge \omega) = \nabla(T)(\omega \wedge \omega) + 2Q(T,T)(\omega \wedge \omega \wedge \omega).$$

Each of R, r, $\nabla(T)$ and Q is a function on $\mathcal{B}$ to the appropriate vector space:

$$R : \mathcal{B} \to \mathfrak{sl}(2) \otimes (\mathcal{V}_4^* \wedge \mathcal{V}_4^*),$$
$$r : \mathcal{B} \to \mathbb{R} \otimes (\mathcal{V}_4^* \wedge \mathcal{V}_4^*),$$
(38)
$$\nabla(T) : \mathcal{B} \to H^{0,2}(\mathfrak{gl}(2)) \otimes \mathcal{V}_4^*, \text{ and}$$
$$Q : \mathcal{B} \to \mathrm{Sym}^2(H^{0,2}(\mathfrak{gl}(2)) \cap (\mathcal{V}_4 \otimes \wedge^3 \mathcal{V}_4^*).$$

Since $\mathcal{B}$ is 2-integrable, T takes values only in $\mathcal{V}_8 \subset H^{0,2}(\mathfrak{gl}(2))$. Using the Clebsch–Gordan decomposition, the irreducible components of these functions are (omitting the domain $\mathcal{B}$ for brevity):

$$R = (R_0^2 + R_2^2 + R_4^2) + (R_4^6 + R_6^6 + R_8^6) \in \mathcal{V}_2 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6),$$
$$r = r_2 + r_6 \in \mathcal{V}_0 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6),$$
(39)
$$\nabla T = S_4 + S_6 + S_8 + S_{10} + S_{12} \in \mathcal{V}_8 \otimes \mathcal{V}_4, \text{ and}$$
$$Q = Q_4 + Q_8 \in \mathrm{Sym}^2(\mathcal{V}_8) \cap (\mathcal{V}_4 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6)).$$

Thus, Equation (37) and Schur's lemma together imply linear relations among the irreducible components listed in Equation (39). To find these relations, one can expand Equation (37) using the Clebsch–Gordan pairing; for example, one of the terms is

(40)
$$\langle \nabla T, \langle \omega, \omega \rangle_1 \rangle_5 = \langle \langle S_4, \omega \rangle_0 + \langle S_6, \omega \rangle_1 + \langle S_8, \omega \rangle_2 + \langle S_{10}, \omega \rangle_3 + \langle S_{12}, \omega \rangle_4, \langle \omega, \omega \rangle_1 \rangle_5.$$

The result is $S_{10} = 0$, $R_8^6 = 33S_8$, $R_6^6 = 45S_6$, $r_6 = 960S_6$, $R_4^6 = -12S_4 - 8Q_4$, $R_4^2 = 42S_4 + 48Q_4$, $r_2 = 0$, and $R_2^2 = 0$. In particular, the only free curvature is R_0^2; all other components of R and r are functions of T and $\nabla(T)$. $\qquad\square$

4. 3-INTEGRABILITY

Theorem 4.1. *If an analytic GL(2)-structure $\pi : \mathcal{B} \to M$ is 3-integrable, then tri-secant 3-folds in M are locally parametrized by three functions of one variable.*

For 3-integrable $GL(2)$-structures that arise from PDEs of hydrodynamic type as in Theorem 1.4 (and are therefore also 2-integrable), this parametrization by three functions of one variable confirms the computation presented in [11].

Proof. This theorem is proven by applying Cartan–Kähler theory to a differential ideal whose integral manifolds are tri-secant 3-folds $N \subset M$ through an arbitrary tri-secant element $E \in Gr_3(\mathbf{T}_pM)$.

As in the proof of Theorem 3.1, consider a local $GL(2)$ co-frame u such that $u(p) = b$. Since E is tri-secant, $b(E)$ is spanned by $(g_1x - h_1y)^4$, $(g_2x - h_2y)^4$, and $(g_3x - h_3y)^4$, distinct, but u can be adapted so that $b(E)$ is spanned by x^4, y^4, and $(x + y)^4$. Therefore
(41)
$$b(E) = \{(A + B)x^4 + B4x^3y + B6x^2y^2 + B4xy^3 + (B + C)y^3 \ : \ A, B, C \in \mathbb{R}\} \subset \mathcal{V}_4.$$

Just as in Theorem 3.1, lift to $\tilde{E} = u_*(E) \in Gr_2(\mathbf{T}_b\mathcal{B})$, which is integral to the linear Pfaffian system $\mathcal{I}$ generated by $\kappa^{-2} = \omega^{-2} - \omega^0$ and $\kappa^2 = \omega^2 - \omega^0$ with the independence condition $\omega^{-4} \wedge \omega^0 \wedge \omega^4 \neq 0$. Again, the projection to M of any integral 3-fold of $\mathcal{I}$ will be a tri-secant 3-fold that passes through E, and every tri-secant 3-fold arises this way (up to a $GL(2)$ frame adaptation).

The tableau and torsion of $\mathcal{I}$ are given by

$$(42) \qquad d\begin{pmatrix} \kappa^{-2} \\ \kappa^2 \end{pmatrix} \equiv \begin{pmatrix} \pi_1 & \pi_3 & 0 \\ 0 & -\pi_1 - \pi_2 - \pi_3 & \pi_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^{-4} \\ \omega^0 \\ \omega^4 \end{pmatrix} + \tau(\omega, \omega),$$

modulo κ^{-2}, κ^2, where $\pi_1 = -2\varphi_2$, $\pi_2 = 2\varphi_{-2}$, and $\pi_3 = 2\varphi_{-2} - 4\varphi_0 + 4\varphi_2$. The apparent torsion can be fully absorbed by redefining $\hat{\pi}_i = \pi_i - p_{i,a}\omega^a$ for the parameters

$$(43) \qquad \begin{aligned} &p_{1,4} = -\tau_{-4,4}^{-2} \\ &p_{2,-4} = \tau_{-4,4}^{2} \\ &p_{3,4} = -\tau_{0,4}^{-2} \\ &p_{3,-4} = -p_{1,-4} - \tau_{-4,4}^{2} - \tau_{-4,0}^{2} \\ &p_{2,4} = -p_{2,0} - p_{3,4} - p_{1,4} + \tau_{0,4}^{2} = -p_{2,0} + \tau_{0,4}^{-2} + \tau_{-4,4}^{-2} + \tau_{0,4}^{2} \\ &p_{1,0} = p_{3,-4} - \tau_{-4,4}^{-2} = -p_{1,-4} - \tau_{-4,4}^{2} - \tau_{-4,0}^{2} - \tau_{-4,4}^{2}. \end{aligned}$$

The integral elements are still free up to arbitrary choice of three functions, $p_{1,-4}$, $p_{2,0}$, and $p_{3,0}$. Since the Cartan characters are $s_1 = 2$, $s_2 = 1$, and $s_3 = 0$, but $s_1 + 2s_2 + 3s_3 = 4 \neq 3$, the tableau is not involutive; prolongation is required.

Let $\mathcal{I}^{(1)}$ be the prolonged ideal, which is differentially generated by the forms κ^{-2} and κ^2 along with

$$\eta^1 = \pi_1 + p_{1,-4}\, \omega^{-4} - (p_{1,-4} + \tau^2_{-4,4} + \tau^2_{-4,0} + \tau^2_{-4,4})\, \omega^0 - \tau^{-2}_{-4,4}\, \omega^4,$$

(44)
$$\eta^2 = \pi_2 + \tau^2_{-4,4}\, \omega^{-4} + p_{2,0}\, \omega^0 + (-p_{2,0} + \tau^{-2}_{0,4} + \tau^{-2}_{-4,4} + \tau^2_{0,4})\, \omega^4, \text{ and}$$

$$\eta^3 = \pi_3 + (-p_{1,-4} - \tau^2_{-4,4} - \tau^2_{-4,0})\, \omega^{-4} + p_{3,0}\, \omega^0 + -\tau^{-2}_{0,4}\, \omega^4.$$

After this prolongation, the tableau and torsion are given by

$$(45) \qquad \mathrm{d}\begin{pmatrix} \kappa^{-2} \\ \kappa^2 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pi_4 & -\pi_4 & 0 \\ 0 & \pi_5 & -\pi_5 \\ -\pi_4 & \pi_6 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^{-4} \\ \omega^0 \\ \omega^4 \end{pmatrix} + \tau^{(1)}(\eta \wedge \eta),$$

modulo $\kappa^{-2}, \kappa^2, \eta^1, \eta^2, \eta^3$. This tableau has Cartan characters $s_1 = 3$, $s_2 = 0$, and $s_3 = 0$. Applying Cartan's test, $s_1 + 2s_2 + 3s_3 = 3$, which matches the dimension of $\mathcal{V}_3(\mathcal{I}^{(1)})$, so the tableau is involutive. When the apparent torsion $\tau^{(1)}$ vanishes or can be absorbed, then (in the analytic category) Cartan's test for involutivity implies that the integral 3-folds locally depend on three functions of one variable.

In fact, one can say more by analyzing the characteristic variety of the involutive linear Pfaffian system presented in Equation (45). Given the highest non-zero Cartan character, $s_1 = 3$, it is clear that the complex characteristic variety has dimension zero and degree three [2, Chapter V]. It is easy to verify that the characteristic variety consists of three real points, so each 3-fold integral to $\mathcal{I}^{(1)}$ is foliated by three families of 2-folds integral to $\mathcal{I}^{(1)}$. A special case of this fact is revisited in Corollary 4.2. $\qquad\qquad\square$

To determine sufficient conditions for *existence* of tri-secant 3-folds, one must study the unabsorbable portion of the remaining torsion, $\tau^{(1)}$. Because $\tau^{(1)}$ is the torsion of the prolonged system $\mathcal{I}^{(1)}$, it will involve second-order invariants of the $GL(2)$-structure $\mathcal{B}$ that appear in the Bianchi identity for $\mathcal{B}$, Equation (37). Since T is *a priori* valued in $H^{0,2}(\mathfrak{gl}(2)) = \mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}$, components of any of the following functions may occur in $\tau^{(1)}$:

$$
\begin{aligned}
R &: \mathcal{B} \to \mathcal{V}_2 \otimes (\wedge^2 \mathcal{V}_4), \\
r &: \mathcal{B} \to \mathcal{V}_0 \otimes (\wedge^2 \mathcal{V}_4), \\
\nabla T &: \mathcal{B} \to (\mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}) \otimes \mathcal{V}_4, \\
Q &: \mathcal{B} \to \mathrm{Sym}^2(\mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}) \cap (\mathcal{V}_4 \otimes \wedge^3 \mathcal{V}_4).
\end{aligned}
$$
(46)

The vanishing of the unabsorbable portion of $\tau^{(1)}$ will place restrictions on the various irreducible representations appearing in Equation (46). The enormity of Q and ∇T makes decomposition of the unabsorbable portion of $\tau^{(1)}$ extremely difficult. Fortunately, one can make a simplifying assumption that is consistent with the motivating PDE theory in Theorem 1.4; henceforth, all theorems discuss only those GL(2)-structures that are both 2-integrable and 3-integrable.

Corollary 4.2. *If $\mathcal{B} \to M$ is a 2,3-integrable GL(2)-structure, then any tri-secant $N^3 \subset M$ is triply foliated by bi-secant surfaces. Moreover, the net defined by $\mathbf{T}N \cap \mathbf{C}$ is a coordinate net.*

The triple foliation by bi-secant surfaces appears to be new, though it is not surprising based on the description of highest-weight polynomial subspaces of $\mathcal{V}_n$ seen in [10]. This foliation shows that the "holonomic characteristic net" condition for tri-secant 3-folds seen in [11] is superfluous since 2-integrability is implied for GL(2)-structures arising from Hessian hydrodynamic PDEs.

Proof. Without loss of generality in a contractible neighborhood in M, one may apply a $GL(2)$ change-of-frame so that $\mathbf{T}N = \{Ax^4 + B(x + y)^4 + Cy^4\}$. The characteristic net, $\mathbf{T}N \cap \mathbf{C}$, is given by its tangent planes

$$
\begin{aligned}
K_1 &= \mathrm{span}\{(x + y)^4, y^4\}, \\
K_2 &= \mathrm{span}\{x^4, y^4\}, \text{ and} \\
K_3 &= \mathrm{span}\{x^4, (x + y)^4\}.
\end{aligned}
\tag{47}
$$

Each of K_1, K_2, and K_3 is clearly bi-secant. It suffices to prove that they are everywhere tangent to the level sets of a coordinate system on N, as these level sets will be bi-secant surfaces.

Recall that N is defined by an integral manifold $\tilde{N}$ of $\mathcal{I}^{(1)}$ from Theorem 4.1. In particular, each $\mathbf{T}_p N$ is the projection of an integral element $\mathbf{T}_b \tilde{N}$ that is defined by the condition

$$
E = \ker(\{\kappa^{-2}, \kappa^2, \eta^1, \eta^2, \eta^3, \pi_4 - P_1(\omega^{-4} - \omega^0), \pi_5 - P_2(\omega^0 - \omega^4), \pi_6 - P_1\omega^{-4} + P_3\omega^0\})
\tag{48}
$$

for parameters P_1, P_2, P_3 that uniquely define $\mathbf{T}_b \tilde{N}$ in $Gr_3(\mathbf{T}_b \mathcal{B})$.

Let $\Upsilon^1 = \omega^{-4} - \omega^0$, $\Upsilon^2 = \omega^0$, and $\Upsilon^3 = \omega^4 - \omega^0$. Therefore, at each basepoint, $K_i \subset \mathbf{T}_p N$ is the projection of the plane $\ker(\tilde{N}^*(\Upsilon^i)) \subset \mathbf{T}_b \tilde{N}$. One can easily compute that $\mathrm{d}(\tilde{N}^*(\Upsilon^i)) \equiv 0$ modulo $\tilde{N}^*(\Upsilon^i)$ for each i; therefore, the Frobenius theorem provides local coordinates (s^1, s^2, s^3) on N such that $K_i = \ker(\mathrm{d}s^i)$. $\qquad\square$

When $\mathcal{B}$ arises from a Hessian hydrodynamic PDE, these coordinates are called the Riemann invariants of the hydrodynamic reduction.

Theorem 4.3 (2,3-integrable GL(2)-structure equations). *A GL(2)-structure $\mathcal{B}$ is 2,3-integrable if and only if the torsion T of $\mathcal{B}$ only takes values in $\mathcal{V}_8$ and the curvature is a function of T. In particular, every 2,3-integrable GL(2)-structure $\mathcal{B}$ has the following structure equations*

$$
\begin{aligned}
\mathrm{d}\omega &= -\langle\varphi, \omega\rangle_1 - \langle\lambda, \omega\rangle_0 + \langle T, \langle\omega, \omega\rangle_1\rangle_5 \\
\mathrm{d}\lambda &= 0 \\
\mathrm{d}\varphi &= -\frac{1}{2}\langle\varphi, \varphi\rangle_1 - 2080\langle\langle T, T\rangle_8, \langle\omega, \omega\rangle_3\rangle_0 + 64\langle\langle T, T\rangle_6, \langle\omega, \omega\rangle_3\rangle_2 \\
&\quad -\frac{88}{7}\langle\langle T, T\rangle_6, \langle\omega, \omega\rangle_1\rangle_4 + \frac{24}{7}\langle\langle T, T\rangle_4, \langle\omega, \omega\rangle_1\rangle_6 \\
\mathrm{d}T &= J(T)\begin{pmatrix}\omega \\ \lambda \\ \varphi\end{pmatrix}
\end{aligned}
\tag{49}
$$

for a 9×9 matrix-valued function J whose entries are linear and quadratic polynomials in the coefficients of T, as provided in [23, 24].

Proof. 2-integrability implies that $T = T_8$, so R, r, $\nabla(T)$, and Q decompose as in Equation (39) with the relations implied in Theorem 3.1. 3-integrability implies the

vanishing of the $GL(2)$ orbit of the unabsorbable portion of $\tau^{(1)}$ in Equation (45). The vanishing of $\tau^{(1)}$ and the Bianchi identity ($\mathrm{d}^2 = 0$) together imply the additional relations $S_{12} = 0$, $7S_8 = 24Q_8 = 24\langle T, T\rangle_4$, $S_6 = 0$, $21S_4 = 8Q_4 = 8\langle T, T\rangle_6$, and $R_0^2 = -2080\langle T, T\rangle_8$.

Therefore, the curvatures, R and r, are quadratic functions of T. The derivative of torsion, $\mathrm{d}T$, is also a quadratic function of T, as expressed in the matrix $J(T)$. Hence, T is the only invariant of any order for 2,3-integrable $GL(2)$-structures. $\square$

Definition 4.1. *The notation $(\mathcal{B}, M, p)$ indicates a smooth 2,3-integrable $GL(2)$-structure $\pi : \mathcal{B} \to M$ such that $p \in M$ and such that M is connected.*

The condition that M is connected is crucial in what follows, and pointedness is technically useful.

Definition 4.2 (Representatives). *$(\mathcal{B}, M, p)$ **represents** $v \in \mathcal{V}_8$ if $v \in T(\pi^{-1}(p))$. More generally, a 2,3-integrable $GL(2)$-structure $\mathcal{B} \to M$ **represents** $v \in \mathcal{V}_8$ if $T(\mathcal{B}) \ni v$. Likewise, a Hessian hydrodynamic PDE **represents** $v \in \mathcal{V}_8$ if the induced 2,3-integrable $GL(2)$-structure over $M = F^{-1}(0)$ represents v.*

Corollary 4.4. *For any $v \in \mathcal{V}_8$, there exists a real-analytic connected 2,3-integrable $GL(2)$-structure $\mathcal{B} \to M$ such that $T(\mathcal{B}) \ni v$. That is, every $v \in \mathcal{V}_8$ is represented by some real-analytic $(\mathcal{B}, M, p)$. Moreover, J has constant rank on any $\mathcal{B}$.*

Proof. Since the structure equations in Theorem 4.3 are closed under exterior derivative, this is a direct application of the existence part of Cartan's structure theorem, which is an generalization of Lie's third fundamental theorem to the intransitive case [7]. See Appendix A of [5] for a clear summary of the special case of Cartan's structure theorem needed here. As used here, Cartan's structure theorem holds in the smooth category; however, because $T \circ T$ and $J(T)$ are real-analytic (in fact, linear and quadratic), one can use the Cartan–Kähler machinery to produce a local real-analytic solution $\mathcal{B}$ of the structure equations. $\square$

Corollary 4.5. *$(\mathcal{B}, M, p)$ and $(\hat{\mathcal{B}}, \hat{M}, \hat{p})$ admit a local $GL(2)$-equivalence $f : M \to \hat{M}$ with $f(p) = \hat{p}$ if and only if $T(\pi^{-1}(p)) \cap \hat{T}(\hat{\pi}^{-1}(\hat{p})) \neq \emptyset$. That is, the value of $T(\mathcal{B}_p)$ uniquely defines $(\mathcal{B}, M, p)$ in a neighborhood of p, and this local $GL(2)$-structure is real-analytic.*

Proof. This is a direct application of the local uniqueness part of Cartan's structure theorem. Again, the theorem applies here in the smooth category, but the real-analyticity of the structure equations implies that any 2,3-integrable $GL(2)$-structure is locally equivalent to the real-analytic $GL(2)$-structure produced in Corollary 4.4. $\square$

Corollaries 4.4 and 4.5 establish the primacy of T in the study 2,3-integrable $GL(2)$-structures and the related PDEs. However, Corollary 4.5 provides only local $GL(2)$-equivalence, so it is useful to have a weaker "chain-wise" notion of equivalence that applies to global (but still connected) $GL(2)$-structures.

Definition 4.3 (Leaf-equivalence). *$(\mathcal{B}_0, M_0, p_0)$ and $(\mathcal{B}_k, M_k, p_k)$ are said to be **leaf-equivalent** if there exist finite sequences $\{(\mathcal{B}_i, M_i, p_i)\}$ and $\{v_i\}$ with $1 \leq i \leq k-1$ such that $(\mathcal{B}_i, M_i, p_i)$ and $(\mathcal{B}_{i+1}, M_{i+1}, p_{i+1})$ both represent v_i for $0 \leq i \leq k-1$.*

Implied in the term "leaf-equivalence" is a foliation of $\mathcal{V}_8$ that separates all possible $(\mathcal{B}, M, p)$'s into equivalence classes by the value of T. This is justified by the following lemma.

Lemma 4.6. *The singular distribution on $\mathcal{V}_8$ defined by the columns of the matrix J is integrable, providing a stratification of $\mathcal{V}_8$ into leaves that are submanifolds. That is, for any $v \in \mathcal{V}_8$, there exists a unique submanifold $\mathcal{O}_J(v)$ such that $\mathbf{T}_v\mathcal{O}_J(v) = \text{range } J(v)$.*

Proof. This is actually an intermediate step of the application of the Cartan structure theorem as used in Corollaries 4.4 and 4.5. Alternatively, one can apply the modern theory of Lie algebroids and smooth groupoids. In particular, the matrix J defines the anchor map of a Lie algebroid over $\mathcal{V}_8$. This Lie algebroid is neither regular nor transitive, but there exists an integrating groupoid over the base, $\mathcal{V}_8$. This groupoid is smooth, but it may not be a Lie groupoid. However, the groupoid is transitive when restricted to each of its orbits, $\mathcal{O}_J(v)$, and each orbit is a submanifold of the base [9, 21, 25]. These orbits may be regarded as the leaves of a (singular) foliation of $\mathcal{V}_8$ such that $\mathbf{T}_v\mathcal{O}_J(v)$ is spanned by the columns of $J(v)$. $\square$

By slight abuse of notation, if $(\mathcal{B}, M, p)$ represents v, then write $\mathcal{O}_J(\mathcal{B})$ to denote $\mathcal{O}_J(v)$. Leaf-equivalence can now be rewritten more succinctly.

Theorem 4.7 (Leaf-equivalence). *$(\mathcal{B}, M, p)$ and $(\hat{\mathcal{B}}, \hat{M}, \hat{p})$ are leaf-equivalent if and only if $\mathcal{O}_J(\mathcal{B}) = \mathcal{O}_J(\hat{\mathcal{B}})$. Moreover, $T : \mathcal{B} \to \mathcal{O}_J(\mathcal{B})$ is a submersion.*

Although $T : \mathcal{B} \to \mathcal{O}_J(\mathcal{B})$ is an open map, it need not be surjective.

5. The Classification

To classify connected 2,3-integrable GL(2)-structures is to explicitly identify the leaves of the foliation of $\mathcal{V}_8$. To identify the leaves of the foliation is to identify connected $GL(2)$-invariant submanifolds of $\mathcal{V}_8$ where the rank of $J(v)$ is constant. This is all ultimately achieved in Theorem 5.3 thanks to the observations in Lemmas 5.1 and 5.2.

Lemma 5.1. *The determinant of $J(v)$ is a scalar multiple of the discriminant of the polynomial $v \in \mathcal{V}_8$. Therefore, $J(v)$ is non-singular if and only if v has eight distinct roots. Moreover, if v is a nontrivial polynomial with k distinct roots, then the rank of $J(v)$ is $k+1$.*

Proof. The determinant statement is verified by computing the g.c.d. of the two polynomials, $\det J(v)$ and $\text{disc}(v)$, in MAPLE. The rank statement is directly verified by writing $v = (g_1 x - h_1 y)(g_2 x - h_2 y) \cdots (g_8 x - h_8 y)$, imposing multiplicity on the h_i's and g_i's and computing the rank of $J(v)$ directly (also using MAPLE). $\square$

As it happens, a simple exercise in linear fractional transformations shows that the multiplicity of the roots is preserved by the $GL(2)$ action on $\mathcal{V}_8$.

Lemma 5.2. *For $v(x,y) \in \mathcal{V}_8$, the multiplicity and complex type of the roots are preserved under the $GL(2, \mathbb{R})$ action.*

Proof. For notational simplicity, instead consider $\mathcal{P}_8$, the space of polynomials of degree at most 8 with real coefficients in the projective variable p. Recall that $\mathcal{V}_8$

and $\mathcal{P}_8$ are isomorphic as $SL(2)$ representations by $v(x, y) \mapsto f(p) = v(p, 1)$. For some distinct $g_1, \ldots, g_k \in \mathbb{CP}^1$, factor $f(p)$ as

$$(50) \qquad f(p) = A \prod_{k=1}^{m} (p + g_k)^{r_k}, \qquad r_1 + r_2 + \cdots + r_m = 8.$$

Fix $G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{R})$. Then

$$(51) \qquad \begin{aligned} f(p) \cdot G &= (\gamma p + \delta)^8 A \prod_{k=1}^{m} \left(\left(\frac{\alpha p + \beta}{\gamma p + \delta} \right) + g_k \right)^{r_k} \\ &= A \prod_{k=1}^{m} \left((\alpha + g_k \gamma) p + (\beta + g_k \delta) \right)^{r_k} \end{aligned}$$

Hence, the root g_k has been moved to $S(g_k) = \frac{\delta g_k + \beta}{\gamma g_k + \alpha} \in \mathbb{CP}^1$. Since $\delta \alpha - \beta \gamma \neq 0$, S is a linear-fractional transformation. The lemma reduces to a few observations from elementary complex analysis. Of course $g_k \neq g_j$ implies $S(g_k) \neq S(g_j)$, so the number and multiplicity of roots is unchanged. Since the action is by $GL(2, \mathbb{R})$, real roots remain real. If there are complex-conjugate roots, $\overline{g_k} = g_j$, then the symmetry principle implies that their images are also complex-conjugate; $\overline{S(g_k)} = S(\overline{g_k}) = S(g_j)$. $\qquad\square$

Bearing in mind this lemma, notational shorthand for the **root-type** of a polynomial $v(x, y) \in \mathcal{V}_8$ is useful: Suppose $v(x, y)$ factors as $(g_1 x - h_1 y)^{r_1} \cdots (g_m x - h_m y)^{r_m}$ such that $r_1 \geq r_2 \geq \cdots \geq r_m$ and $r_1 + r_2 + \cdots + r_m = 8$. These exponents define a partition of 8 that is written as $\{r_1, r_2, \ldots, r_m\}$. If the roots g_k/h_k and g_{k+1}/h_{k+1} are complex-conjugate, then denote this by enclosing their exponents in square-braces: $\{r_1, \ldots, [r_k, r_{k+1}], \ldots, r_m\}$. Denote the root-type containing $v(x, y)$ by $[v(x, y)]$. For example, $[x^4(x + iy)^2(x - iy)^2] = \{4, [2, 2]\} \subset \mathcal{V}_8$. Let $\{0\}$ denote the trivial root-type, the zero polynomial. There are 54 non-trivial root-types in $\mathcal{V}_8$, as represented in Figure 1.

In Figure 1, arrows mean "closure contains." Shaded nodes represent root-types that contain exactly one $GL(2)$ orbit. Oval nodes represent open root-types. The square node represents the nearly-closed root-type, $\{8\}$, which (when 0 is included) is the rational normal cone in $\mathcal{V}_8$. Hexagonal nodes represent root classes that are neither closed nor open. Note that strictly real root-types actually have two connected components (for example, $\{8\}$ is comprised of the two ends of the rational normal cone), but this is another artifact of the projective nature of the projective action, as the two ends are in the same $GL(2)$ orbit by the $-I_9$ action.

Theorem 5.3 (Leaf-Equivalence Classes)**.** *The leaf-equivalence classes of connected 2,3-integrable $GL(2)$-structures are exactly the root-types in $\mathcal{V}_8$. That is, for all $v \in \mathcal{V}_8$, $\mathcal{O}_J(v) = [v]$.*

Proof. For each v, both $\mathcal{O}_J(v)$ and $[v]$ are smooth submanifolds of $\mathcal{V}_8$, and by Lemma 5.1 they have the same dimension. To prove that $\mathcal{O}_J(v) = [v]$, it suffices to prove that $\mathbf{T}_v \mathcal{O}_J(v) = \mathbf{T}_v[v]$ for all $v \in \mathcal{V}_8$.

Fix a root-type $[v]$ and an arbitrary $v \in [v]$. Note that

$$\mathbf{T}_v \mathcal{O}_J(\mathcal{B}) = DT_b(\mathbf{T}_b \mathcal{B}) = \text{range } J(v).$$
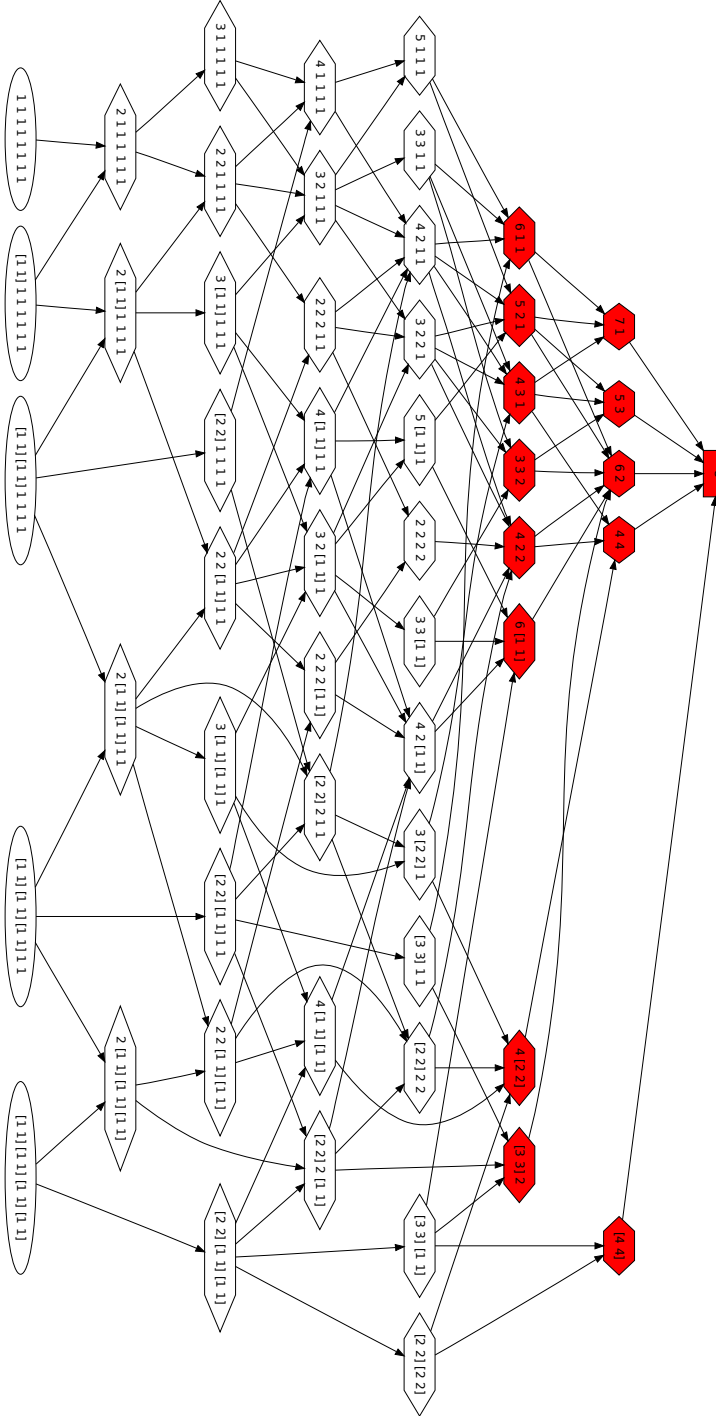
FIGURE 1. The stratification of $\mathcal{V}_8$ into the 55 root-types, sorted by dimension. Shaded nodes indicate root-types that contain exactly one $GL(2)$ orbit. The square node is closed (if 0 is included); the oval nodes are open, and the hexagonal nodes are neither closed nor open.

For each column $J_i(v)$ of the matrix $J(v)$, solve $D(v) = J_i(v)$ for $D(v)$, an arbitrary tangent to $[v]$ at v. As it happens, these equations are easily solvable at arbitrary points in all 54 non-trivial root-types and for all columns of J. Because both the leaf-equivalence classes and the root-types partition $\mathcal{V}_8$ by smooth submanifolds, and because $\mathbf{T}_v \mathcal{O}_J(v) = \mathbf{T}_v[v]$ for all $v \in \mathcal{V}_8$, the partitions must be identical.

To illustrate the computations, consider $v = (gx + hy)^8$, where $[v] = \{8\}$. An arbitrary element of $\mathbf{T}_v[v]$ looks like

$$(52) \qquad D(v) = 8(Gx + Hy)(gx + hy)^7 = \begin{pmatrix} 8Gg^7 \\ Hg^7 + 7Ghg^6 \\ 2Hhg^6 + 6Gh^2g^5 \\ 3Hh^2g^5 + 5Gg^4h^3 \\ 4Gg^3h^4 + 4Hg^4h^3 \\ 5Hg^3h^4 + 3Gh^5g^2 \\ 6Hh^5g^2 + 2Gh^6g \\ 7Hh^6g + Gh^7 \\ 8Hh^7 \end{pmatrix}.$$

Therefore, one must solve $D(v) = J_i(v)$ for G and H in each of the cases $i = 1, \ldots, 9$. In this case,

$$(53) \qquad J(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & g^8 & -16hg^7 & 16g^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & hg^7 & -14h^2g^6 & 12hg^7 & 2g^8 \\ 0 & 0 & 0 & 0 & 0 & h^2g^6 & -12h^3g^5 & 8h^2g^6 & 4hg^7 \\ 0 & 0 & 0 & 0 & 0 & h^3g^5 & -10h^4g^4 & 4h^3g^5 & 6h^2g^6 \\ 0 & 0 & 0 & 0 & 0 & h^4g^4 & -8h^5g^3 & 0 & 8h^3g^5 \\ 0 & 0 & 0 & 0 & 0 & h^5g^3 & -6h^6g^2 & -4h^5g^3 & 10h^4g^4 \\ 0 & 0 & 0 & 0 & 0 & h^6g^2 & -4h^7g & -8h^6g^2 & 12h^5g_1{}^3 \\ 0 & 0 & 0 & 0 & 0 & h^7g & -2h^8 & -12h^7g & 14h^6g^2 \\ 0 & 0 & 0 & 0 & 0 & h^8 & 0 & -16h^8 & 16h^7g \end{pmatrix}$$

For each column $J_i(v)$, it is easy to find the solutions G, H.

For the other root-types, the computations are similar but somewhat more complicated. All that matters is the fact that they can be solved for arbitrary v. $\qquad \square$

5.1. **Symmetry Reduction.** Geometrically, it is interesting to reduce all symmetry from a structure. Fix $(\mathcal{B}, M, p)$ and $v \in T(\mathcal{B})$. Let $\mathcal{B}^v = \{b \in \mathcal{B} : T(b) = v\}$. This is a sub-bundle of $\mathcal{B}$ with fiber group $\mathrm{Stab}(v) \subset GL(2)$. If $\dim[v] \leq 3$, then the stabilizer group is smooth; however, the stabilizer groups must be discrete for all the larger root types. The stabilizer group, either smooth or discrete, must always appear in the well-known list of $GL(2, \mathbb{C})$-stabilizers of polynomials [1]. In the discrete case, the fiber must be therefore be either a cyclic group or a dihedral group [24].

The structure equations for $\mathcal{B}^v$ show no dependence on T (as it has been fixed), so a neighborhood in $\mathcal{B}^v$ is a local Lie group of dimension $9 - \dim[v]$, and M is locally the symmetric space $\mathcal{B}^v/\mathrm{Stab}(v)$. The relations defined by $\ker dT(v) = \ker J(v)$ determine the pull-backs of ω, λ, and φ to $\mathcal{B}^v$, so one may explicitly reduce Equation (49) to obtain structure equations for the local Lie group $\mathcal{B}^v$.

If $v = 0$, then $\mathrm{Stab}(v) = GL(2)$, so the local flat 2,3-integrable GL(2)-structure is the local Lie group of dimension 9 obtained by setting $T = 0$ in Equation (49).

The root type $\{8\}$ is a single $GL(2)$ orbit, so any representative will generate all representatives. Suppose $v = x^8$. Then $dT_{-8}(x^8) = \lambda + 16\varphi_0$ and $dT_{-6}(x^8) = 2\varphi_2$ while $dT_k = 0$ for $k > -6$. Since dT is a vertical 1-form on $\mathcal{B}$, the value of T only varies in the fiber of $\mathcal{B}$. The reduced structure equations are

$$
\begin{aligned}
d\omega^{-4} &= 24\ \varphi_0 \wedge \omega^{-4} - 8\ \varphi_{-2} \wedge \omega^{-2} + 2 \cdot 322560\ \omega^0 \wedge \omega^4, \\
d\omega^{-2} &= 20\ \varphi_0 \wedge \omega^{-2} - 6\ \varphi_{-2} \wedge \omega^0 + 322560\ \omega^2 \wedge \omega^4, \\
d\omega^0 &= 16\ \varphi_0 \wedge \omega^0 - 4\ \varphi_{-2} \wedge \omega^2, \\
d\omega^2 &= 12\ \varphi_0 \wedge \omega^2 - 2\ \varphi_{-2} \wedge \omega^4, \\
d\omega^4 &= 8\ \varphi_0 \wedge \omega^4, \\
d\varphi_0 &= 0, \quad d\varphi_{-2} = 4\ \varphi_0 \wedge \varphi_{-2}.
\end{aligned}
$$

(54)

These equations can be easily integrated, so $\mathcal{B}^{x^8}$ has coordinates ξ^{-4}, ξ^{-2}, ξ^0, ξ^2, ξ^4, a, and b such that

$$
\begin{aligned}
\varphi_0 &= a^{-1}\ da \\
\varphi_{-2} &= a^4\ db, \\
\omega^4 &= a^8\ d\xi^4, \\
\omega^2 &= a^{12}\left(d\xi^2 - 2b\ d\xi^4\right), \\
\omega^0 &= a^{16}\left(d\xi^0 - 4b\ d\xi^2 + 4b^2\ d\xi^4\right), \\
\omega^{-2} &= a^{20}\left(d\xi^{-2} - 6b\ d\xi^0 + 12b^2\ d\xi^2 - 8b^3\ d\xi^4 - 322560\xi^4\ d\xi^2\right), \\
\omega^{-4} &= a^{24}\big(d\xi^{-4} - 8b\ d\xi^{-2} + 24b^2\ d\xi^0 - 32b^3\ d\xi^2 + 16b^4\ d\xi^4 + \\
&\qquad 8(322560)\xi^4 b\ d\xi^2 - 2(322560)\xi^4\ d\xi^0\big).
\end{aligned}
$$

(55)

The field of rational normal cones can now be written in these local coordinates, since $\mathbf{C}_p = \{\langle u_p(v), u_p(v)\rangle_2\ :\ v \in \mathbf{T}_p M\}$ for a local section u of $\mathcal{B}$.

Each of the 3-dimensional root-types is a single $GL(2)$-orbit. Again, $T(\mathcal{B}) = [v]$ for any v in the root-type, so an arbitrary representative v may be chosen for any $(\mathcal{B}, M, p)$. Each v has a 1-dimensional stabilizer, which is a Lie subgroup of $GL(2)$. The corresponding Lie algebras are easy to compute [24, 1]. In all cases, dT has both vertical and semi-basic components, so the embedding of the stabilizer fiber group varies over M. The reduced structure $\mathcal{B}^v$ is a 6-dimensional Lie group.

If $\dim[v] = 4$, then $\mathcal{B}^v$ is a finite cover of M, but there are only eight such structures, since each root-type with three roots is a single $GL(2)$-orbit. This contrasts with the case $\dim[v] \geq 5$. If $\dim[v] \geq 5$, then $\mathcal{B}^v$ is a finite cover of M, but there is no reason to believe that $T(\mathcal{B}) \subset [v]$ implies $T(\mathcal{B}) = [v]$ when $\dim[v] \geq 5$. Consider $[v] = \{2, 2, 2, 2\}$. The quotient space $\{2, 2, 2, 2\}/GL(2)$ has orbifold singularities; for example, the point $x^2 y^2 (x + y)^2 (x - y)^2$ has an eight-element stabilizer group, but all nearby points have trivial stabilizer groups. The existence of these orbifold singularities implies that there cannot be a surjective smooth map $M \to [v]/GL(2)$. If $T(\mathcal{B})$ is a proper subset of $[v]$ for all $\mathcal{B}$ representing v, then (trivially by the definition of $[v]$) a finite sequence of 2,3-integrable GL(2)-structures connects any two points in the leaf, but infinitely many locally distinct GL(2)-structures are required to cover the entire leaf. This behavior is closely related to the topology of $\mathbb{CP}^n$ with k marked points, up to $GL(2, \mathbb{C})$ action, which is a difficult and well-known problem in complex algebraic geometry. The real case

encountered here is both harder and less well-known than the classical complex case. The most relevant result is [8], which shows that the orbifold singularities present in the larger leaves of $\mathcal{V}_8$ are so bad that they preclude the existence of Riemannian metrics on the leaves.

In the open case, $\dim[v] = 9$, so $J(T)$ has maximum rank, and T itself provides local coordinates on $\mathcal{B}$.

6. GL(2) PDEs

This section contains various conclusions regarding Hessian hydrodynamic PDEs that can be inferred from Theorem 1.4 and Theorem 5.3. Recall that a PDE $F = 0$ is said to have k symmetries if the Lie algebra of point symmetries of $F^{-1}(0)$ has dimension k.

Lemma 6.1. *A Hessian hydrodynamic PDE $F(u_{ij}) = 0$ representing $v \in \mathcal{V}_8$ has k symmetries if and only if $\dim[v] = 9 - k$.*

Proof. The contact transformations by $CSp(3)$ induce the local automorphisms on $\mathcal{B}$, but an automorphism $\varphi : \mathcal{B} \to \mathcal{B}$ near b is a symmetry if and only if the structure equations are preserved by φ^*. This can happen if and only if $T(b)$ is preserved near $\varphi(b)$. In particular, $\varphi^*(\mathrm{d}T) = \mathrm{d}T$ if and only if φ^* is the identity on the range of $\mathrm{d}T$, which has dimension $\dim[v]$. $\qquad\square$

Corollary 6.2. *There is no Hessian hydrodynamic PDE with exactly eight symmetries.*

Proof. There is no root-type of dimension one. $\qquad\square$

The classification provides even more bountiful information about the Hessian hydrodynamic PDEs, because Theorem 4.3 allows a converse of Theorem 1.4.

Theorem 6.3. *Every $(\mathcal{B}, M, p)$ is realized, locally near p, by a hydrodynamic PDE of Hessian type.*

Proof. The structure $(\mathcal{B}, M, p)$ is realized, locally near p, by a Hessian hydrodynamic PDE as in Section 1 if and only if there exists a neighborhood $M' \subset M$ of p and an embedding $i : M' \to \Lambda^o$ such that the distribution of rational-normal cones $i_*(\mathbf{C}(M'))$ over $i(M')$ is fixed by the fiber action of the bundle $\Pi : CSp(3)^o \to \Lambda^o$.

Since the fibers $\mathcal{B}$ are exactly the symmetries of $\mathbf{C}$, it suffices to establish a bundle immersion $h : \mathcal{B}(M') \to CSp(3)^o$ covering an embedding $i : M' \to \Lambda^o$.

Let μ denote the Maurer–Cartan form of $CSp(3)$, which is of the form

$$(56) \qquad \mu = \begin{pmatrix} \beta & \gamma \\ \alpha & -\beta^t \end{pmatrix}, \ \alpha = \alpha^t, \ \gamma = \gamma^t.$$

(One may assume that the conformal scaling has been incorporated into β, as it is below.)

Recall the Fundamental Lemma of Lie Groups [20, Theorem 1.6.10]: If there exists $\eta : \mathbf{T}\mathcal{B} \to \mathfrak{csp}(3)$ such that $\mathrm{d}\eta + \eta \wedge \eta = 0$, then for any $b \in \mathcal{B}$, there exists a neighborhood $\mathcal{B}'$ of b and a map $h : \mathcal{B}' \to CSp(3)$ such that $h^*(\mu) = \eta$. Moreover, if $h^*(\alpha)$ is semi-basic, then the fibers of $\mathcal{B}'$ immerse into the fibers of $CSp(3)$.

Therefore, it suffices to construct an $\mathfrak{csp}(3)$-valued Maurer–Cartan form η on $\mathcal{B}$ such that the component $h^*(\alpha)$ is semi-basic on $\mathcal{B}$. For brevity, the pull-backs $h^*(\cdot)$ are dropped from the notation henceforth.

The condition that α is semi-basic is that $\alpha_{ij} = A_{ija}\omega^a$. Note that $\alpha : \mathcal{V}_4 \to \mathrm{Sym}^2(\mathbb{R}^3) = \mathrm{Sym}^2(\mathcal{V}_2)$, and recall that for $u, v \in \mathcal{V}_2$ the symmetric tensor is given by

(57)
$$\begin{pmatrix} u^{-2} \\ u^0 \\ v^2 \end{pmatrix} \circ \begin{pmatrix} v^{-2} \\ v^0 \\ v^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u^{-2}v^{-2} + v^{-2}u^{-2} & u^{-2}v^0 + v^{-2}u^0 & u^{-2}v^2 + v^{-2}u^2 \\ u^0v^{-2} + v^0u^{-2} & u^0v^0 + v^0u^0 & u^0v^2 + v^0u^2 \\ u^2v^{-2} + v^2u^{-2} & u^2v^0 + v^2u^0 & u^2v^2 + v^2u^2 \end{pmatrix}.$$

Therefore, to respect the weights in the $GL(2)$ representation, α must have the following form for constants A_{-4}, A_{-2}, A_0, A'_0, A_2, and A_4:

(58)
$$\alpha = \begin{pmatrix} A_{-4}\,\omega^{-4} & A_{-2}\,\omega^{-2} & A_0\,\omega^0 \\ A_{-2}\,\omega^{-2} & A'_0\,\omega^0 & A_2\,\omega^2 \\ A_0\,\omega^0 & A_0\,\omega^2 & A_4\,\omega^4 \end{pmatrix}.$$

Writing $\beta = \beta_\varphi(\varphi) + \beta_\lambda(\lambda) + \beta_T(\omega)$, it is apparent that β_φ must be the representation $\mathfrak{sl}(2) \to M_{3\times 3}(\mathbb{R})$ such that the natural action of $M_{3\times 3}(\mathbb{R})$ on $\mathrm{Sym}^2(\mathcal{V}_2)$ is induced by the natural action of $\mathfrak{sl}(2)$ on $\omega \in \mathcal{V}_4$. One can now easily verify the following formulas:

(59)
$$\alpha = \begin{pmatrix} \omega^{-4} & \omega^{-2} & \omega^0 \\ \omega^{-2} & \omega^0 & \omega^2 \\ \omega^0 & \omega^2 & \omega^4 \end{pmatrix},$$
$$\beta = \begin{pmatrix} 4\varphi_0 & 2\varphi_2 & 0 \\ -4\varphi_{-2} & 0 & 4\varphi_2 \\ 0 & -2\varphi_{-2} & -4\varphi_0 \end{pmatrix} - \frac{1}{2}\lambda I_3 + \beta_T(\omega).$$

Here, the $-\frac{1}{2}\lambda I_3$ component of β is simply the scaling action of $GL(2)$ as represented by the scaling action in $CSp(3)$. If $PGL(2)$ and $Sp(3)$ were used instead, it would not appear.

All that remains is to find β_T and γ such that $\mathrm{d}\eta + \eta \wedge \eta = 0$. This is merely arithmetic, and solutions exist. The simplest η (the one where the undetermined coefficients in β_T are set to 0) is provided in [23]. $\qquad\square$

Theorem 6.3 allows explicit construction of several of the most-symmetric Hessian hydrodynamic PDEs. Each of the root-types with three or fewer roots is itself closed under $GL(2)$; hence, the torsion of any GL(2)-structure covers the entire root-type. This allows a first step at constructing all Hessian hydrodynamic PDEs. However, Theorem 6.3 does not say that there are only 55 Hessian hydrodynamic PDEs up to $CSp(3)$ actions. There is no reason to believe that the torsion map $T : \mathcal{B} \to \mathcal{V}_8$ is surjective for the larger root-types. When T fails to be surjective, there cannot be a single PDE that represents all possible torsions in its root-type. None-the-less, the leaves of dimension at most 4 can be used to fully describe the most symmetric Hessian hydrodynamic PDEs, as in the following theorems.

Theorem 6.4. *The root-type $\{0\}$ is represented by the wave equation,*

(60)
$$u_{22} = u_{13}$$

and this representation is unique up to $CSp(3)$. Therefore, the wave equation is the unique Hessian hydrodynamic PDE with nine symmetries.

Theorem 6.5. *The root-type $\{8\}$ is represented by the first flow of the dKP equation,*

$$(61) \qquad u_{22} = u_{13} - \frac{1}{2} \left(u_{33} \right)^2$$

and this representation is unique up to $CSp(3)$. Therefore, the first flow of the dKP equation is the unique hydrodynamic PDE with seven symmetries.

Theorem 6.6. *The root-type $\{7,1\}$ is represented by the PDE*

$$(62) \qquad u_{22} = u_{13} - \frac{1}{48} u_{33} + \frac{1}{2} u_{33} u_{23}$$

and this representation is unique up to $CSp(3)$.

Theorem 6.7. *The root-type $\{6,2\}$ is represented by the PDE*

$$(63) \qquad u_{22} = u_{13} + \frac{7\ u_{23}}{5 u_{33} - 14}$$

and this representation is unique up to $CSp(3)$.

Theorem 6.8. *The root-type $\{6,1,1\}$ is represented by the PDE*
(64)

$$u_{22} = u_{13} + \frac{7 u_{23} \left(u_{23} - u_{33} \right)}{\left(5 u_{33} - 14 \right)} + \frac{49(-(u_{33})^2 + 14 u_{33} - 28)}{12(5 u_{33} - 14)} - \frac{49}{6} \left(\frac{-(5 u_{33} - 14)}{14} \right)^{2/5}$$

and this representation is unique up to $CSp(3)$.

Theorems 6.4 through 6.8 are proven via the same technique. For illustration, consider the simplest non-trivial case, Theorem 6.5.

Proof. Consider a 2,3-integrable GL(2)-structure $\pi : \mathcal{B} \to M$ with $T(b) = x^8 \frac{1}{322560}$, so $[T(\mathcal{B})] = \{8\}$. The constant is chosen for the aesthetic appeal of the resulting PDE. The goal is to describe Theorem 6.3's embedded hyper-surface $i(M') = \{\Pi(h(q)) \in \Lambda^o : q \text{ near } b\}$ as the locus of a single equation $F(U) = 0$.

As $CSp(3)^o$ is a matrix group, the open set $\{g = h(q) : q \text{ near } b\}$ can be described as $\{\exp_I(\eta_b(v)) : v \in \mathbf{T}_b \mathcal{B}\}$. Moreover, only $v \in \ker(\varphi, \lambda)$ need be considered, since the fibers of $\pi : \mathcal{B} \to M$ immerse into the fibers of $\Pi : CSp(3)^o \to \Lambda^o$ Fix an arbitrary $v \in \ker(\varphi, \lambda) \subset \mathbf{T}_b \mathcal{B}$ and write v in components using the tautological 1-form, $v \lrcorner \omega = (v_{-4}, v_{-2}, v_0, v_2, v_4) \in \mathbb{R}^5$. Of course, $v \lrcorner \omega$ does not actually provide local coordinates on $\mathcal{B}$ or M; however, the matrix $\eta_b(v)$ still represents a generic point in $h_*(\mathbf{T}_b(\mathcal{B}))$, as seen here:

$$(65) \qquad \eta_b(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -v_4 & 0 & 0 & 0 & 0 & 0 \\ v_{-4} & v_{-2} & v_0 & 0 & 0 & v_4 \\ v_{-2} & v_0 & v_2 & 0 & 0 & 0 \\ v_0 & v_2 & v_4 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$(66) \qquad \exp_I(\eta_b(v)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -v_4 & 0 & 1 & 0 & 0 & 0 \\ v_{-4} - \frac{1}{6}v_4^3 & v_{-2} + \frac{1}{2}v_4v_2 & v_0 + \frac{1}{2}v_4^2 & 1 & 0 & v_4 \\ v_{-2} - \frac{1}{2}v_4v_2 & v_0 & v_2 & 0 & 1 & 0 \\ v_0 - \frac{1}{2}v_4^2 & v_2 & v_4 & 0 & 0 & 1 \end{pmatrix}.$$

So, using Equation (14), a generic point $U \in i(M')$ looks like

$$(67)$$
$$U = \Pi(\exp_I(\eta_p(v))) = \begin{pmatrix} v_{-4} - \frac{1}{6}v_4^3 + (v_0 + \frac{1}{2}v_4^2)v_4 & v_{-2} + \frac{1}{2}v_4v_2 & v_0 + \frac{1}{2}v_4^2 \\ v_{-2} + \frac{1}{2}v_4v_2 & v_0 & v_2 \\ v_0 + \frac{1}{2}v_4^2 & v_2 & v_4 \end{pmatrix}.$$

There is a single relation between the entries of such U:

$$(68) \qquad\qquad\qquad U_{22} = U_{13} - \frac{1}{2}(U_{33})^2.$$

When U is interpreted as the Hessian of $u : \mathbb{R}^3 \to \mathbb{R}$, Equation (68) is the first flow of the dKP equation, a well-known example of a Hessian hydrodynamic equation. $\square$

The only change for the other root-types is that η is more complicated; hence, its exponential is (immensely) more difficult to compute, and the relation $F(U) = 0$ is more difficult to recognize. Note also that $U = 0$ is always in the locus of the equation obtained by this procedure. Therefore, PDEs such as the Boyer–Finley equation, $u_{xx} + u_{yy} = e^{u_{tt}}$ (which has six symmetries and must represent one of the 3-dimensional root-types [11]), will not directly appear as representatives via this procedure. None-the-less, every $CSp(3)$ equivalence class of Hessian hydrodynamic PDEs must arise this way.

6.1. **Hyperbolic Planar PDEs.** This section presents some preliminary but intriguing observations regarding the hyperbolic linear Pfaffian system $\mathcal{I}$ describing bi-secant surfaces in Theorem 3.1 and its relation to hyperbolic second-order planar PDEs,

$$(69) \qquad\qquad f(\xi^1, \xi^2, z, z_1, z_2, z_{11}, z_{12}, z_{22}) = 0.$$

Equation (69) defines a 7-dimensional manifold $\Sigma_f = f^{-1}(0) \subset \mathbb{J}^2(\mathbb{R}^2, \mathbb{R})$ whose structure equations are obtained by pulling back the contact system [15, 27], such Σ_f admit a point-wise classification into Monge–Ampère, Goursat, or generic-type equations.

Theorem 6.9. *Consider a $(\mathcal{B}, M, p)$ with $T(b) = v$ for some $b \in \mathcal{B}_p$. Over a neighborhood M' of p, there a bundle $W \to M'$ with 7-dimensional fiber and a submersion $f : \mathcal{B}(U) \to W$ such that the ideal $\mathcal{I}$ from Theorem 3.1 describing the existence of bi-secant surfaces through p is the pull-back of a hyperbolic linear Pfaffian ideal $\bar{\mathcal{I}}$ on W. This W admits a co-frame $(\beta^1, \ldots, \beta^7)$ such that*

$$(70) \qquad \begin{aligned} \mathrm{d}\beta^1 &\equiv \beta^2 \wedge \beta^4 + \beta^3 \wedge \beta^6, & \mod \beta^1 \\ \mathrm{d}\beta^2 &\equiv U_1\, \beta^3 \wedge \beta^7 + \beta^4 \wedge \beta^5, & \mod \beta^1, \beta^2 \\ \mathrm{d}\beta^3 &\equiv U_2\, \beta^2 \wedge \beta^5 + \beta^6 \wedge \beta^7, & \mod \beta^1, \beta^3. \end{aligned}$$

for $U_1 = -645120T_{-8}\frac{(\mu_3)^3}{\mu_2\mu_1}$ and $U_2 = -645120T_8\frac{(\mu_2)^3}{\mu_3\mu_1}$ for some non-zero functions μ_1, μ_2, and μ_3 on W.

Moreover, there is some second-order hyperbolic planar PDE f and a diffeomorphism $\varphi : \Sigma_f \to W$ such that these structure equations on W pull back via φ^* to the contact-induced structure equations on Σ_f.

Proof. Consider a 2,3-integrable GL(2)-structure $(\mathcal{B}, M, p)$. As in the general case of Theorem 3.1, the linear Pfaffian system describing the existence of bi-secant surfaces through p is differentially generated by ω^{-2}, ω^0, and ω^2, and it has tableau given by

$$
(71) \qquad \mathrm{d}\begin{pmatrix} \omega^{-2} \\ \omega^0 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} \pi_1 & 0 \\ 0 & 0 \\ 0 & \pi_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^{-2} \\ \omega^4 \end{pmatrix}
$$

for $\pi_1 = 2\varphi_2 - 322560T_{-2}\omega^4$ and $\pi_2 = -2\varphi_{-2} - 322560T_2\omega^{-4}$. The Lie algebra $A(\mathcal{I})$ of Cauchy characteristics of $\mathcal{I}$ is spanned by the duals of λ and φ^0. Therefore, the retracting space $C(\mathcal{I}) = A(\mathcal{I})^\perp$ is a rank 7 Frobenius system on $\mathcal{B}$, so $\mathcal{B}$ admits a foliation by 2-dimensional Cauchy characteristic surfaces [2, Section II§2] [20, Section 6.1]. Let W denote the 7-dimensional (local) leaf space for this foliation, so there is a submersion $\tilde{\pi} : \mathcal{B} \to W$. Since the ω is semi-basic for the submersion $\tilde{\pi}$, W also admits a submersion onto a neighborhood of $p \in M$. It remains to find the structure equations for a coframing on W.

Write $\alpha^1 = \omega^0$, $\alpha^2 = \omega^{-2}$, $\alpha^3 = \omega^2$, $\alpha^4 = \pi_1$, $\alpha^5 = \omega^{-4}$, $\alpha^6 = \pi_2$, $\alpha^7 = \omega^4$, $\alpha^8 = \varphi_0$, and $\alpha^9 = \lambda$ as a co-frame for $\mathcal{B}$. To simplify the notation, fix the index convention $1 \le i, j, k \le 7$ and $8 \le r, s \le 9$, so the forms α^i are semi-basic for the bundle $\mathcal{B} \to W$, and the forms α^r are vertical for the bundle $\mathcal{B} \to W$. Write $\mathrm{d}\alpha^i = -\frac{1}{2}C^i_{jk}\alpha^j \wedge \alpha^k - C^i_{jr}\alpha^j \wedge \alpha^r$, where $C^i_{jk} = -C^i_{kj}$ and C^i_{ja} are functions of T as determined by Theorem 4.3. Note that $C^i_{ja} = 0$ if $i \ne j$, and

$$
(72) \qquad C^1_{18} = 0, C^2_{28} = -4, C^3_{38} = 4, C^4_{48} = 4, C^5_{58} = -8, C^6_{68} = -4, C^7_{78} = 8,
$$

$$
(73) \qquad C^1_{19} = 1, C^2_{29} = 1, C^3_{39} = 1, C^4_{48} = 0, C^5_{58} = 1, C^6_{68} = 0, C^7_{78} = 1.
$$

Corollary 2.3 of [2] and implies that there exist functions μ_i on $\mathcal{B}$ such that $\frac{1}{\mu_i}\alpha^i$ is basic. Define a new co-framing $(\tilde{\alpha}^i)$ for $\mathcal{B}$ by setting $\tilde{\alpha}^i = \frac{1}{\mu_i}\alpha^i$ (no sum) and $\tilde{\alpha}^r = \alpha^r$. Thus, $\mathbf{T}^*\mathcal{B}$ is (locally) split into basic 1-forms and vertical 1-forms with respect to the bundle $\tilde{\pi} : \mathcal{B} \to W$. Of course, $\tilde{\alpha}^i = \tilde{\pi}^*(\beta^i)$ for some independent 1-forms β^i, and $(\beta^1, \ldots, \beta^7)$ is the desired co-framing for W. Since $\tilde{\alpha}^i$ is basic, the structure equations of $(\tilde{\alpha}^i)$ and (β^i) are identical.

There is a lot of freedom in the designation of μ_i. Write $\mathrm{d}\mu_i = \mu_{i,j}\tilde{\alpha}^j + \mu_{i,r}\tilde{\alpha}^r$. The condition that $\tilde{\alpha}^i$ is basic is equivalent to $\mu_{i,r} = \mu_i C^i_{ir}$ for all $1 \le i \le 7$ and $8 \le r \le 9$, but $\mu_{i,j}$ is otherwise free, so any non-zero function f_i on W may be lifted to $\mathcal{B}$ in local coordinates $(x^1, \ldots, x^9)$ by setting $\mu_i = f_i \exp(C^i_{i8}x^8 + C^i_{i9}x^9)$.

At this point, the co-frame (β^i) satisfies

$$
(74) \qquad
\begin{aligned}
\mathrm{d}\beta^1 &\equiv 0, \\
\mathrm{d}\beta^2 &\equiv \frac{\mu_4\mu_5}{\mu_2}\beta^4 \wedge \beta^5, \quad \mod \{\beta^1, \beta^2, \beta^3\} \\
\mathrm{d}\beta^3 &\equiv \frac{\mu_6\mu_7}{\mu_3}\beta^6 \wedge \beta^7,
\end{aligned}
$$

so one may effectively eliminate μ_5 and μ_7 by redefining β^5 as $\frac{\mu_4\mu_5}{\mu_2}\beta^5$ and β^7 as $\frac{\mu_6\mu_7}{\mu_3}\beta^7$. Thus, the structure equations for W satisfy

$$
\begin{aligned}
\mathrm{d}\beta^1 &\equiv 0, \\
(75) \qquad\qquad \mathrm{d}\beta^2 &\equiv \beta^4 \wedge \beta^5, \quad \mod \{\beta^1,\beta^2,\beta^3\} \\
\mathrm{d}\beta^3 &\equiv \beta^6 \wedge \beta^7.
\end{aligned}
$$

One may now re-label the co-frame (β^i) following the procedure given in Appendix A of [27] to obtain Equation (70).

Although U_1 and U_2 as written in the theorem are not explicitly functions on W, this is easily amended. As noted above, μ_i may be taken as the lift of f_i on W. Also, the action of α^8 and α^9 is diagonal on T:

$$
(76) \qquad
\mathrm{d}\begin{pmatrix} T_{-8} \\ T_{-6} \\ T_{-4} \\ T_{-2} \\ T_0 \\ T_2 \\ T_4 \\ T_6 \\ T_8 \end{pmatrix}
\equiv
\begin{pmatrix} T_{-8} \\ T_{-6} \\ T_{-4} \\ T_{-2} \\ T_0 \\ T_2 \\ T_4 \\ T_6 \\ T_8 \end{pmatrix}\alpha^8 +
\begin{pmatrix} 16T_{-8} \\ 12T_{-6} \\ 8T_{-4} \\ 4T_{-2} \\ 0 \\ -4T_2 \\ -8T_4 \\ -12T_6 \\ -16T_8 \end{pmatrix}\alpha^9, \quad \mod \alpha^1,\ldots,\alpha^7.
$$

So, while there is no natural map from W to $\mathcal{O}_J(\mathcal{B})$, the local Lie group G generated by the fiber actions of H and I_9 (corresponding to α^8 and α^9) induce a map $W \to \mathcal{O}_J(\mathcal{B})/G$ that one could also call T.

Theorem 11.1.1 of [26] implies that any 7-dimensional manifold with structure equations of this form must be diffeomorphic to Σ_f for some hyperbolic planar PDE f. $\qquad\square$

Corollary 6.10. *For any Hessian hydrodynamic PDE, the level sets of the Riemann invariants are the solutions of planar hyperbolic PDEs.*

Proof. The manifold W arises from the ideal describing bi-secant surfaces in M, and Corollary 4.2 shows that these are the level sets of the Riemann invariants of the Hessian hydrodynamic PDE defining $\mathcal{B}$. $\qquad\square$

This corollary is not at all surprising, since the entire point of hydrodynamic reduction is to reduce a hyperbolic PDE in three variables to a family of hyperbolic planar PDEs defined by the Riemann invariants.

In the case of 2,3-integrability, Corollary 6.10 also provides a more explicit justification for the claim in Theorem 3.1 that the PDE defining bi-secant surfaces can be solved in the smooth category with smooth initial data. One naturally asks "to which Σ_f is W equivalent?"

Corollary 6.11. *Let $\Sigma_f = f^{-1}(0) \subset \mathbb{J}^2(\mathbb{R}^2,\mathbb{R})$ for a hyperbolic planar PDE f. Suppose there is a local diffeomorphism $\psi : \Sigma_f \to W$ with $\tilde{\pi}(b) \in \psi(\Sigma_f)$.*

    (1) *If $x = 0$ and $y = 0$ are both roots of $T(b)$, then Σ_f is of the Monge–Ampère type at $\psi^{-1}(\tilde{\pi}(b))$,*

    (2) *If exactly one of $x = 0$ or $y = 0$ is a root of $T(b)$, then Σ_f is of the Goursat type at $\psi^{-1}(\tilde{\pi}(b))$,*

(3) *If neither $x = 0$ nor $y = 0$ is a root of $T(b)$, then Σ_f is of the generic type at $\psi^{-1}(\tilde{\pi}(b))$.*

Proof. These are the three pointwise types of planar hyperbolic PDEs, and for structure equations of the form in Equation (70), they are determined by whether U_1 and U_2 vanish [15] [27]. Since $U_1 \sim T_{-8}$, $U_1 = 0$ if and only if $y = 0$ is a root of $T(b) \in \mathcal{V}_8$. Similarly, $U_2 = 0$ if and only if $x = 0$ is a root of $T(b) \in \mathcal{V}_8$. $\square$

Corollary 6.11 puts interesting restrictions on which GL(2)-structures can yield which planar PDEs. In particular, $x = 0$ and $y = 0$ can both be roots of $v = T(b)$ if and only if $[v]$ is a root-type having two distinct real roots, and the strictly complex root types cannot have $x = 0$ or $y = 0$ as roots. Notably, the type of W can change. For example, suppose $T(b) = x^7 y$, so $\psi^{-1}(\tilde{\pi}(b))$ is of the Monge–Ampére type but nearby $\psi^{-1}(\tilde{\pi}(b'))$ is of the generic type, since $T(b') = (x+\varepsilon_1 y)^7(\varepsilon_2 x+y)$. This type-changing does not occur for the flat structure, and it is easy to compute a change-of-frame from the structure equations of the flat W to the structure equations of Σ_f for the planar wave equation $z_{12} = 0$.

Corollary 6.12. *Let $(\mathcal{B}, M, p)$ be a flat 2,3-integrable GL(2)-structure. Then W is isomorphic to $\{z_{12} = 0\} \subset \mathbb{J}^2(\mathbb{R}^2, \mathbb{R})$.*

7. Concluding Remarks

The main results of this article are summarized in Figure 2. In short, Hessian hydrodynamic PDEs in three independent variables are equivalent to local 2,3-integrable GL(2)-structures of degree 4, and both objects admit a geometric, coordinate-free classification by the foliation of $\mathbb{R}^9$ shown in the figure.

Lemma 5.1 seems to be a miraculous coincidence. The bluntness of this relationship between v and $J(v)$ prompted me to investigate relationships between the roots of v and the structure of the leaf $\mathcal{O}_J(v)$, yielding this project's main result, Theorem 5.3. It appears that no such relationship holds for 2,3-integrable GL(2)-structures of degree $n \geq 5$, even though versions of Theorem 4.3 and Lemma 4.7 exist in those cases. In general, intransitive groupoids and pseudo-groups are very poorly understood, and it is generally impossible to explicitly write down the integral manifolds of any given singular distribution. If it were not for the coincidence in this case, the leaf-equivalence classes would have little utility in understanding the Hessian hydrodynamic PDEs.

Of course, Hessian hydrodynamic PDEs on $u : \mathbb{R}^3 \to \mathbb{R}$ are not the only integrable PDEs of interest in mathematics and physics. Two generalizations are important to consider:

Q1 Integrability should be a contact-invariant property of a PDE. The Hessian-only form $F(u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0$ is not preserved under the full family of contact transformations. However, the associated GL(2)-structures apparently relied on this form and its associated $CSp(3)$ transformations. How can this classification be extended to second-order PDEs in three independent variables that also include lower derivatives? How does the GL(2) geometry generalize to these PDEs? The observations and computations of some recent articles may prove very useful [6] [12].

FIGURE 2. The leaf-classification of all $(\mathcal{B}, M, p)$. The number of sides on a node is the dimension of the reduced bundle. Representative PDEs for the shaded nodes are included in Section 6.

Q2 Very few integrable PDEs are known to exist in more than three independent variables, but equations of the form $F(u_{11}, \ldots u_{NN}) = 0$ can sometimes yield distributions of rational normal cones of degree n on hypersurfaces $M^n = F^{-1}(0) \subset \mathrm{Sym}^2(\mathbb{R}^N)$ with $n = \frac{1}{2}N(N+1) - 2$. Results similar to those in Section 4 are known for 2,3-integrability for degrees 5 through 20, but the foliation by groupoid orbits of $\mathcal{V}_{n+4}$ is not understood, and k-integrability is extremely restrictive [24]. What can integrable GL(2) geometry say about the existence of integrable PDEs in more variables?

On a more detailed level, it would be interesting to study the foliation that appears in the present case. Despite the significant computational difficulties, it is important both to produce representative PDEs for each root-type and to find the root-type of the well-known Hessian hydrodynamic PDEs. For example, if one can produce Hessian hydrodynamic PDEs as well as contact-equivalent PDEs that involve lower-order terms, such computations could open the door to Q1, above. Additionally, the lack of surjectivity of $T : \mathcal{B} \to \mathcal{O}_J(\mathcal{B})$ is irritating. What is the exact relationship between two PDEs that are leaf-equivalent but do not have overlapping torsion?

Finally, the relationship between W, which describes bi-secant surfaces in M, and Σ_f, which arises from a planar hyperbolic PDE, is worth pursuing. What is is the nature of the correspondence? Can every hyperbolic planar PDE appear this way? Can this correspondence provide any new information about the Riemann invariants of the Hessian hydrodynamic PDE?

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McGill University, Department of Mathematics and Statistics, Montreal, Quebec H3A 2K6, Canada
  E-mail address: adsmith@msri.org
  URL: http://www.math.mcgill.ca/adsmith