Integrable discretizations of the short pulse equation

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Abstract. In the present paper, we propose integrable semi-discrete and full-discrete analogues of the short pulse (SP) equation. The key of the construction is the bilinear forms and determinant structure of solutions of the SP equation. We also give the determinant formulas of N-soliton solutions of the semi-discrete and full-discrete analogues of the SP equations, from which the multi-loop and multi-breather solutions can be generated. In the continuous limit, the full-discrete SP equation converges to the semi-discrete SP equation, then to the continuous SP equation. Based on the semi-discrete SP equation, an integrable numerical scheme, i.e., a self-adaptive moving mesh scheme, is proposed and used for the numerical computation of the short pulse equation.

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1. Introduction

Most recently, the short pulse (SP) equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \tag{1.1}$$

was derived as a model equation for the propagation of ultra-short optical pulses in nonlinear media [1, 2]. Here, $u = u(x,t)$ represents the magnitude of the electric field, the subscripts t and x denote partial differentiation. Apart from the context of nonlinear optics, the SP equation has also been derived as an integrable differential equation associated with pseudospherical surfaces [3]. The SP equation has been shown to be completely integrable [3, 4, 5, 6, 7]. The loop soliton solutions as well as smooth soliton solutions of the SP equation were found in [8, 9, 10]. The connection between the SP equation and the sine-Gordon equation through the hodograph transformation was clarified in [11], and then the N-soliton solutions including multi-loop and multi-breather ones were given by using the Hirota bilinear method.

Integrable discretizations of soliton equations have received considerable attention recently [12, 13, 14, 15]. In our recent work, the authors proposed an integrable semi-discrete

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analogue of the Camassa-Holm (CH) equation and apply it as a numerical scheme, i.e., a self-adaptive moving mesh scheme [16, 17]. The key of the discretization is an introduction of an nonuniform mesh, which plays a role of the hodograph transformation as in the continuous case.

In the present paper, we attempt to construct integrable semi-discrete and full-discretizations of the SP equation by the same approach used in the CH equation. We also attempt to use the semi-discrete analogue of the SP equation as a self-adaptive moving mesh scheme to perform numerical simulations.

The rest of the present paper is organized as follows. In Section 2, we review the bilinear equations and determinant solutions of the SP equation. In Section 3, we propose an integrable semi-discrete analogue of the SP equation, whose N-soliton solutions are also constructed in terms of determinant form. By using the semi-discrete analogue of the SP equation as a self-adaptive moving mesh scheme, the numerical results for one- and two-loop solutions are also presented. In Section 4, the full-discrete analogues of the SP equation are proposed. The paper is concluded by Section 5.

2. Bilinear equations and determinant solutions of the short pulse equation

In this section, the results in [11] regarding the bilinear equations and the solutions of the SP equation will be briefly reviewed.

First, by introducing the new dependent variable

$$r^2 = 1 + u_x^2\,, \tag{2.1}$$

the SP equation is rewritten as

$$r_t = \left(\frac{1}{2}u^2 r\right)_x\,. \tag{2.2}$$

Introducing the hodograph transformation

$$dy = rdx + \frac{1}{2}u^2 r dt\,, \quad ds = dt\,, \tag{2.3}$$

i.e.,

$$\frac{\partial}{\partial t} = \frac{1}{2}u^2 r\frac{\partial}{\partial y} + \frac{\partial}{\partial s}\,, \quad \frac{\partial}{\partial x} = r\frac{\partial}{\partial y}\,,$$

we obtain

$$r_s = r^2 u u_y\,, \tag{2.4}$$

where

$$r^2 = 1 + r^2 u_y^2\,.$$

The equation (2.4) can also be cast into a form of

$$\left(\frac{1}{r}\right)_s = -\left(\frac{1}{2}u^2\right)_y\,. \tag{2.5}$$

Introducing new variables

$$r = \frac{1}{\cos\phi}\,, \quad u = \phi_s\,, \tag{2.6}$$

Eq.(2.5) leads to the sine-Gordon equation

$$\phi_{ys} = \sin\phi\,. \tag{2.7}$$

Moreover, as is shown in [18, 22], upon the dependent variable transformation

$$\phi(y,s) = 2\mathrm{i}\,\ln\frac{F^*(y,s)}{F(y,s)}\,,$$

the sine-Gordon equation (2.7) leads to the following bilinear equations

$$FF_{ys} - F_yF_s = \frac{1}{4}(F^2 - F^{*2})\,, \tag{2.8}$$

$$F^*F^*_{ys} - F^*_yF^*_s = \frac{1}{4}(F^{*2} - F^2)\,, \tag{2.9}$$

where F^* is the complex conjugate of F. Henceforth, the solutions of the SP equation are obtained by F and F^* through the dependent variable transformation

$$u(y,s) = \frac{\partial}{\partial s}\phi(y,s) = \frac{\partial}{\partial s}\left(2\mathrm{i}\,\ln\frac{F^*(y,s)}{F(y,s)}\right)\,. \tag{2.10}$$

In what follows, we will show that the bilinear equations (2.8)–(2.9) are actually obtained as the 2-reduction of the two-dimensional Toda lattice (2DTL) equations: [19, 20, 21, 22]

$$\frac{1}{2}D_YD_S\tau_n\cdot\tau_n = {\tau_n}^2 - \tau_{n+1}\tau_{n-1}\,, \tag{2.11}$$

i.e.,

$$\tau_n\frac{\partial^2\tau_n}{\partial Y\partial S} - \frac{\partial\tau_n}{\partial Y}\frac{\partial\tau_n}{\partial S} = {\tau_n}^2 - \tau_{n+1}\tau_{n-1}\,, \tag{2.12}$$

where D_x is the Hirota D-operator which is defined as

$$D_x^n f\cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^n f(x)g(y)|_{y=x}\,.$$

Applying the 2-reduction $\tau_{n-1} = \alpha^{-1}\tau_{n+1}$ (α is a constant), we obtain

$$\tau_n\frac{\partial^2\tau_n}{\partial Y\partial S} - \frac{\partial\tau_n}{\partial Y}\frac{\partial\tau_n}{\partial S} = {\tau_n}^2 - \tau_{n+1}^2\,, \tag{2.13}$$

where the gauge transformation $\tau_n \to \alpha^{\frac{n}{2}}\tau_n$ is used. Letting $f = \tau_0$ and $\bar{f} = \tau_1$, we have

$$f f_{YS} - f_Y f_S = f^2 - \bar{f}^2\,, \tag{2.14}$$

$$\bar{f}\bar{f}_{YS} - \bar{f}_Y\bar{f}_S = \bar{f}^2 - f^2\,. \tag{2.15}$$

Under the independent variable transformation $y = 2Y$, $s = 2S$, we obtain

$$f f_{ys} - f_y f_s = \frac{1}{4}(f^2 - \bar{f}^2)\,, \tag{2.16}$$

$$\bar{f}\bar{f}_{ys} - \bar{f}_y\bar{f}_s = \frac{1}{4}(\bar{f}^2 - f^2)\,, \tag{2.17}$$

which are bilinear equations of the SP equation.

Next, we give the Casorati determinant (N-soliton) solution of the SP equation. It is known that the Casorati determinant solution of the 2DTL equation is of the form [21, 22]:

$$\tau_n(Y,S) = \left|\psi_i^{(n+j-1)}(Y,S)\right|_{1\le i,j\le N}\,, \tag{2.18}$$

where $\psi_i^{(n)}(Y,S)$ satisfies linear dispersive relations

$$\frac{\partial\psi_i^{(n)}}{\partial Y} = \psi_i^{(n+1)}, \quad \frac{\partial\psi_i^{(n)}}{\partial S} = \psi_i^{(n-1)}\,. \tag{2.19}$$

For example, a particular choice of $\psi_i^{(n)}(Y,S)$

$$\psi_i^{(n)}(Y,S) = c_{i,1}p_i^n e^{p_iY+\frac{1}{p_i}S+\eta_{0i}} + c_{i,2}q_i^n e^{q_iY+\frac{1}{q_i}S+\eta'_{0i}}\,, \tag{2.20}$$

with $c_{i,1}$ and $c_{i,2}$ being constants, satisfies the linear dispersive relations and gives the N-soliton solutions.

Applying the 2-reduction $q_i = -p_i$ and the change of variables $y = 2Y$ and $s = 2S$, we obtain the determinant solution of bilinear equations (2.16) and (2.17):

$$f(y,s) = \tau_0(y,s), \quad \bar{f}(y,s) = \tau_1(y,s)\,,$$
$$\tau_n(y,s) = \left|\psi_i^{(n+j-1)}(y,s)\right|_{1\le i,j\le N}\,, \tag{2.21}$$

where

$$\psi_i^{(n)}(y,s) = c_{i,1}p_i^n e^{\frac{1}{2}p_iy+\frac{1}{2p_i}s+\eta_{0i}} + c_{i,2}(-p_i)^n e^{-\frac{1}{2}p_iy-\frac{1}{2p_i}s+\eta'_{0i}}\,. \tag{2.22}$$

Since u is real and the dependent variable transformation u includes the imaginary number, we must consider the reality condition of u. Let us introduce α and β such that $F^* = \alpha\bar{f}$ and $F = \beta f$, where F and F^* are complex conjugate of each other. Note that F and F^* also satisfies the bilinear equations (2.16) and (2.17) because of

$$u = \frac{\partial}{\partial s}\left(2\mathrm{i}\ln\frac{F^*}{F}\right) = \frac{\partial}{\partial s}\left(2\mathrm{i}\ln\frac{\alpha\bar{f}}{\beta f}\right) = \frac{\partial}{\partial s}\left(2\mathrm{i}\ln\frac{\bar{f}}{f} + 2\mathrm{i}\ln\frac{\alpha}{\beta}\right) = \frac{\partial}{\partial s}\left(2\mathrm{i}\ln\frac{\bar{f}}{f}\right)\,. \tag{2.23}$$

Thus a set of F and F^* gives solutions of the SP equation as well as a set of f and $\bar{f}$. By choosing phase constants appropriately, the functions f and $\bar{f}$ can be made to be complex conjugate of each other to keep the reality and regularity of u. For example, the following choice

$$\psi_i^{(n)} = p_i^n e^{\frac{1}{2}p_iy+\frac{1}{2p_i}s+\eta_{0i}-\mathrm{i}\pi/4} + (-p_i)^n e^{-\frac{1}{2}p_iy-\frac{1}{2p_i}s+\eta'_{0i}+\mathrm{i}\pi/4}\,, \tag{2.24}$$

guarantees the reality and regularity of the solution.

Summarizing the above results, the determinant (N-soliton) solution of the SP equation is given by

$$u(y,s) = \frac{\partial}{\partial s}\left(2\mathrm{i}\ln\frac{\bar{f}(y,s)}{f(y,s)}\right)\,, \tag{2.25}$$
$$x = y - 2\mathrm{i}(\ln\bar{f}f)_t, \quad t = s\,,$$
$$f(y,s) = \tau_0(y,s)\,, \quad \bar{f}(y,s) = \tau_1(y,s)\,,$$

$$\tau_n(y,s) = \left|\psi_i^{(n+j-1)}(y,s)\right|_{1\le i,j\le N} ,$$

where

$$\psi_i^{(n)} = p_i^n e^{\frac{1}{2}p_i y + \frac{1}{2p_i}s + \eta_{0i} - \mathrm{i}\pi/4} + (-p_i)^n e^{-\frac{1}{2}p_i y - \frac{1}{2p_i}s + \eta'_{0i} + \mathrm{i}\pi/4} .$$

3. An integrable semi-discretization of the short pulse equation and numerical computations

Based on the above fact, we construct the integrable spatial-discretization of the SP equation. Consider the following Casorati determinant:

$$\tau_n(k,S) = \left|\psi_i^{(n+j-1)}(k,S)\right|_{1\le i,j\le N} , \tag{3.1}$$

where $\psi_i^{(n)}$ satisfies the dispersion relations

$$\Delta_k \psi_i^{(n)} = \psi_i^{(n+1)} , \tag{3.2}$$

$$\partial_S \psi_i^{(n)} = -\psi_i^{(n-1)} . \tag{3.3}$$

Here Δ_k is the backward difference operator with the spacing constant a

$$\Delta_k f(k) = \frac{f(k) - f(k-1)}{a} .$$

Particularly, one can choose

$$\psi_i^{(n)}(k,S) = c_{i,1} p_i^n (1 - a p_i)^{-k} e^{\frac{1}{p_i}S + \xi_{i0}} + c_{i,2} q_i^n (1 - a q_i)^{-k} e^{\frac{1}{q_i}S + \eta_{i0}} , \tag{3.4}$$

which automatically satisfies the dispersion relations (3.2) and (3.3). The above Casorati determinant satisfies the bilinear form of the semi-discrete 2DTL equation (the Bäcklund transformation of the bilinear equation of 2DTL equation) [23, 22]

$$\left(\frac{1}{a} D_S - 1\right) \tau_n(k+1) \cdot \tau_n(k) + \tau_{n+1}(k+1)\tau_{n-1}(k) = 0 . \tag{3.5}$$

Applying 2-reduction

$$q_i = -p_i ,$$

and letting

$$f_k = \tau_0(k) , \quad \bar{f}_k = \tau_1(k) = \left(\prod_{i=1}^{N} p_i^2\right) \tau_{-1}(k) ,$$

we obtain

$$\frac{1}{a} D_S f_{k+1} \cdot f_k - f_{k+1} f_k + \bar{f}_{k+1} \bar{f}_k = 0 , \tag{3.6}$$

$$\frac{1}{a} D_S \bar{f}_{k+1} \cdot \bar{f}_k - \bar{f}_{k+1} \bar{f}_k + f_{k+1} f_k = 0 , \tag{3.7}$$

where the gauge transformation $\tau_n \to \left(\prod_{i=1}^{N} p_i\right)^n \tau_n$ is used. Note that f and $\bar{f}$ can be made complex conjugate of each other by choosing the phase constants properly. Under the change of independent variable $s = 2S$, Eq.(3.5) implies the following two bilinear equations

$$\frac{2}{a} D_s f_{k+1} \cdot f_k - f_{k+1} f_k + \bar{f}_{k+1} \bar{f}_k = 0\,, \tag{3.8}$$

$$\frac{2}{a} D_s \bar{f}_{k+1} \cdot \bar{f}_k - \bar{f}_{k+1} \bar{f}_k + f_{k+1} f_k = 0\,, \tag{3.9}$$

which can be readily shown to be equivalent to

$$-\left(\frac{2}{a}\left(\ln \frac{f_{k+1}}{f_k}\right)_s - 1\right) = \frac{\bar{f}_{k+1} \bar{f}_k}{f_{k+1} f_k}\,, \tag{3.10}$$

$$-\left(\frac{2}{a}\left(\ln \frac{\bar{f}_{k+1}}{\bar{f}_k}\right)_s - 1\right) = \frac{f_{k+1} f_k}{\bar{f}_{k+1} \bar{f}_k}\,. \tag{3.11}$$

Subtracting the above two equations, one obtains

$$\frac{2}{a}\left(\left(\ln \frac{\bar{f}_{k+1}}{\bar{f}_k}\right)_s - \left(\ln \frac{f_{k+1}}{f_k}\right)_s\right) = \frac{\bar{f}_{k+1} \bar{f}_k}{f_{k+1} f_k} - \frac{f_{k+1} f_k}{\bar{f}_{k+1} \bar{f}_k}\,. \tag{3.12}$$

Introducing the dependent variable transformation $\phi_k(s) = 2\mathrm{i} \ln\left(\frac{\bar{f}_k(s)}{f_k(s)}\right)$, one arrives at

$$\frac{\phi_{k+1,s} - \phi_{k,s}}{2a} = \sin\left(\frac{\phi_{k+1} + \phi_k}{2}\right)\,, \tag{3.13}$$

which is nothing but an integrable semi-discretization of the sine-Gordon equation. Note that this is also known as the Bäcklund transformation of the sine-Gordon equation [24, 25].

It is obvious that, from the semi-discrete sine-Gordon equation (3.13), the equation

$$\left(\cos\left(\frac{\phi_{k+1} + \phi_k}{2}\right)\right)_s = -\frac{\phi_{k+1,s}^2 - \phi_{k,s}^2}{4a}\,, \tag{3.14}$$

is implied. By introducing the dependent variable transformation

$$u_k = \frac{d\phi_k}{ds} = 2\mathrm{i} \ln\left(\frac{\bar{f}_k(s)}{f_k(s)}\right)_s\,, \quad \delta_k = \cos\left(\frac{\phi_{k+1} + \phi_k}{2}\right)\,, \tag{3.15}$$

it then follows

$$\frac{d\delta_k}{ds} = -\frac{u_{k+1}^2 - u_k^2}{4}\,, \tag{3.16}$$

which is the first equation of a semi-discrete analogue of the SP equation

From the facts

$$\cos^2\left(\frac{\phi_{k+1} + \phi_k}{2}\right) + \sin^2\left(\frac{\phi_{k+1} + \phi_k}{2}\right) = 1\,, \tag{3.17}$$

$$\sin\left(\frac{\phi_{k+1} + \phi_k}{2}\right) = \frac{u_{k+1} - u_k}{2a}\,, \tag{3.18}$$

and

$$\frac{1}{r_k} = \frac{\delta_k}{a} = \cos\left(\frac{\phi_{k+1} + \phi_k}{2}\right)\,, \tag{3.19}$$

it follows

$$\frac{\delta_k^2}{a^2}+\frac{(u_{k+1}-u_k)^2}{4a^2}=1\,,$$

i.e.,

$$\delta_k^2=a^2-\frac{(u_{k+1}-u_k)^2}{4}\,, \tag{3.20}$$

which becomes another equation of a semi-discrete analogue of the SP equation.

Summarizing the above results, we obtained an integrable semi-discrete analogue of the SP equation and its solutions

$$(u_{k+1}-u_k)^2=4(a^2-\delta_k^2)\,, \tag{3.21}$$

$$\frac{d\delta_k}{ds}=-\frac{u_{k+1}^2-u_k^2}{4}\,, \tag{3.22}$$

where the x-coordinate of the k-th lattice point is given by $X_k=X_0+\sum_{l=0}^{k-1}\delta_l$. From the construction, the semi-discrete analogue of the SP equation has the following Casorati determinant solution:

$$u_k(s)=\frac{d}{ds}\left(2\mathrm{i}\ln\frac{\bar{f}_k}{f_k}\right)\,,\quad \delta_k=\frac{a}{2}\left(\frac{\bar{f}_{k+1}\bar{f}_k}{f_{k+1}f_k}+\frac{f_{k+1}f_k}{\bar{f}_{k+1}\bar{f}_k}\right)\,, \tag{3.23}$$

$$X_k=X_0+\sum_{l=0}^{k-1}\delta_l\,,$$

$$f_k(s)=\tau_0(k,s)\,,\quad \bar{f}_k(s)=\tau_1(k,s)\,,$$

$$\tau_n(k,s)=\left|\psi_i^{(n+j-1)}(k,s)\right|_{1\le i,j\le N}\,,$$

where $\psi_i^{(n)}(k,s)$ satisfies

$$\psi_i^{(n)}(k,s)=p_i^n(1-ap_i)^{-k}e^{\frac{1}{2p_i}s+\xi_{i0}-\mathrm{i}\pi/4}+(-p_i)^n(1+ap_i)^{-k}e^{-\frac{1}{2p_i}s+\eta_{i0}+\mathrm{i}\pi/4}\,,$$

and the phase constants $\pm\mathrm{i}\pi/4$ play a role of keeping the reality and regularity.

Note that a^2 must be always greater than or equal to δ_k^2 because $(u_{k+1}-u_k)^2\ge 0$. This can be easily verified by

$$|\delta_k|=\left|a\cos\left(\frac{\phi_{k+1}+\phi_k}{2}\right)\right|\le|a|\,. \tag{3.24}$$

The mesh size of self-adaptive mesh $|\delta_k|$ is always chosen as less than $|a|$.

We can rewrite the semi-discrete SP equation in an alternative form which converges to the SP equation in the continuous limit $\delta_k\to 0$. Multiplying Eq.(3.22) by $2\delta_k$, we have

$$\frac{d\delta_k^2}{ds}=-\delta_k\frac{u_{k+1}^2-u_k^2}{2}\,. \tag{3.25}$$

Eliminating δ_k^2 using Eq.(3.21), this leads to

$$\frac{d(u_{k+1}-u_k)}{ds}=\delta_k(u_{k+1}+u_k)\,. \tag{3.26}$$

Since

$$\frac{d}{ds}\left(\frac{u_{k+1}-u_k}{\delta_k}\right)=\frac{1}{\delta_k}\frac{d(u_{k+1}-u_k)}{ds}-\frac{u_{k+1}-u_k}{\delta_k^2}\frac{d\delta_k}{ds}\,, \tag{3.27}$$

it follows that

$$\frac{d}{ds}\left(\frac{u_{k+1}-u_k}{\delta_k}\right)=u_{k+1}+u_k+\frac{u_{k+1}+u_k}{4}\left(\frac{u_{k+1}-u_k}{\delta_k}\right)^2\,, \tag{3.28}$$

by using Eqs.(3.26) and (3.22). Equation (3.28) gives another form of the semi-discrete SP equation. In the continuous limit $a\to 0$ ($\delta_k\to 0$), we have

$$\frac{u_{k+1}-u_k}{\delta_k}\to\frac{du}{dx}\,,\qquad \frac{u_{k+1}+u_k}{2}\to u\,,$$

$$\frac{\partial x}{\partial s}=\frac{\partial x_0}{\partial s}+\sum_{j=0}^{k-1}\frac{d\delta_j}{ds}=-\frac{1}{4}\sum_{j=0}^{k-1}(u_{j+1}^2-u_j^2)\to-\frac{1}{4}u^2\,,$$

$$\partial_s=\partial_t+\frac{\partial x}{\partial s}\partial_x\to\partial_t-\frac{1}{4}u^2\partial_x\,,$$

Consequently, Eq.(3.28) converges to

$$\left(\partial_t-\frac{1}{4}u^2\partial_x\right)u_x=2u+\frac{1}{2}uu_x^2.$$

By the scaling transformation $2x\to x$, one arrives

$$u_{xt}=u+u(u_x)^2+\frac{1}{2}u^2u_{xx}\,,$$

which turns out to be the SP equation

$$u_{xt}=u+\frac{1}{6}(u^3)_{xx}\,.$$

In a similar way employed in [16, 17], the semi-discrete analogue of the SP equation can be used as a novel numerical scheme, i.e., the so-called self-adaptive moving mesh method, to perform numerical computations for the SP equation. However, the first equation (3.21) has ambiguity for determining the sign even if the non-uniform mesh δ_k is solved from the second equation (3.22). To avoid this difficulty, we introduce an intermediate variable $\bar{\phi}_k=(\phi_{k+1}+\phi_k)/2$, and employ the following scheme,

$$\begin{cases} (u_{k+1}-u_k)=2a\sin(\bar{\phi}_k),\\ \dfrac{d\bar{\phi}_k}{ds}=\dfrac{u_{k+1}+u_k}{2}. \end{cases} \tag{3.29}$$

which can be derived from Eqs.(3.18) and (3.15). Equations (3.29) are equivalent to the integrable semi-discrete analogue of the SP equation, and the relation between the non-uniform mesh δ_k and $\bar{\phi}_k$ is $\delta_k=a\cos(\bar{\phi}_k)$. Figures 1 and 2 are numerical results for one-loop and two-loop soliton solutions, respectively. The time stepsize is $\Delta t=0.01$ and the number of grid points is $N=200$. The detailed numerical results by using the integrable semi-discrete SP equation will be reported somewhere else.

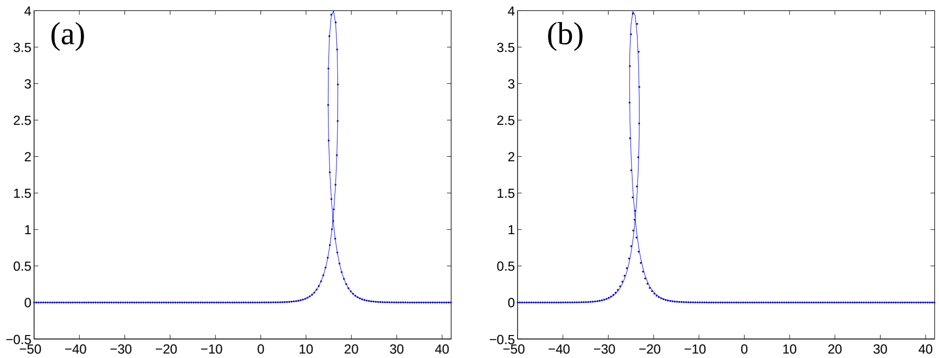

Figure 1. Numerical solutions for one-loop soliton solution with (a) $t = 0.0$; (b) $t = 10.0$. The parameters of the initial condition are $p_1 = 0.5$

4. Full-discretizations of the short pulse equation

To construct a full-discrete analogue of the SP equation, we introduce one more discrete variable l which corresponds to the discrete time variable.

It is known that the τ-function

$$\tau_n(k,l) = \left|\psi_i^{(n+j-1)}(k,l)\right|_{1\le i,j\le N}\,, \tag{4.1}$$

with

$$\psi_i^{(n)}(k,l) = c_{i,1}p_i^n(1-ap_i)^{-k}\left(1-b\frac{1}{p_i}\right)^{-l}e^{\frac{1}{2p_i}s+\xi_{i0}} + c_{i,2}q_i^n(1-aq_i)^{-k}\left(1-b\frac{1}{q_i}\right)^{-l}e^{\frac{1}{2q_i}s+\eta_{i0}}\,,$$

satisfies bilinear equations [23]

$$\left(\frac{2}{a}D_s - 1\right)\tau_n(k+1,l)\cdot\tau_n(k,l) + \tau_{n+1}(k+1,l)\tau_{n-1}(k,l) = 0\,, \tag{4.2}$$

and

$$(2bD_s - 1)\tau_n(k,l+1)\cdot\tau_{n+1}(k,l) + \tau_n(k,l)\tau_{n+1}(k,l+1) = 0\,. \tag{4.3}$$

Applying the 2-reduction $\tau_{n-1} = \left(\prod_{i=1}^N p_i^2\right)^{-1}\tau_{n+1}$, i.e., adding constraints $q_i = -p_i$ to the N-soliton solution, we obtain

$$\left(\frac{2}{a}D_s - 1\right)\tau_n(k+1,l)\cdot\tau_n(k,l) + \tau_{n+1}(k+1,l)\tau_{n+1}(k,l) = 0\,, \tag{4.4}$$

and

$$(2bD_s - 1)\tau_n(k,l+1)\cdot\tau_{n+1}(k,l) + \tau_n(k,l)\tau_{n+1}(k,l+1) = 0\,, \tag{4.5}$$

where the gauge transformation $\tau_n \to \left(\prod_{i=1}^N p_i\right)^n\tau_n$ is used. Letting

$$f_{k,l} = \tau_0(k,l)\,, \quad \bar{f}_{k,l} = \tau_1(k,l)\,,$$

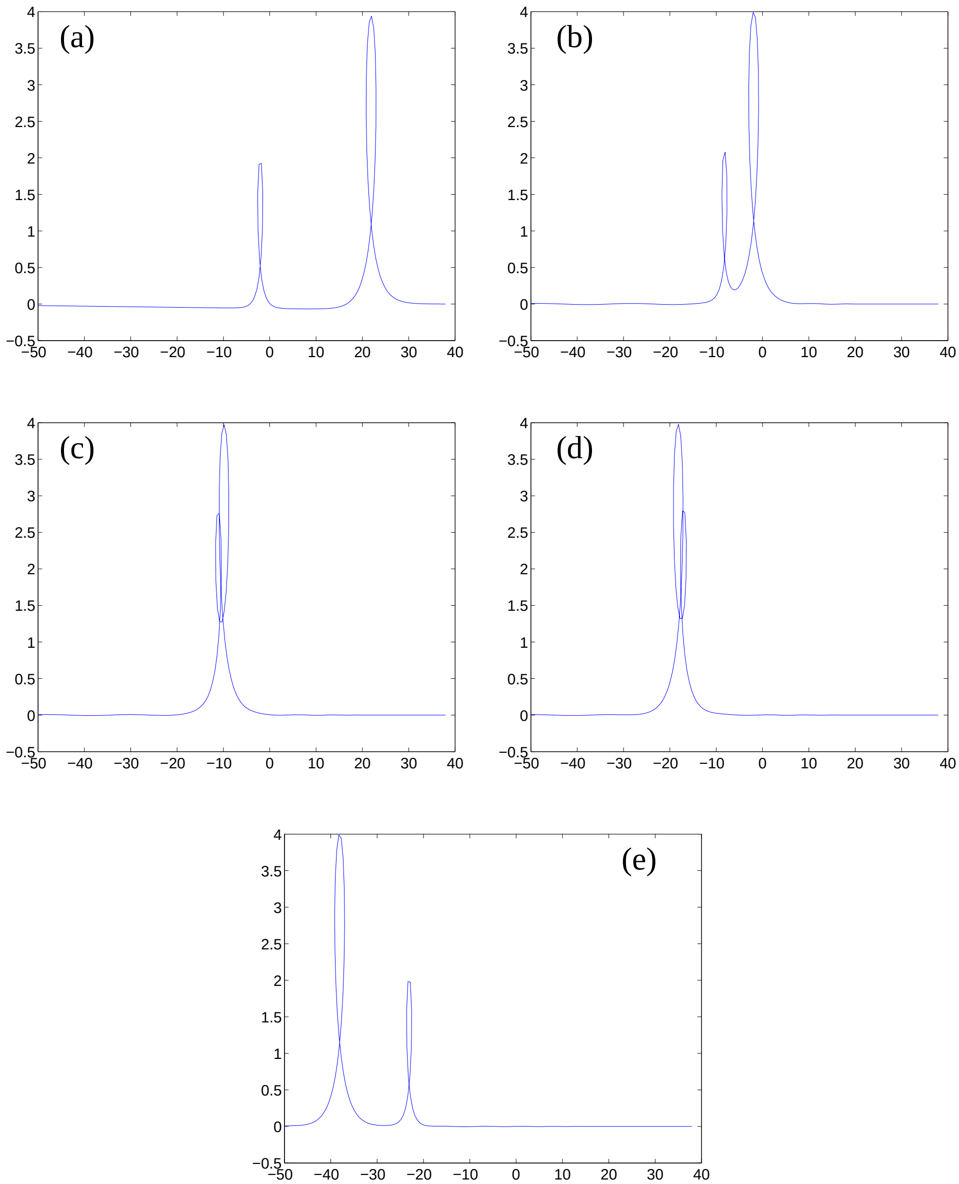

Figure 2. Numerical solutions for the collision of two-loop soliton solution with (a) $t = 0.0$; (b) $t = 6.0$; (c) $t = 8.0$; (d) $t = 10.0$; (e) $t = 15.0$. The parameters of the initial condition are $p_1 = 0.5$, $p_2 = 1.0$.

the bilinear equations (4.4) and (4.5) imply the following four equations

$$\left(\frac{2}{a}D_s - 1\right) f_{k+1,l} \cdot f_{k,l} + \bar{f}_{k+1,l}\,\bar{f}_{k,l} = 0\,, \tag{4.6}$$

$$\left(\frac{2}{a}D_s - 1\right) \bar{f}_{k+1,l} \cdot \bar{f}_{k,l} + f_{k+1,l}\,f_{k,l} = 0\,, \tag{4.7}$$

$$(2bD_s - 1)\, f_{k,l+1} \cdot \bar{f}_{k,l} + f_{k,l}\,\bar{f}_{k,l+1} = 0\,, \tag{4.8}$$

$$(2bD_s - 1)\, \bar{f}_{k,l+1} \cdot f_{k,l} + \bar{f}_{k,l}\,f_{k,l+1} = 0\,, \tag{4.9}$$

which are actually equivalent to

$$\frac{2}{a}\left(\ln\frac{f_{k+1,l}}{f_{k,l}}\right)_s - 1 + \frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} = 0\,, \tag{4.10}$$

$$\frac{2}{a}\left(\ln\frac{\bar{f}_{k+1,l}}{\bar{f}_{k,l}}\right)_s - 1 + \frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}} = 0\,, \tag{4.11}$$

$$2b\left(\ln\frac{f_{k,l+1}}{f_{k,l}}\right)_s - 1 + \frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} = 0\,, \tag{4.12}$$

$$2b\left(\ln\frac{\bar{f}_{k,l+1}}{\bar{f}_{k,l}}\right)_s - 1 + \frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}} = 0\,. \tag{4.13}$$

Note that f and $\bar{f}$ can be made complex conjugate of each other by choosing the phase constants properly. By introducing

$$u_{k,l} = \left(2\mathrm{i}\ln\frac{\bar{f}_{k,l}}{f_{k,l}}\right)_s\,, \tag{4.14}$$

and

$$X_{k,l} = ka - (\ln \bar{f}_{k,l}\,f_{k,l})_s\,, \tag{4.15}$$

where $X_{k,l}$ is the x-coordinate of the k-th lattice point at time l, we find the following relations

$$u_{k+1,l} - u_{k,l} = \mathrm{i}a\left(\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} - \frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}\right)\,, \tag{4.16}$$

$$u_{k,l+1} + u_{k,l} = \frac{\mathrm{i}}{b}\left(\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} - \frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}}\right)\,, \tag{4.17}$$

$$X_{k+1,l} - X_{k,l} = \frac{a}{2}\left(\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} + \frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}\right)\,, \tag{4.18}$$

$$X_{k,l+1} - X_{k,l} = -\frac{1}{b} + \frac{1}{2b}\left(\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} + \frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}}\right)\,. \tag{4.19}$$

It is straightforward to derive

$$(u_{k+1,l} - u_{k,l})^2 = 4(a^2 - \delta_{k,l}^2)\,, \tag{4.20}$$

from Eqs.(4.16) and (4.18) and

$$(u_{k,l+1} + u_{k,l})^2 = 4\left(\frac{1}{b^2} - \left(X_{k,l+1} - X_{k,l} + \frac{1}{b}\right)^2\right)\,. \tag{4.21}$$

from Eqs.(4.17) and (4.19), where $\delta_{k,l} = X_{k+1,l} - X_{k,l}$. Equations (4.20) and (4.21) give a full-discrete analogue of the SP equation.

Let us consider another full-discrete analogue of the SP equation. From Eqs.(4.16)-Eqs.(4.19), we obtain

$$\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} = \frac{1}{a}\left(X_{k+1,l} - X_{k,l} - \mathrm{i}\frac{u_{k+1,l} - u_{k,l}}{2}\right)\,, \tag{4.22}$$

$$\frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}} = \frac{1}{a}\left(X_{k+1,l} - X_{k,l} + \mathrm{i}\frac{u_{k+1,l} - u_{k,l}}{2}\right)\,, \tag{4.23}$$

$$\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} = b\left(X_{k,l+1} - X_{k,l} + \frac{1}{b} - \mathrm{i}\frac{u_{k,l+1}+u_{k,l}}{2}\right), \tag{4.24}$$

$$\frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}} = b\left(X_{k,l+1} - X_{k,l} + \frac{1}{b} + \mathrm{i}\frac{u_{k,l+1}+u_{k,l}}{2}\right). \tag{4.25}$$

From the relations (4.22)-(4.25), we have

$$\begin{aligned}&\frac{X_{k+1,l+1} - X_{k,l+1} - \mathrm{i}\frac{u_{k+1,l+1}-u_{k,l+1}}{2}}{X_{k+1,l} - X_{k,l} - \mathrm{i}\frac{u_{k+1,l}-u_{k,l}}{2}} \\ &\qquad = \frac{X_{k+1,l+1} - X_{k+1,l} + \frac{1}{b} - \mathrm{i}\frac{u_{k+1,l+1}+u_{k+1,l}}{2}}{X_{k,l+1} - X_{k,l} + \frac{1}{b} + \mathrm{i}\frac{u_{k,l+1}+u_{k,l}}{2}}.\end{aligned} \tag{4.26}$$

Equating the real part and imaginary part respectively, we have

$$\begin{aligned}&(X_{k+1,l+1} - X_{k,l+1})\left(X_{k,l+1} - X_{k,l} + \frac{1}{b}\right) + \frac{u_{k+1,l+1}-u_{k,l+1}}{2}\frac{u_{k,l+1}+u_{k,l}}{2} \\ &\quad = \left(X_{k+1,l+1} - X_{k+1,l} + \frac{1}{b}\right)(X_{k+1,l} - X_{k,l}) - \frac{u_{k+1,l+1}+u_{k+1,l}}{2}\frac{u_{k+1,l}-u_{k,l}}{2},\end{aligned} \tag{4.27}$$

$$\begin{aligned}&\left(X_{k,l+1} - X_{k,l} + \frac{1}{b}\right)(u_{k+1,l+1} - u_{k,l+1}) - (X_{k+1,l+1} - X_{k,l+1})(u_{k,l+1} + u_{k,l}) \\ &\quad = \left(X_{k+1,l+1} - X_{k+1,l} + \frac{1}{b}\right)(u_{k+1,l} - u_{k,l}) + (X_{k+1,l} - X_{k,l})(u_{k+1,l+1} + u_{k+1,l}),\end{aligned} \tag{4.28}$$

which can be rearranged into the following simpler form:

$$\begin{aligned}&(X_{k+1,l+1} - X_{k+1,l} - X_{k,l+1} + X_{k,l})\left(\frac{1}{b} - X_{k+1,l} + X_{k,l+1}\right) \\ &\qquad = -\frac{u_{k+1,l+1}+u_{k+1,l}-u_{k,l+1}-u_{k,l}}{2}\frac{u_{k+1,l}+u_{k,l+1}}{2},\end{aligned} \tag{4.29}$$

$$\begin{aligned}&(u_{k+1,l+1} - u_{k+1,l} - u_{k,l+1} + u_{k,l})\left(\frac{2}{b} + X_{k+1,l+1} - X_{k+1,l} + X_{k,l+1} - X_{k,l}\right) \\ &\quad = (X_{k+1,l+1} + X_{k+1,l} - X_{k,l+1} - X_{k,l})(u_{k+1,l+1} + u_{k+1,l} + u_{k,l+1} + u_{k,l}).\end{aligned} \tag{4.30}$$

Equations (4.29) and (4.30) constitute another form of integrable full-discretization of the SP equation. Taking the continuous limit $b \to 0$ in time, we obtain

$$(X_{k+1} - X_k)_s = -\frac{1}{4}(u_{k+1} - u_k)(u_{k+1} + u_k), \tag{4.31}$$

and

$$(u_{k+1} - u_k)_s = (X_{k+1} - X_k)(u_{k+1} + u_k), \tag{4.32}$$

which are nothing but the semi-discrete analogue of the SP equation (3.21) and (3.22). Here we used $\frac{F_{l+1}-F_l}{2b} \to \partial_s F$ as $b \to 0$.

From the construction of the full-discrete analogue of the SP equation, the determinant solution of the full-discrete SP equation is

$$u_{k,l} = \mathrm{i}\left(\frac{\bar{g}_{k,l}}{\bar{f}_{k,l}} - \frac{g_{k,l}}{f_{k,l}}\right) = \frac{\partial}{\partial s}\left(2\mathrm{i}\ln\frac{\bar{f}_{k,l}}{f_{k,l}}\right), \tag{4.33}$$

$$X_{k,l} = ka - \frac{1}{2}\left(\frac{\bar{g}_{k,l}}{\bar{f}_{k,l}} + \frac{g_{k,l}}{f_{k,l}}\right) = ka - \frac{\partial}{\partial s}(\ln \bar{f}_{k,l}\,f_{k,l}), \tag{4.34}$$

$$f_{k,l} = \tau_0(k,l)\,, \quad \bar{f}_{k,l} = \tau_1(k,l)\,,$$
$$g_{k,l} = \rho_0(k,l)\,, \quad \bar{g}_{k,l} = \rho_1(k,l)\,,$$
$$\tau_n(k,l) = \begin{vmatrix} \psi_1^{(n)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \cdots & \vdots & \vdots \\ \psi_2^{(n)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \end{vmatrix},$$
$$\rho_n(k,l) = \begin{vmatrix} \psi_1^{(n-1)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n-1)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \cdots & \vdots & \vdots \\ \psi_2^{(n-1)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \end{vmatrix},$$

where $\psi_i^{(n)}(k,l)$ satisfies

$$\psi_i^{(n)}(k,l) = p_i^n(1-ap_i)^{-k}\left(1-b\frac{1}{p_i}\right)^{-l} e^{\frac{1}{2p_i}s+\xi_{i0}-\mathrm{i}\pi/4} + (-p_i)^n(1+ap_i)^{-k}\left(1+b\frac{1}{p_i}\right)^{-l} e^{-\frac{1}{2p_i}s+\eta_{i0}+\mathrm{i}\pi/4}\,,$$

and the phase constants $\pm\mathrm{i}\pi/4$ play a role of keeping the reality and regularity. s is an auxiliary parameter. Note that ρ_n^m can be expressed as $\rho_n^m = 2\partial_s\tau_n(k,l)$ because the auxiliary parameter s works on elements of the above determinant by $2\partial_s\psi_i^{(n)}(k,l) = \psi_i^{(n-1)}(k,l)$. In the lattice KdV and lattice Boussinesq equations, one of τ-functions is also expressed by the derivative of another τ-function with respect to an auxiliary parameter [26, 27]. This is a common property of discrete soliton equations which are directly connected to the Bäcklund transformations of continuous soliton equations.

Let us consider Eqs.(4.20) and (4.21) again. Rewriting Eqs.(4.20) and (4.21), we have

$$\left(\frac{u_{k+1,l}-u_{k,l}}{2}\right)^2 + \delta_{k,l}^2 = a^2\,, \tag{4.35}$$
$$\left(\frac{u_{k,l+1}+u_{k,l}}{2}\right)^2 + \left(X_{k,l+1}-X_{k,l}+\frac{1}{b}\right)^2 = \frac{1}{b^2}\,. \tag{4.36}$$

These equations actually give conserved quantities because a^2 and $1/b^2$ are constants.

Introducing

$$I_{k,l} \equiv \left(\frac{u_{k+1,l}-u_{k,l}}{2}\right)^2 + \delta_{k,l}^2\,, \tag{4.37}$$
$$J_{k,l} \equiv \left(\frac{u_{k,l+1}+u_{k,l}}{2}\right)^2 + \left(X_{k,l+1}-X_{k,l}+\frac{1}{b}\right)^2, \tag{4.38}$$

Eqs. (4.35) and (4.36) imply the following conserved quantities

$$I_{k,l} = a^2\,, \quad J_{k,l} = \frac{1}{b^2}\,, \tag{4.39}$$

for arbitrary integer values of k and l. Hence, we have

$$I_{k,l+1}-I_{k,l}=0\,,\quad J_{k+1,l}-J_{k,l}=0\,. \tag{4.40}$$

A substitution of the corresponding conserved quantities leads to

$$\left(\frac{u_{k+1,l+1}+u_{k+1,l}-u_{k,l+1}-u_{k,l}}{2}\right)\left(\frac{u_{k+1,l+1}-u_{k+1,l}-u_{k,l+1}+u_{k,l}}{2}\right)$$
$$=-(X_{k+1,l+1}+X_{k+1,l}-X_{k,l+1}-X_{k,l})(X_{k+1,l+1}-X_{k+1,l}-X_{k,l+1}+X_{k,l})\,, \tag{4.41}$$

$$\left(\frac{u_{k+1,l+1}+u_{k+1,l}+u_{k,l+1}+u_{k,l}}{2}\right)\left(\frac{u_{k+1,l+1}+u_{k+1,l}-u_{k,l+1}-u_{k,l}}{2}\right)$$
$$=-\left(X_{k+1,l+1}-X_{k+1,l}+X_{k,l+1}-X_{k,l}+\frac{2}{b}\right)(X_{k+1,l+1}-X_{k+1,l}-X_{k,l+1}+X_{k,l})\,. \tag{4.42}$$

It can be readily shown that the difference of Eq.(4.42) and Eq.(4.41) gives Eq.(4.29), whereas, the quotient is nothing but Eq.(4.30). In summary, Eqs.(4.35) and (4.36), which imply conserved quantities, can also be derived from the full-discrete analogue of the SP equation (4.29) and (4.30).

5. Conclusions

In the present paper, we proposed integrable semi-discrete and full-discrete analogues of the short pulse equation. The N-soliton solutions of both the continuous and discrete SP equations were formulated in the form of Casorati determinants, which include multi-loop soliton and multi-breather solutions. Based on the semi-discrete SP equation, a self-adaptive moving mesh method is proposed and used for the numerical solutions of the SP equation. The examples of one- and two-loop soliton solutions shows the potential of this novel method for the numerical study of the short pulse equation.

References

[1] Schäfer T and Wayne C E 2004 *Physica D* **196**, 90–105

[2] Chung Y, Jones C K R T, Schäfer T and Wayne C E 2005 *Nonlinearity* **18**, 1351–1374

[3] Robelo M L 1989 *Stud. Appl. Math.* **81**, 221–248

[4] Beals R, Rabelo M and Tenenblat K 1989 *Stud. Appl. Math.* **81**, 125–151

[5] Sakovich A and Sakovich S 2005 *J. Phys. Soc. Jpn.* **74**, 239–241

[6] Brunelli J C 2005 *J. Math. Phys.* **46**, 123507

[7] Brunelli J C 2006 *Phys. Lett. A* **353**, 475–478

[8] Sakovich A and Sakovich S 2006 *J. Phys. A* **39**, L361–367

[9] Kuetche V K, Bouetou T B and Kofane T C 2007 *J. Phys. Soc. Jpn.* **76**, 024004

[10] Kuetche V K, Bouetou T B and Kofane T C 2007 *J. Phys. A* **40**, 5585–5596

[11] Matsuno Y 2007 *J. Phys. Soc. Jpn.* **76**, 084003

[12] Levi D and Ragnisco O (Eds.) 1998 *SIDE III–Symmetries and integrability of difference equations*, CRM Proceedings and Lecture Notes 25, AMS, Montreal

[13] Grammaticos B, Kosmann-Schwarzbach Y and Tamizhmani T (Eds.) 2004 *Discrete Integrable Systems*, Lecture Notes in Physics 644, Springer-Verlag, Berlin

[14] Suris Y B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach*, Progress in Mathematics 219, Birkháuser Verlag, Basel-Boston-Berlin

[15] Bobenko A I and Suris Y B 2008 *Discrete Differential Geometry*, Graduate Studies in Mathematics 98, AMS, Rhode Island
[16] Ohta Y, Maruno K and Feng B F 2008 *J. Phys. A* **41**, 355205
[17] Feng B F, Ohta Y and Maruno K 2009 *arXiv:0905.2693*
[18] Hirota R 1972 *J. Phys. Soc. Jpn.* **33**, 1459–1463
[19] Mikhailov A V 1979 *JETP Lett.* **30**, 414–418
[20] Hirota R 1981 *J. Phys. Soc. Jpn.* **50**, 3785–3791
[21] Hirota R, Ito M and Kako F 1988 *Prog. Theor. Phys. Suppl.* **94**, 42–58
[22] Hirota R, 2004 *The Direct Method in Soliton Theory*, Cambridge University Press.
[23] Ohta Y, Kajiwara K, Matsukidaira J and Satsuma J 1993 *J. Math. Phys.* **34**, 5190–5204
[24] Bäcklund A V 1880 *Math. Ann.* **17**, 285–328
[25] Hirota R 1977 *J. Phys. Soc. Jpn.* **43**, 2079–2086
[26] Kajiwara K and Ohta Y 2008 *J. Phys. Soc. Jpn.* **77**, 054004
[27] Maruno K and Kajiwara K 2009 *arXiv:0908.1800*