LEMNISCATES DO NOT SURVIVE LAPLACIAN GROWTH

D. KHAVINSON, M. MINEEV-WEINSTEIN, M. PUTINAR, AND R. TEODORESCU

1. Introduction

Many moving boundary processes in the plane, e.g., solidification, electrodeposition, viscous fingering, bacterial growth, etc., can be mathematically modeled by the so-called Laplacian growth [9,13]. In a nutshell, it can be described by the equation

$$V(z) = \partial_n g_{\Omega(t)}(z, \zeta), \tag{1.1}$$

where V is the normal component of the velocity of the boundary $\partial\Omega(t)$ of the moving domain $\Omega(t) \subset \mathbb{R}^2 \simeq \mathbb{C}$, $z \in \partial\Omega(t)$, t is time, $\dfrac{\partial}{\partial n}$ denotes the normal derivative on $\partial\Omega(t)$ and $g_{\Omega(t)}(z, \zeta)$ is the Green function for the Laplace operator in the domain $\Omega(t)$ with a unit source at the point $\zeta \in \Omega(t)$. Equation (1.1) can be elegantly rewritten as the area-preserving diffeomorphism

$$\Im\left(\bar{z}_t z_\theta\right) = 1, \tag{1.2}$$

where $\Im$ denotes the imaginary part of a complex number, $\partial\Omega(t) := \{z := z\,(t, \theta)\}$ is the moving boundary parametrized by $w = e^{i\theta}$ on the unit circle and the conformal mapping from, say, the exterior of the unit disk $\mathbb{D}^+ := \{|w| > 1\}$ onto $\Omega(t)$ with the normalization $z(\infty) = \zeta, z'(\infty) > 0$.

The equation (1.2), named *Laplacian growth* or the *Polubarinova - Galin* equation in modern literature, was first derived by Polubarinova-Kochina [11] and Galin [7] in 1945, as a description of secondary oil recovery processes.

This equation is known to be integrable [10], and as such possesses an infinte number of conserved quantities. More precisely, it admits

Part of this work was done during the first and third authors' visit to LANL and was supported by the LDRD project 20070483 "Minimal Description of Complex 2D Shapes" at LANL. Also, D. Khavinson and M. Putinar gratefully acknowledge partial support by the National Science Foundation.

conserved moments $c_n = \int_{\Omega(t)} z^n dx\, dy$, where n runs over either all non-negative or all non-positive integers depending on whether domains $\Omega(t)$ are finite or infinite. At the same time (1.2) admits an impressive number of closed-form solutions.

For the background, history, generalizations, references, connections to the theory of quadrature domains and other branches of mathematical physics we refer the reader to [4, 8–10, 12, 13] and the references therein.

In section §2 of this paper, we show that any continuous chain of polynomial lemniscates of order n: $\Gamma_t := \{|P(z,t)| = 1\}$, $P(z,t) = a(t)\prod_{j=1}^{n}(z - \lambda_j(t))$, where $a(t)$ is real-valued, is destroyed instantly under the Laplacian growth process described in (1.1), with $\Omega(t) = \{|P(z,t)| > 1\}$, $\zeta = \infty$, unless $n = 1$, $\lambda_1(t) = \text{const}$ and $\{\Gamma_t = \partial\Omega(t)\}$ is simply a family of concentric circles. Here the roots $\lambda_j(t)$ of $P(z,t)$ are all assumed to be inside $\Omega'_t := \{|P(z,t)| < 1\}$, so Ω and Ω' are simply connected.

This result shows that unlike quadrature domains (cf. [4,12]) that are preserved under the Laplacian growth process, lemniscates for which all the roots of the defining polynomial are in Ω'_t are instantly destroyed, except for the trivial case of concentric circles. This, incidentally, agrees with a well-known fact — cf. [5] — that lemniscates which are also quadrature domains must be circles. The proof of the theorem for the case of Laplacian growth is given in §2.

In §3 we extend the result of §2 to all the growth processes that are invariant under time-reversal and for which the boundary velocity is given by

$$V(z) = \chi(z)\partial_n g_{\Omega(t)}(z,\zeta), \tag{1.3}$$

with $\chi(z)$ is a bounded, real, positive function on Γ_t. Invariance under time-reversal is defined here in the following way: if the boundary Γ_{t+dt} is the image of Γ_t under a map $f_{(t,dt)} : z_t \in \Gamma_t \mapsto z_{t+dt} \in \Gamma_{t+dt}$, then $f_{(t+dt,-dt)} \circ f_{(t,dt)} = \mathbb{I}$.

We conclude with a few remarks in §4.

2. Destruction of Lemniscates

Theorem 2.1. *Suppose that a family of moving boundaries Γ_t, (where $t > 0$ is time), produced by a Laplacian growth process, is a family of polynomial lemniscates $\{|P(z,t)| = 1\}$, where $P(z,t) = a(t)\prod_{j=1}^{n}[z - \lambda_j(t)]$,* *and all $\lambda_j(t)$ are assumed to be inside Γ_t. Then, $n = 1$ and $\lambda_1 = \text{const}$, i.e., Γ_t is a family of concentric circles.*

Proof. Let $\Omega_t = \{z : |P(z,t)| > 1\}$, $\mathbb{D}^+ = \{|w| > 1\}$. The function $\varphi(t) : \Omega_t \to \mathbb{D}^+$, $w = \varphi(z,t) = \sqrt[n]{P(z,t)}$, where we choose the branch for the $n-$th root so that $\varphi'(t,\infty) > 0$, maps Ω_t conformally onto $\mathbb{D}_+$, $\varphi(t,\infty) = \infty$. It is useful to note that on Γ_t, $P(z,t) = w^n$, $|w| = 1$ and does not depend on t. This is because for any two moments of time t, τ, we have $w(t)(.) = w(\tau) \circ \kappa(t,\tau)(.)$, where κ is a Möbius automorphism of the disk. In our case, $\kappa(t,\tau)(\infty) = \infty$, so $\kappa(t,\tau)(z) = e^{i\alpha}z, \alpha \in \mathbb{R}$, but since it also fixes the argument at ∞, κ is the identity.

Therefore, we have (where, as is customary, we denote the partial t-derivative by a "dot"):

$$\dot{P} + P'_z \dot{z} = 0. \tag{2.1}$$

Since $\varphi(t)$ maps Γ_t onto the unit circle, we have $z(t) = \Psi(t,w)$, where $\Psi(t,w) = \varphi^{-1}(t,z)$. We also have on Γ_t, by differentiating $P(z(w),t) = w^n$ with respect to w,

$$P'_z \cdot z_w = nw^{n-1} \tag{2.2}$$

or

$$wz_w = \frac{nw^n}{P'_z} = \frac{nP}{P'_z}. \tag{2.3}$$

From (2.1), conjugating, we infer

$$\bar{\dot{z}} = -\frac{\overline{\dot{P}}}{\overline{P'_z}}. \tag{2.4}$$

Parametrize the unit circle by $w = e^{i\theta}$, $0 \le \theta \le 2\pi$. Then, from (2.3), it follows that we have on Γ_t (since $(z(t) = z(w,t) = z(w(\theta,t))))$,

$$\frac{1}{i}\, z_\theta := \frac{\partial z}{i\partial\theta} = z_w w = \frac{nP}{P'_z}. \tag{2.5}$$

Combining (2.4) and (2.5) yields ($\Re$ stands for the real part):

$$\Re\left(\bar{\dot{z}}\frac{1}{i}\, z_\theta\right) = \Re\left(-\frac{\overline{\dot{P}}}{\overline{P'_z}} \cdot \frac{nP}{P'_z}\right). \tag{2.6}$$

Also,

$$\Re\left(\bar{\dot{z}}\frac{\partial z}{i\partial\theta}\right) = \Im\left(\bar{\dot{z}} z_\theta\right) = 1, \tag{2.7}$$

where in the last equality we used the hypothesis that the lemniscates $\Gamma_t := \{z(t,\theta)\}$ satisfy the main equation (1.2) of Laplacian growth processes— cf. [8, §4]. Hence, (2.7), (2.4) and (2.5) imply that

$$\frac{1}{n}\Re\left(\frac{\dot{P}}{P'_z}\overline{iz_\theta}\right) = \Re\left(\frac{\dot{P}}{P'_z}\frac{\overline{P}}{\overline{P'_z}}\right) = \frac{-1}{n}. \tag{2.8}$$

Or, we can rewrite (2.8) as

$$\Re\left(\dot{P}\overline{P}\right) = -\frac{1}{n}\left|P'_z\right|^2. \tag{2.9}$$

Thus, we are finally arriving at

$$\frac{d}{dt}\left(|P|^2\right) = -\frac{1}{2n}\left|P'_z\right|^2. \tag{2.10}$$

Therefore, (2.10) holds on the lemniscates $\Gamma_t = \{|P(z,t)| = 1\}$ that are assumed to be interfaces of a Laplacian growth process. Now the theorem follows from the following.

Lemma 2.1. *Let t be the time variable, $P(z,t) = a(t)\prod_1^n(z-\lambda_i(t))$, be a "flow" of n-degree polynomials. Assume that the lemniscates $\Gamma_t := \{|P(z,t)| = 1\}$ all have connected interiors $\{|P(z,t)| < 1\}$ and a generalized equation* (2.10) *holds on Γ_t; i.e.,*

$$\frac{d}{dt}\left(|P(z,t)|^2\right) - c(t)\left|P'_z(z,t)\right|^2 = 0, \tag{2.11}$$

where the function $c(t)$ is real-valued, depends on t only and, hence, is a constant on Γ_t. Then, $n = 1$, $\lambda_1 = \lambda_1(t) = \text{const}$ and Γ_t is a family of concentric circles centered at λ_1.

Proof of the Lemma. Our hypothesis implies that all polynomials $|P(z,t)|^2 - 1$ are irreducible. Hence, using Hilbert's Nullstellensatz (e.g., c.f. [2], Proposition 3.3.2), we infer from (2.11) that

$$\frac{d}{dt}\left(|P(z,t)|^2\right) - c(t)\left|P'_z(z,t)\right|^2 = B(t)\left(|P(z,t)|^2 - 1\right). \tag{2.12}$$

Equation (2.12) holds for all $z \in \mathbb{C}$ and for an interval of time t, and for each t, both sides are real-analytic functions in z and $\overline{z}$. Hence, we can "polarize" (2.12), i.e., replace $\overline{z}$ by an independent complex variable ξ. (This is due to a simple observation: real-analytic functions of two variables are nothing else but restrictions of holomorphic functions in z, ξ-variables to the plane $\{\xi = \overline{z}\}$. Hence, if two real-analytic functions coincide on that plane, they coincide in $\mathbb{C}^2$ as well.) Denoting by $P^\#$ the polynomial whose coefficients are obtained from P by complex conjugation, we have (2.12) in a "polarized" form holding for $(z,\xi)\in\mathbb{C}^2$:

$$\frac{d}{dt}\left(P(z,t)P^{\#}(\xi,t)\right)-c(t)\left(P_z'(z,t)\cdot\left(P^{\#}\right)_\xi'(\xi,t)\right) = B(t)\left(P(z,t)P^{\#}(\xi,t)-1\right). \tag{2.13}$$

Now let us denote by k_j the multiplicity of the root $\lambda_j(t)$ of the polynomial $P(z,t)$, so that there are $m\le n$ distinct roots and $\sum_{j=1}^m k_j=n$. Since

$$P(z,t)=a(t)\prod_1^m (z-\lambda_j(t))^{k_j},$$
$$P^{\#}(\xi,t)=\bar a(t)\prod_1^m \left(\xi-\overline{\lambda_j(t)}\right)^{k_j},$$

dividing by $P(z,t)P^{\#}(\xi,t)$ we obtain:

$$2\Re\left(\frac{\dot a}{a}\right)-\sum_1^m\left(\frac{k_j\dot\lambda_j(t)}{z-\lambda_j(t)}+\frac{k_j\dot{\overline{\lambda}}_j(t)}{\xi-\overline{\lambda_j(t)}}\right) - c(t)\left[\sum_1^m\frac{k_j}{z-\lambda_j(t)}\right]\cdot\left[\sum_1^m\frac{k_j}{\xi-\overline{\lambda_j(t)}}\right] = B(t)\left(1-\frac{1}{P(z,t)P^{\#}(\xi,t)}\right). \tag{2.14}$$

Integrating (2.14) along a small circle centered at $\lambda_j(t)$, so that it does not enclose other zeros of P, yields for all ξ:

$$-k_j\dot\lambda_j(t)-c(t)\left(\sum_1^m\frac{k_ik_j}{\xi-\bar\lambda_i(t)}\right)=-\frac{B(t)}{P^{\#}(\xi,t)}q_j, \tag{2.15}$$

where $q_j=\frac{1}{(k_j-1)!}\left(\frac{\partial}{\partial z}\right)^{k_j-1}\left[\frac{(z-\lambda_j)^{k_j}}{P(z,t)}\right]_{z=\lambda_j}$. Letting $\xi\to\infty$ in (2.15) implies that $\dot\lambda_j(t)=0$ for all $j=1,\dots,n$. In other words, the "nodes" $\lambda_j(t)$ of all the lemniscates Γ_t are fixed, i.e. do not move with time. So,

$$P(z,t)=a(t)\prod_1^n(z-\lambda_j)=a(t)\,Q(z). \tag{2.16}$$

Substituting (2.16) into (2.13), we obtain

$$\frac{d}{dt}\left(|a|^2\right) Q(z)Q^{\#}(\xi) - c(t)|a|^2 Q'_z \left(Q^{\#}\right)'_\xi = B(t)\left(|a|^2 Q(z)Q^{\#}(\xi) - 1\right). \tag{2.17}$$

Comparing the leading terms (i.e., the coefficients at $z^n\xi^n$) in (2.17) yields

$$\frac{d}{dt}\left(|a|^2\right) = B(t)\,|a|^2. \tag{2.18}$$

Therefore,

$$c(t)\,|a|^2 Q'_z \left(Q^{\#}\right)'_\xi = B(t), \tag{2.19}$$

and thus $\deg Q'_z = 0$, i.e., $n = \deg P = 1$. The proofs of the Lemma and the Theorem are now complete. □

3. Extending the theorem to growth processes invariant under time reversal

First, let us note that any boundary Γ_t is an equipotential line of the logarithmic potential

$$\Phi(z) = \log |P_n(z, \lambda_i(t))|^2. \tag{3.1}$$

The boundary velocity of the general growth process defined in (1.3) can now be expressed as $\vec{V}(z) = \chi(z)\vec{\nabla}\Phi, z \in \Gamma_t, \chi(z) \in \mathbb{R}_+$.

As indicated in the Introduction, invariance under time-reversal is defined here in the following way: if the boundary Γ_{t+dt} is the image of Γ_t under a map $f_{(t,dt)} : z_t \in \Gamma_t \mapsto z_{t+dt} \in \Gamma_{t+dt}$, then $f_{(t+dt,-dt)} \circ f_{(t,dt)} = \mathbb{I}$. That means that the normal at $z_{t+dt} \in \Gamma_{t+dt}$ must be parallel to the normal at $z_t \in \Gamma_t$, which shows that Γ_{t+dt} is perpendicular at every point to gradient lines of Φ, and is therefore a level line of Φ. The displacement of the point z_t becomes

$$z_{t+dt} - z_t = \chi(z)\vec{\nabla}\Phi(z_t)dt.$$

Denoting by

$$\vec{E} = \vec{\nabla}\Phi = 2\bar{\partial}\Phi = 2 \cdot \overline{\frac{P'_n(z, \lambda_i(t))}{P_n(z, \lambda_i(t))}}$$

the gradient of the logarithmic potential and by $\vec{r} = z_t$, conservation of the normal (or gradient) direction becomes

$$\vec{E}(\vec{r} + \chi\vec{E}(\vec{r})dt) = \mu(z)\vec{E}(\vec{r}),$$

where $\mu(z) = 1 + m(z)dt, m(z) = O(1), m(z) \in \mathbb{R}$, so after expanding in the infinitesimal time interval dt,

$$(\vec{E} \cdot \vec{\nabla})\vec{E}(\vec{r}) = \frac{m(z)}{\chi(z)}\vec{E}(\vec{r}).$$

Remark 3.1. The proportionality relation indicated above carries also the following physical significance: the dynamical system that we study is of *frictional* type, where the *acceleration* field (proportional to the force, or gradient of Green's function) is also proportional to the *velocity*. In other words, the transport derivative (or Lie derivative) of the velocity field must be parallel to the velocity itself:

$$\mathcal{L}_{\vec{V}}\vec{V} = [i_{\vec{V}} \circ \mathrm{d} - \mathrm{d} \circ i_{\vec{V}}]\vec{V} = (\vec{V} \cdot \vec{\nabla})\vec{V} = \chi[\chi(\vec{E} \cdot \vec{\nabla})\vec{E} + (\vec{E} \cdot \vec{\nabla}\chi)\vec{E}]$$

is parallel to $\vec{V}$ and therefore, to $\vec{E}$.

In complex notation, using the fact that $(\vec{E} \cdot \vec{\nabla}) = \bar{E}\bar{\partial} + E\partial$, we obtain

$$\overline{\frac{P_n'(z,\lambda_i(t))}{P_n(z,\lambda_i(t))}} = \delta(z)\frac{P_n'(z,\lambda_i(t))}{P_n(z,\lambda_i(t))} \cdot \overline{\left(\frac{P_n'(z,\lambda_i(t))}{P_n(z,\lambda_i(t))}\right)'}, \quad \delta(z) \in \mathbb{R},$$

which (after multiplying both sides by $E(z)$) reduces to

$$\left[\frac{P_n'(z,\lambda_i(t))}{P_n(z,\lambda_i(t))}\right]^{-2}\left(\frac{P_n'(z,\lambda_i(t))}{P_n(z,\lambda_i(t))}\right)' \in \mathbb{R},$$

or

$$\Im\left\{\left[\frac{P_n(z,\lambda_i(t))}{P_n'(z,\lambda_i(t))}\right]'\right\} = 0, \quad (\forall) z \in \Gamma_t. \tag{3.2}$$

We note that, since $E(z) = 2\bar{P}_n'/\bar{P}_n$ is the gradient of the Green's function for Ω_t and Ω_t is simply connected, it cannot vanish anywhere in $\Omega_t \cup \Gamma_t$, so all the zeros of $P_n'(z)$, denoted by $\xi_k, k = 1, 2, \ldots, n-1$, are found inside the domain Ω_t'. Then

$$\left[\frac{P_n(z,\lambda_i(t))}{P_n'(z,\lambda_i(t))}\right]' = \frac{1}{n} + \sum_{k=1}^{n-1}\frac{A_k}{(z-\xi_k)^2}, \quad \xi_k \in \Omega_t',$$

with A_k constants. The imaginary part of this expression coincides with the imaginary part of an analytic function in Ω_t, that is bounded there, so the condition (3.2) can only be satisfied if the function is a constant. Since at $z \to \infty$ it vanishes, it follows that

$$\left[\frac{P_n(z,\lambda_i(t))}{P_n'(z,\lambda_i(t))}\right]' = \frac{1}{n},$$

which means that boundaries Γ_t can only be concentric circles.

4. Concluding Remarks

(1) It is plausible that the result can be extended to rational lemniscates $\Gamma_t := \{|R(z,t)| = 1\}$, where $R(z,t)$ are rational functions of degree n where all the zeros are inside Γ, while all poles are in the unbounded component of $\mathbb{C} \setminus \Gamma_t$.

(2) It is well-known that arbitrary "shapes", i.e. Jordan curves can be arbitrarily close approximated by both lemniscates (Hilbert's theorem – cf. [14]) and quadrature domains [1]. At the same time our results imply that there are fundamental differences between these two classes of curves. We think it is interesting to pursue these observations in greater depth.

(3) From the argument in §3 we can extract more. Suppose a family of Jordan curves, $\{\Gamma_t\}_{t>0}$ evolves by the flow along the velocity field $V(z)$ according to (1.3). Assuming the invariance under time-reversal, the argument of §3 can be used to prove that $\chi =$ const, i.e. the process is that of Laplacian growth. Invoking now well-known results on standard Hele-Shaw flows, we can at once conclude, e.g., that the process (1.3) continues for all times $t > 0$, i.e., the curves $\{\Gamma_t\}$ move out to infinity such that $\cup_{t>t_0}\Gamma_t = \mathbb{C} \setminus \overline{\Omega}_{t_0}$, if and only if the initial curve Γ_0 is an ellipse and all the curves $\{\Gamma_t\}$ are also ellipses homotetic with Γ_0 - cf. [3], also cf. [6].

References

[1] S. R. Bell, *Density of quadrature domains in one and several variables*, Compl. Var. Elliptic Equations **54** (2009), no. 3-4, 165–171.

[2] N. Bourbaki, *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers.*, Actualités Scientifiques et Industrielles, No. 1308, Hermann, Paris, 1964.

[3] Emmanuele DiBenedetto and Avner Friedman, *Bubble growth in porous media*, Indiana Univ. Math. J. **35** (1986), no. 3, 573–606.

[4] Peter Ebenfelt, Björn Gustafsson, Dmitry Khavinson, and Mihai Putinar (eds.), *Quadrature domains and their applications*, Operator Theory: Advances and Applications, vol. 156, Birkhäuser Verlag, Basel, 2005.

[5] Peter Ebenfelt, Dmitry Khavinson, and Harold S. Shapiro, *An inverse problem for the double layer potential*, Comput. Methods Funct. Theory **1** (2001), no. 2, 387–401.

[6] Avner Friedman and Makoto Sakai, *A characterization of null quadrature domains in* $\mathbf{R}^N$, Indiana Univ. Math. J. **35** (1986), no. 3, 607–610.

[7] L. A. Galin, Dokl. Acad. Nauk SSSR **47** (1945), no. 1-2, 250–3.

[8] Dmitry Khavinson, Mark Mineev-Weinstein, and Mihai Putinar, *Planar elliptic growth*, Compl. Anal. Oper. Theory **3** (2009), no. 2, 425–451.

[9] M. Mineev-Weinstein, M. Putinar, and R. Teodorescu, *Random matrices in 2D, Laplacian growth and operator theory*, Journal of Physics A Mathematical General **41** (2008), no. 26, 263001.

[10] M. Mineev-Weinstein, P.B. Wiegmann, and A. Zabrodin, *Integrable structure of interface dynamics*, Physical Review Letters **84** (2000), 5106.

[11] P. Ya. Polubarinova-Kochina, Dokl. Acad. Nauk SSSR **47** (1945), 254–7.

[12] Harold S. Shapiro, *The Schwarz function and its generalization to higher dimensions*, University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 9, John Wiley & Sons Inc., New York, 1992.

[13] A. N. Varchenko and P. I. Ètingof, *Why the boundary of a round drop becomes a curve of order four*, University Lecture Series, vol. 3, American Mathematical Society, Providence, RI, 1992.

[14] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publ., vol. XX, American Mathematical Society, Providence, R.I., 1965.

Department of Mathematics & Statistics, University of South Florida, Tampa, FL 33620-5700, USA

E-mail address: dkhavins@cas.usf.edu

Los Alamos National Laboratory, MS-365, Los Alamos, NM 87545, USA

E-mail address: mariner@lanl.gov

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

E-mail address: mputinar@math.ucsb.edu

Department of Mathematics & Statistics, University of South Florida, Tampa, FL 33620-5700, USA

E-mail address: razvan@cas.usf.edu