

Rate of Convergence in Recursive Parameter Estimation procedures.

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Abstract

We consider estimation procedures which are recursive in the sense that each successive estimator is obtained from the previous one by a simple adjustment. We study rate of convergence of recursive estimation procedures for the general statistical model.

Keywords: recursive estimation, estimating equations, stochastic approximation.

1 Introduction

Let X_1, \dots, X_n be random variables, with a joint distribution depending on a real unknown parameter θ . Then an M -estimator of θ is defined as a solution of the estimating equation

$$(1.1) \quad \sum_{i=1}^n \psi_i(v) = 0,$$

where $\psi_i(v) = \psi_i(X_{i-k}^i; v)$ ($i = 1, 2, \dots, n$) are suitably chosen functions and $X_{i-k}^i = (X_{i-k}, \dots, X_i)$ is the a vector of past and present observations at step (time) i . For instance, if X_i 's are observations from a discrete time Markov process, then one can assume that $k = 1$. If observations are i.i.d., then we take $k = 0$ so that $\psi_i(v) = \psi_i(X_i; v)$. In general, if no restrictions are made on the dependence structure of the process X_i , one may need to consider ψ -functions depending on the vector of all past and present observations of the process (that is, $k = i - 1$). If the conditional probability density function (or

probability function) of the observation X_i , given X_{i-k}, \dots, X_{i-1} , is $f_i(x, \theta) = f_i(x, \theta | X_{i-k}, \dots, X_{i-1})$, then one can obtain a MLE (maximum likelihood estimator) on choosing $\psi_i(v) = f'_i(X_i, v)/f_i(X_i, v)$. Besides MLEs, the class of M -estimators includes estimators with special properties such as robustness. Under certain regularity and ergodicity conditions it can be proved that there exists a consistent sequence of solutions of (1.1) which has the property of local asymptotic linearity (See e.g., Serfling (1980), Huber (1981), Lehman (1983). A comprehensive bibliography can be found in Launer and Wilkinson (1979), Hampel et al (1986), Rieder (1994), and Jurečková and Sen (1996).)

If ψ -functions are nonlinear, it is rather difficult to work with the corresponding estimating equations. In this paper we consider estimation procedures which are recursive in the sense that each successive estimator is obtained from the previous one by a simple adjustment. In particular, we consider a class of estimators

$$(1.2) \quad \hat{\theta}_n = \hat{\theta}_{n-1} + \Gamma_n^{-1}(\hat{\theta}_{n-1})\psi_n(\hat{\theta}_{n-1}), \quad n \geq 1,$$

where ψ_n is a suitably chosen vector process, Γ_n is a (possibly random) normalizing matrix process and $\hat{\theta}_0 \in \mathbb{R}^m$ is some initial point. (See the introduction in Sharia (2006) for a detailed discussion and a heuristic justification of this estimation procedure.)

In i.i.d. models, estimating procedures similar to (1.2) have been studied by a number of authors using methods of stochastic approximation theory (see, e.g., Khas'minskii and Nevelson (1972), Fabian (1978), Ljung and Soderstrom (1987), Ljung, Pflug and Walk (1992), and references therein). Some work has been done for non i.i.d. models as well. In particular, Englund, Holst, and Ruppert (1989) give an asymptotic representation results for certain type of X_n processes. In Sharia (1998) theoretical results on convergence, rate of convergence and the asymptotic representation are given under certain regularity and ergodicity assumptions on the model, in the one-dimensional case with $\psi_n(x, \theta) = \frac{\partial}{\partial \theta} \log f_n(x, \theta)$ (see also Campbell (1982), Sharia (1997), Lazrieva and Toronjadze (1987)).

In Sharia (2006), imposing “global” restrictions on the processes ψ and Γ , we study “global” convergence of the recursive estimators (1.2), that is, convergence for an arbitrary starting point $\hat{\theta}_0$. In the present paper, we present results on rate of the convergence and demonstrate the use of these results on some examples.

2 Notation and preliminaries

Let X_t , $t = 1, 2, \dots$, be observations taking values in a measurable space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ equipped with a σ -finite measure μ . Suppose that the distribution of the process X_t depends on an unknown parameter $\theta \in \Theta$, where Θ is an open subset of the m -dimensional Euclidean space \mathbb{R}^m . Suppose also that for each $t = 1, 2, \dots$, there exists a regular conditional probability density of X_t given values of past observations of X_{t-1}, \dots, X_2, X_1 , which will be denoted by

$$f_t(\theta, x_t \mid x_1^{t-1}) = f_t(\theta, x_t \mid x_{t-1}, \dots, x_1),$$

where $f_1(\theta, x_1 \mid x_1^0) = f_1(\theta, x_1)$ is the probability density of the random variable X_1 . Without loss of generality we assume that all random variables are defined on a probability space (Ω, \mathcal{F}) and denote by $\{P^\theta, \theta \in \Theta\}$ the family of the corresponding distributions on (Ω, \mathcal{F}) .

Let $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$ be the σ -field generated by the random variables X_1, \dots, X_t . By $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ we denote the m -dimensional Euclidean space with the Borel σ -algebra $\mathcal{B}(\mathbb{R}^m)$. Transposition of matrices and vectors is denoted by T . By (u, v) we denote the standard scalar product of $u, v \in \mathbb{R}^m$, that is, $(u, v) = u^T v$.

Suppose that h is a real valued function defined on $\Theta \subset \mathbb{R}^m$. We denote by $\dot{h}(\theta)$ the row-vector of partial derivatives of $h(\theta)$ with respect to the components of θ , that is,

$$\dot{h}(\theta) = \left(\frac{\partial}{\partial \theta^1} h(\theta), \dots, \frac{\partial}{\partial \theta^m} h(\theta) \right).$$

If for each $t = 1, 2, \dots$, the derivative $\dot{f}_t(\theta, x_t \mid x_1^{t-1})$ w.r.t. θ exists, then we can define the function

$$l_t(\theta, x_t \mid x_1^{t-1}) = \frac{1}{f_t(\theta, x_t \mid x_1^{t-1})} \dot{f}_t^T(\theta, x_t \mid x_1^{t-1})$$

with the convention $0/0 = 0$.

The *one step conditional Fisher information matrix* for $t = 1, 2, \dots$ is defined as

$$i_t(\theta \mid x_1^{t-1}) = \int l_t(\theta, z \mid x_1^{t-1}) l_t^T(\theta, z \mid x_1^{t-1}) f_t(\theta, z \mid x_1^{t-1}) \mu(dz).$$

We shall use the notation

$$f_t(\theta) = f_t(\theta, X_t \mid X_1^{t-1}), \quad l_t(\theta) = l_t(\theta, X_t \mid X_1^{t-1}),$$

$$i_t(\theta) = i_t(\theta \mid X_1^{t-1}).$$

Note that the process $i_t(\theta)$ is “predictable”, that is, the random variable $i_t(\theta)$, is \mathcal{F}_{t-1} measurable for each $t \geq 1$.

Note also that by definition, $i_t(\theta)$ is a version of the conditional expectation w.r.t. \mathcal{F}_{t-1} , that is,

$$i_t(\theta) = E_\theta \{l_t(\theta)l_t^T(\theta) \mid \mathcal{F}_{t-1}\}.$$

Everywhere in the present work conditional expectations are meant to be calculated as integrals w.r.t. the conditional probability densities.

The *conditional Fisher information* at time t is

$$I_t(\theta) = \sum_{s=1}^t i_s(\theta), \quad t = 1, 2, \dots$$

If the X_t 's are independent random variables, $I_t(\theta)$ reduces to the standard Fisher information matrix. Sometimes $I_t(\theta)$ is referred as the incremental expected Fisher information. Detailed discussion of this concept and related work appears in Barndorff-Nielsen and Sorensen (1994), and Prakasa-Rao (1999) Ch.3.

We say that $\psi = \{\psi_t(\theta, x_t, x_{t-1}, \dots, x_1)\}_{t \geq 1}$ is a sequence of estimating functions and write $\psi \in \Psi$, if for each $t \geq 1$, $\psi_t(\theta, x_t, x_{t-1}, \dots, x_1) : \Theta \times \mathbf{X}^t \rightarrow \mathbb{R}^m$ is a Borel function.

Note that $\{l_t(\theta, x_t \mid x_1^{t-1})\}_{t \geq 1} \in \Psi$ and a ML recursive procedure is given by

$$\hat{\theta}_t = \hat{\theta}_{t-1} + I_t^{-1}(\hat{\theta}_{t-1})l_t(\hat{\theta}_{t-1}), \quad t \geq 1.$$

Convention *Everywhere in the present work $\theta \in \mathbb{R}^m$ is an arbitrary but fixed value of the parameter. Convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. A sequence of random variables $(\xi_t)_{t \geq 1}$ has some property eventually if for every ω in a set Ω^θ of P^θ probability 1, ξ_t has this property for all t greater than some $t_0(\omega) < \infty$.*

3 Main results

Suppose that $\psi \in \Psi$ and $\Gamma_t(\theta)$, for each $\theta \in \mathbb{R}^m$, is a predictable $m \times m$ matrix process with $\det \Gamma_t(\theta) \neq 0$, $t \geq 1$. Consider the estimator $\hat{\theta}_t$ defined by

$$(3.1) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + \Gamma_t^{-1}(\hat{\theta}_{t-1})\psi_t(\hat{\theta}_{t-1}), \quad t \geq 1,$$

where $\hat{\theta}_0 \in \mathbb{R}^m$ is arbitrary initial point.

Let $\theta \in \mathbb{R}^m$ be an arbitrary but fixed value of the parameter and for any $u \in \mathbb{R}^m$ define

$$b_t(\theta, u) = E_\theta \{ \psi_t(\theta + u) \mid \mathcal{F}_{t-1} \}.$$

Lemma 3.1 *Let $\{C_t(\theta)\}$ be a symmetric predictable $m \times m$ matrix process such that $C_t(\theta)$ is non-negative definite for $t = 1, 2, \dots$. Denote $\Delta_t = \hat{\theta}_t - \theta$, $V_t(u) = (C_t(\theta)u, u)$ and $\Delta V_t(u) = V_t(u) - V_{t-1}(u)$. Suppose that*

$$(3.2) \quad \sum_{t=1}^{\infty} (1 + V_{t-1}(\Delta_{t-1}))^{-1} [\mathcal{K}_t(\theta)]^+ < \infty, \quad P^\theta\text{-a.s.},$$

where

$$(3.3) \quad \mathcal{K}_t(\theta) = \Delta V_t(\Delta_{t-1}) + 2(C_t(\theta)\Delta_{t-1}, \Gamma_t^{-1}(\theta + \Delta_{t-1})b_t(\theta, \Delta_{t-1})) \\ + E_\theta \left\{ [\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1})]^T C_t(\theta)\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1}) \mid \mathcal{F}_{t-1} \right\}.$$

Then $V_t(\Delta_t)$ converges (P^θ -a.s.) to a finite limit.

Proof. As always (see the convention in Section 2), convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. To simplify notation we drop the argument or the index θ in some of the expressions below. Rewrite (3.1) in the form

$$\Delta_t = \Delta_{t-1} + \Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1}).$$

By the Taylor expansion,

$$V_t(\Delta_t) = V_t(\Delta_{t-1}) + \dot{V}_t(\Delta_{t-1})\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1}) \\ + \frac{1}{2} [\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1})]^T \ddot{V}_t(\tilde{\Delta}_t)\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1}),$$

Since $\dot{V}_t(u) = 2u^T C_t$ and $\ddot{V}_t(u) = 2C_t$ we obtain

$$V_t(\Delta_t) = V_t(\Delta_{t-1}) + 2(C_t\Delta_{t-1}, \Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1})) \\ + [\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1})]^T C_t\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1}).$$

Since

$$V_t(\Delta_{t-1}) = V_{t-1}(\Delta_{t-1}) + \Delta V_t(\Delta_{t-1}),$$

we have

$$E_\theta \{ V_t(\Delta_t) \mid \mathcal{F}_{t-1} \} = V_{t-1}(\Delta_{t-1}) + \mathcal{K}_t.$$

Then, using the obvious decomposition $\mathcal{K}_t = [\mathcal{K}_t]^+ - [\mathcal{K}_t]^-$, the previous inequality can be rewritten as

$$E_\theta \{V_t(\Delta_t) \mid \mathcal{F}_{t-1}\} = V_{t-1}(\Delta_{t-1})(1 + B_t) + B_t - [\mathcal{K}_t]^-,$$

where $B_t = (1 + V_{t-1}(\Delta_{t-1}))^{-1} [\mathcal{K}_t]^+$. Since, by (3.2), $\sum_{t=1}^\infty B_t < \infty$, the assertion of the lemma follows immediately on application of Lemma A1 in Appendix A (with $X_n = V_n(\Delta_n)$, $\beta_{n-1} = \xi_{n-1} = B_n$ and $\zeta_{n-1} = [\mathcal{K}_n]^-$). \diamond

Corollary 3.1 *Let $\{a_t(\theta)\}$ be a predictable non-decreasing scalar process such that $a_t(\theta) \rightarrow \infty$ as $t \rightarrow \infty$. Denote $\Delta a_t(\theta) = a_t(\theta) - a_{t-1}(\theta)$ and suppose that*

(R1)

$$\lim_{t \rightarrow \infty} \frac{\Delta a_t(\theta)}{a_{t-1}(\theta)} = 0, \quad P^\theta\text{-a.s.};$$

(R2) *there exist a symmetric and non-negative definite matrix C_θ and a predictable non-negative scalar process \mathcal{P}_t such that*

$$(3.4) \quad \begin{aligned} 2(C_\theta \Delta_{t-1}, \Gamma_t^{-1}(\theta + \Delta_{t-1})b_t(\theta, \Delta_{t-1})) + \mathcal{P}_t \leq \\ -\lambda_t(\theta)(C_\theta \Delta_{t-1}, \Delta_{t-1}), \end{aligned}$$

eventually, where $\{\lambda_t(\theta)\}$ is a predictable scalar process, satisfying

$$(3.5) \quad \sum_{s=1}^\infty \left[\frac{\Delta a_t(\theta)}{a_t(\theta)} - \lambda_t(\theta) \right]^+ < \infty, \quad P^\theta\text{-a.s.};$$

(R3) *for each $0 < \varepsilon < 1$ and the process \mathcal{P}_t defined in (R2),*

$$\sum_{s=1}^\infty a_t^\varepsilon(\theta) [E_\theta \{ \|\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1})\|^2 \mid \mathcal{F}_{t-1} \} - \mathcal{P}_t]^+ < \infty, \quad P^\theta\text{-a.s.}$$

Then $a_t(\theta)^{2\delta}(\hat{\theta}_t - \theta)^T C_\theta(\hat{\theta}_t - \theta) \rightarrow 0$ (P^θ -a.s.) for any $\delta \in]0, 1/2[$.

Proof. As always (see the convention in Section 2), convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. Let us check the conditions of Lemma 3.1 for $C_t(\theta) = C_\theta(a_t(\theta))^{2\delta}$, $\delta \in]0, 1/2[$. To simplify notation we drop the fixed argument or the index θ in some of the expressions below. Denote

$$r_t = (\Delta a_t^{2\delta} - a_t^{2\delta} \lambda_t) / a_{t-1}^{2\delta}$$

and

$$\tilde{\mathcal{P}}_t = a_t^{2\delta} (\mathcal{E}_t - \mathcal{P}_t)$$

where

$$\mathcal{E}_t = E_\theta \left\{ [\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1})]^T C [\Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1})] \mid \mathcal{F}_{t-1} \right\}.$$

By (R2), for \mathcal{K}_t defined in (3.3) we have

$$\begin{aligned} \mathcal{K}_t &= \Delta a_t^{2\delta} (C\Delta_{t-1}, \Delta_{t-1}) + 2a_t^{2\delta} (C\Delta_{t-1}, \Gamma_t^{-1}(\theta + \Delta_{t-1})b_t(\theta, \Delta_{t-1})) + \\ &\hspace{20em} a_t^{2\delta} \mathcal{P}_t + \tilde{\mathcal{P}}_t \\ &\leq (\Delta a_t^{2\delta} - a_t^{2\delta} \lambda_t) (C\Delta_{t-1}, \Delta_{t-1}) + \tilde{\mathcal{P}}_t \\ &\leq r_t (a_{t-1}^{2\delta} C\Delta_{t-1}, \Delta_{t-1}) + \tilde{\mathcal{P}}_t. \end{aligned}$$

Since C is non-negative definite,

$$(1 + V_{t-1}(\Delta_{t-1}))^{-1} [\mathcal{K}_t]^+ = (1 + (a_{t-1}^{2\delta} C\Delta_{t-1}, \Delta_{t-1}))^{-1} [\mathcal{K}_t]^+ \leq [r_t]^+ + [\tilde{\mathcal{P}}_t]^+.$$

By (R3), $\sum_{t=1}^{\infty} [\tilde{\mathcal{P}}_t]^+ < \infty$ which implies that (3.2) is equivalent to $\sum_{t=1}^{\infty} [r_t]^+ < \infty$. Since $\Delta a_t^{2\delta} = a_t^{2\delta} - a_{t-1}^{2\delta}$, we can rewrite r_t as

$$r_t = (a_t a_{t-1}^{-1})^{2\delta} (1 - \lambda_t) - 1.$$

Also, since $(1 + x)^{2\delta} = 1 + 2\delta x + O(x^2)$, we have

$$(a_t a_{t-1}^{-1})^{2\delta} = \left(1 + \frac{\Delta a_t}{a_{t-1}}\right)^{2\delta} = 1 + 2\delta \frac{\Delta a_t}{a_{t-1}} + \delta_t^{(1)},$$

where, by (R1), $\delta_t^{(1)} = O(\Delta a_t/a_{t-1})^2 \rightarrow 0$ as $t \rightarrow \infty$. Denote

$$\eta_t = \Delta a_t/a_t - \lambda_t.$$

Then simple calculations show that

$$\begin{aligned} r_t &\leq (a_t a_{t-1}^{-1})^{2\delta} \left(1 + \eta_t^+ - \frac{\Delta a_t}{a_t}\right) - 1 \\ &= -(1 - 2\delta) \frac{\Delta a_t}{a_{t-1}} + \delta_t^{(1)} + \eta_t^+ + 2\delta \eta_t^+ \frac{\Delta a_t}{a_{t-1}} + \eta_t^+ \delta_t^{(1)} + \\ &\hspace{15em} (1 - 2\delta) \frac{\Delta a_t}{a_t} \frac{\Delta a_t}{a_{t-1}} - \frac{\Delta a_t}{a_t} \delta_t^{(1)} \\ &= \frac{\Delta a_t}{a_{t-1}} \left(- (1 - 2\delta) + \delta_t^{(2)}\right) + \delta_t^{(3)} \end{aligned}$$

where

$$\begin{aligned}\delta_t^{(2)} &= \left(\frac{\Delta a_t}{a_{t-1}}\right)^{-1} \delta_t^{(1)} \left(1 - \frac{\Delta a_t}{a_t}\right) + (1 - 2\delta) \frac{\Delta a_t}{a_t}, \\ \delta_t^{(3)} &= \eta_t^+ + 2\delta \eta_t^+ \frac{\Delta a_t}{a_{t-1}} + \eta_t^+ \delta_t^{(1)}.\end{aligned}$$

From (R1) and (R2), $\delta_t^{(2)} \rightarrow 0$ and $\sum_{t=1}^{\infty} |\delta_t^{(3)}| < \infty$. Then, since $1 - 2\delta > 0$, we obtain that $[r_t]^+ \leq |\delta_t^{(3)}|$. It therefore follows that the conditions of Lemma 3.1 are satisfied implying that $a_t^{2\delta} \|\hat{\theta}_t - \theta\|^2$ converges to a finite limit. Finally, since this holds for an arbitrary $\delta \in]0, 1/2[$ and $a_t \rightarrow \infty$, the result follows. \diamond

Remark 3.1 Note that the first term in the left hand side of (3.4) is usually negative and assuming that $\mathcal{P}_t = 0$ the positive parts in (3.5) are usually zero (or quite small) in many examples. On the other hand, the choice $\mathcal{P}_t = 0$ means that (R3) becomes more restrictive imposing stronger probabilistic restrictions on the model. The choice $\mathcal{P}_t = 0$ is natural in the iid case since all the required probabilistic conditions are in this case automatically satisfied. (see also Remark 3.2). Now, if the first term in the left hand side of (3.4) is negative with a “high enough” absolute value, then it may be possible to introduce a non-zero \mathcal{P}_t without jeopardising (3.5). One possibility might be $\mathcal{P}_t = \|\Gamma_t^{-1}(\theta + \Delta_{t-1})b_t(\theta, \Delta_{t-1})\|^2$. Also, in this case, since $b_t(\theta, u) = E_\theta\{\psi_t(\theta + u) \mid \mathcal{F}_{t-1}\}$ and $\Gamma_t^{-1}(\theta + u)$ are predictable processes, the condition in (R3) can be rewritten as

$$\sum_{s=1}^{\infty} a_t^\varepsilon(\theta) E_\theta \left\{ \|\Gamma_t^{-1}(\theta + \Delta_{t-1}) \{\psi_t(\theta + \Delta_{t-1}) - b_t(\theta, \Delta_{t-1})\}\|^2 \mid \mathcal{F}_{t-1} \right\} < \infty.$$

Remark 3.2 Consider the i.i.d. case with

$$f_t(\theta, z \mid x_1^{t-1}) = f(\theta, z), \quad \psi_t(\theta) = \psi(\theta, z)|_{z=X_t},$$

where $\int \psi(\theta, z) f(\theta, z) \mu(dz) = 0$ and $\Gamma_t(\theta) = t\gamma(\theta)$ for some invertible non-random matrix $\gamma(\theta)$. Then

$$b_t(\theta, u) = b(\theta, u) = \int \psi(\theta + u, z) f(\theta, z) \mu(dz),$$

implying that $b_t(\theta, 0) = 0$. Denote $\Delta_t = \hat{\theta}_t - \theta$ and rewrite (3.1) in the form

$$(3.6) \quad \Delta_t = \Delta_{t-1} + \frac{1}{t} \left(\gamma^{-1}(\theta + \Delta_{t-1}) b(\theta, \Delta_{t-1}) + \varepsilon_t^\theta \right),$$

where

$$\varepsilon_t^\theta = \gamma^{-1}(\theta + \Delta_{t-1}) \{ \psi(\theta + \Delta_{t-1}, X_t) - b(\theta, \Delta_{t-1}) \}.$$

Equation (3.6) defines a Robbins-Monro stochastic approximation procedure that converges to the solution of the equation

$$R^\theta(u) := \gamma^{-1}(\theta + u)b(\theta, u) = 0,$$

when the values of the function $R^\theta(u)$ can only be observed with zero expectation errors ε_t^θ . Note that in general, recursion (3.1) cannot be considered in the framework of classical stochastic approximation theory (see Lazrieva, Sharia, and Toronjadze (1997, 2003) for the generalized Robbins-Monro stochastic approximations procedures). For the i.i.d. case, conditions of Corollary 3.1 can be written as **(B1)** and **(B2)** in Corollary 4.1 (see also Remark 4.1), which are standard assumptions for stochastic approximation procedures of type (3.6) (see, e.g., Robbins and Monro (1951), Gladyshev (1965), Khas'minskii and Nevelson (1972), Ljung and Soderstrom (1987), Ljung, Pflug and Walk (1992)).

4 SPECIAL MODELS AND EXAMPLES

1. The i.i.d. scheme. Consider the classical scheme of i.i.d. observations X_1, X_2, \dots , with a common probability density/mass function $f(\theta, x)$, $\theta \in \mathbb{R}^m$. Suppose that $\psi(\theta, z)$ is an estimating function with

$$\int \psi(\theta, z) f(\theta, z) \mu(dz) = 0.$$

Let us define the recursive estimator $\hat{\theta}_t$ by

$$(4.1) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t} \gamma^{-1}(\hat{\theta}_{t-1}) \psi(\hat{\theta}_{t-1}, X_t), \quad t \geq 1,$$

where $\gamma(\theta)$ is a non-random matrix such that $\gamma^{-1}(\theta)$ exists for any $\theta \in \mathbb{R}^m$ and $\hat{\theta}_0 \in \mathbb{R}^m$ is any initial value.

Corollary 4.1 *Suppose that $\hat{\theta} \rightarrow \theta$ (P^θ -a.s.) and*

(B1) *there exists a symmetric and non-negative definite matrix C_θ such that*

$$(C_\theta u, \gamma^{-1}(\theta + u) E^\theta \psi(\theta + u, X_1)) \leq -\frac{1}{2} (C_\theta u, u),$$

for small u 's;

$$(B2) \quad E_\theta \|\gamma^{-1}(\theta + u)\psi(\theta + u)\|^2 = O(1) \text{ as } u \rightarrow 0.$$

Then $t^\delta(\hat{\theta}_t - \theta)^T C_\theta(\hat{\theta}_t - \theta) \rightarrow 0$ (P^θ -a.s.) for any $\delta \in]0, 1/2[$.

Proof. The result follows immediately if we take $a_t(\theta) = t$, $\mathcal{P}_t = 0$ and $\lambda_t(\theta) = 1/t$ in Corollary 3.1. \diamond

Remark 4.1 As it was mentioned in Remark 3.2, for the i.i.d. case the recursive procedures can be studied in the framework of stochastic approximation theory. For stochastic approximation procedures of this type, conditions which guarantee a good rate of convergence are expressed in terms of stability of matrices. Recall that a matrix A is called stable if the real parts of its eigenvalues are negative. A standard requirement in stochastic approximation theory is the existence of the representation (see Remark 3.1 for the notation)

$$(4.2) \quad R^\theta(u) = B^\theta u + o(\|u\|) \text{ as } u \rightarrow 0,$$

where the matrix $S^\theta = B^\theta + \frac{1}{2}\mathbf{1}$ is stable. It is easy to see that this assumption implies (B1). Indeed, it follows from the stability of S^θ that the maximum of the real parts of the eigenvalues of B^θ is less than $-1/2$. This implies (see, e.g., Khas'minskii and Nevelson (1972), Ch.6, §3, Corollary 3.1), that there exists a symmetric and positive definite matrix C_θ such that

$$(C_\theta u, B_\theta u) < -\frac{1}{2}(C_\theta u, u),$$

which, together with (4.2), implies (B1).

As a particular example, consider

$$f(\theta, x) = \frac{1}{\pi(1 + (x - \theta)^2)},$$

the probability density function of the Cauchy distribution with mean θ . Simple calculations show that

$$\frac{\dot{f}}{f}(\theta, x) = \frac{2(x - \theta)}{1 + (x - \theta)^2} \quad \text{and} \quad \frac{\partial^2}{\partial \theta^2} \log f(\theta, x) = \frac{2(x - \theta)^2 - 2}{(1 + (x - \theta)^2)^2}.$$

Now, using tables of standard integrals, it is easy to check that

$$i(\theta) = - \int \frac{\partial^2}{\partial \theta^2} \log f(\theta, x) f(\theta, x) dx = \frac{1}{2}.$$

So, a ML recursive procedure is

$$\hat{\theta}_t = \hat{\theta}_{t-1} - \frac{1}{t} \frac{2(X_t - \hat{\theta}_{t-1})}{1 + (X_t - \hat{\theta}_{t-1})^2}, \quad t \geq 1.$$

Using tables of standard integrals and simple algebra,

$$b(\theta, u) = \frac{2}{\pi} \int \frac{x - u}{1 + (x - u)^2} \frac{1}{1 + x^2} dx = -\frac{2u}{4 + u^2},$$

and

$$\int \left(\frac{\dot{f}}{f}(\theta + u, x) \right)^2 f(\theta, x) dx = \frac{4}{\pi} \int \left(\frac{x - u}{1 + (x - u)^2} \right)^2 \frac{1}{1 + x^2} dx = \frac{2(4 + 3u^2)}{(4 + u^2)^2}.$$

Now, it is easy to check that conditions (I) and (II) of Corollary 4.1 in Sharia (2006) (or in Sharia (1998)) are satisfied, implying that $\hat{\theta}_t \rightarrow \theta$ (P^θ -a.s.). Let us check the conditions of Corollary 4.1. It follows from the above calculations that (B2) holds. Then, for arbitrary $0 < \varepsilon < 1/2$ we have

$$\frac{i^{-1}(\theta)b(\theta, u)}{u} = -\frac{4}{4 + u^2} = -1 + \frac{u^2}{4 + u^2} \leq -1 + \varepsilon$$

for small u 's, which yields that (B1) is satisfied with $C_\theta = 1$. Therefore, $t^\delta(\hat{\theta}_t - \theta) \rightarrow 0$ (P^θ -a.s.) for any $0 < \delta < 1/2$.

2 Exponential family of Markov processes Consider a conditional exponential family of Markov processes in the sense of Feigin (1981) (see also Barndorf-Nielsen (1988)). This is a time homogeneous Markov chain with the one-step transition density

$$f(y; \theta, x) = h(x, y) \exp(\theta^T m(y, x) - \beta(\theta; x)),$$

where $m(y, x)$ is a m -dimensional vector and $\beta(\theta; x)$ is one dimensional. Then in our notation $f_t(\theta) = f(X_t; \theta, X_{t-1})$ and

$$l_t(\theta) = \frac{d}{d\theta} \log f_t(\theta) = m(X_t, X_{t-1}) - \dot{\beta}^T(\theta; X_{t-1}).$$

It follows from standard exponential family theory (see, e.g., Feigin (1981)) that $l_t(\theta)$ is a martingale-difference and the conditional Fisher information is

$$I_t(\theta) = \sum_{s=1}^t \ddot{\beta}(\theta; X_{s-1}).$$

So, a maximum likelihood type recursive procedure can be defined as

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \left(\sum_{s=1}^t \ddot{\beta}(\hat{\theta}_{s-1}; X_{s-1}) \right)^{-1} \left(m(X_t, X_{t-1}) - \dot{\beta}^T(\hat{\theta}_{t-1}; X_{t-1}) \right), \quad t \geq 1.$$

Let us find the functions appearing in the conditions of our theorems for the case $\psi_t = l_t$ and $\Gamma_t = I_t$. Since $E_\theta \{l_t(\theta) \mid \mathcal{F}_{t-1}\} = 0$ we have

$$E_\theta \{m(X_t, X_{t-1}) \mid \mathcal{F}_{t-1}\} = \dot{\beta}^T(\theta; X_{t-1})$$

and also,

$$\begin{aligned} \ddot{\beta}(\theta; X_{t-1}) &= i_t(\theta) = E_\theta \{l_t(\theta)l_t^T(\theta) \mid \mathcal{F}_{t-1}\} \\ &= E_\theta \{m(X_t, X_{t-1})m^T(X_t, X_{t-1}) \mid \mathcal{F}_{t-1}\} - \dot{\beta}^T(\theta; X_{t-1})\dot{\beta}(\theta; X_{t-1}), \end{aligned}$$

which implies that

$$(4.3) \quad E_\theta \{m(X_t, X_{t-1})m^T(X_t, X_{t-1}) \mid \mathcal{F}_{t-1}\} = \ddot{\beta}(\theta; X_{t-1}) + \dot{\beta}^T(\theta; X_{t-1})\dot{\beta}(\theta; X_{t-1}).$$

Now, it is a simple matter to check that

$$(4.4) \quad b_t(\theta, u) = E_\theta \{l_t(\theta + u) \mid \mathcal{F}_{t-1}\} = \dot{\beta}^T(\theta; X_{t-1}) - \dot{\beta}^T(\theta + u; X_{t-1}).$$

Using (4.3) (since $\text{trace}(vv^T) = v^T v$ and $\text{trace}(A + B) = \text{trace} A + \text{trace} B$),

$$(4.5) \quad \begin{aligned} E_\theta \{ \|l_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1} \} &= \text{trace} \ddot{\beta}(\theta; X_{t-1}) + \|\dot{\beta}^T(\theta; X_{t-1}) - \dot{\beta}^T(\theta + u; X_{t-1})\|^2 \\ &= \text{trace} \ddot{\beta}(\theta; X_{t-1}) + \|b_t(\theta, u)\|^2. \end{aligned}$$

Using these expressions one can check conditions of the relevant theorems for different choices of functions m and β .

Now suppose that θ is one dimensional and consider the class of conditionally additive exponential families, that is,

$$f(y; \theta, x) = h(x, y) \exp(\theta m(y, x) - \beta(\theta; x)),$$

with

$$(4.6) \quad \beta(\theta; x) = \gamma(\theta)h(x)$$

where $h(\cdot) \geq 0$ and $\ddot{\gamma}(\cdot) \geq 0$ (see Feigin (1981)). Then,

$$I_t(\theta) = \ddot{\gamma}(\theta)H_t \quad \text{where} \quad H_t = \sum_{s=1}^t h(X_{s-1}).$$

Assuming that $\ddot{\gamma}(\theta) \neq 0$, the likelihood recursive procedure is

$$(4.7) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{\ddot{\gamma}(\hat{\theta}_{t-1})H_t} \left(m(X_t, X_{t-1}) - \dot{\gamma}(\hat{\theta}_{t-1})h(X_{t-1}) \right).$$

The following result gives sufficient conditions for the convergence of (4.7).

Proposition 4.1 *Suppose that $H_t \rightarrow \infty$ (P^θ -a.s.) and either $\dot{\gamma}$ is a linear function, or the following conditions are satisfied:*

(M1)

$$\frac{h(X_{t-1})}{H_t} \rightarrow 0, \quad P^\theta\text{-a.s.};$$

(M2) *for any finite a and b ,*

$$0 < \inf_{u \in [a,b]} \ddot{\gamma}(u) \leq \sup_{u \in [a,b]} \ddot{\gamma}(u) < \infty;$$

(M3) *there exists a constant B such that*

$$\frac{1 + \dot{\gamma}^2(u)}{\ddot{\gamma}^2(u)} \leq B(1 + u^2)$$

for each $u \in \mathbb{R}$.

Then $\hat{\theta}_t$ defined by (4.7) is strongly consistent (i.e., $\hat{\theta}_t \rightarrow \theta$ P^θ -a.s.) for any initial value $\hat{\theta}_0$.

Proof. See Appendix B.

In the next statement we assume that the recursive procedure converges and study the rate of convergence.

Corollary 4.2 *Suppose that $\hat{\theta}_t$ defined by (4.7) is strongly consistent (i.e., $\hat{\theta}_t \rightarrow \theta$ P^θ -a.s.). Suppose also that*

(1) $H_t \rightarrow \infty$, P^θ -a.s.;

(2)

$$\frac{h(X_t)}{H_t} \rightarrow 0, \quad P^\theta\text{-a.s.};$$

(3) $\ddot{\gamma}(\cdot)$ *is a continuous positive function.*

Then $H_t^\delta(\hat{\theta}_t - \theta) \rightarrow 0$ (P^θ -a.s.) for any $\delta \in]0, 1/2[$.

Proof. As always (see the convention in Section 2), convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. By (4.4),

$$(4.8) \quad b_t(\theta, u) = h(X_{t-1})(\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u)).$$

Let us check that the conditions of Corollary 3.1 are satisfied with $\psi_t(\theta) = l_t(\theta) = m(X_t, X_{t-1}) - \dot{\gamma}(\theta)h(X_{t-1})$, $\Gamma_t(\theta) = I_t(\theta) = H_t\ddot{\gamma}(\theta)$, $a_t(\theta) = H_t$, $C_\theta = 1$ and $\mathcal{P}_t = H_t^{-2}\ddot{\gamma}^{-2}(\theta + \Delta_{t-1})b_t^2(\theta, \Delta_{t-1})$. Since $\Delta H_t = h(X_{t-1})$, (R1) is obviously translated into (2). Since $\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u) = -\ddot{\gamma}(\theta + \tilde{u})u$ where $|\tilde{u}| \leq |u|$, the left hand side of (3.4) is

$$-2\frac{h(X_{t-1})}{H_t}\frac{\ddot{\gamma}(\theta + \tilde{\Delta}_{t-1})}{\ddot{\gamma}(\theta + \Delta_{t-1})}\Delta_{t-1}^2 + \frac{h^2(X_{t-1})}{H_t^2}\left(\frac{\ddot{\gamma}(\theta + \tilde{\Delta}_{t-1})}{\ddot{\gamma}(\theta + \Delta_{t-1})}\right)^2\Delta_{t-1}^2$$

Since $\ddot{\gamma}(\cdot)$ is continuous and $\Delta_{t-1} = \hat{\theta}_t - \theta \rightarrow 0$, for any small $\tilde{\varepsilon} > 0$ (which may depend on θ), $1 - \tilde{\varepsilon} < \ddot{\gamma}(\theta + \tilde{\Delta}_{t-1})/\ddot{\gamma}(\theta + \Delta_{t-1}) < 1 + \tilde{\varepsilon}$ for large t 's. So, (3.4) holds with

$$\lambda_t(\theta) = 2(1 - \tilde{\varepsilon})\frac{h(X_{t-1})}{H_t} - (1 + \tilde{\varepsilon})^2\frac{h^2(X_{t-1})}{H_t^2}.$$

To check (3.5), consider

$$(4.9) \quad \frac{h(X_{t-1})}{H_t} - \lambda_t(\theta) = \frac{h(X_{t-1})}{H_t}\left(-1 + 2\tilde{\varepsilon} + (1 + \tilde{\varepsilon})^2\frac{h(X_{t-1})}{H_t}\right)$$

Now, since $\tilde{\varepsilon}$ is arbitrary, we can assume that $-1 + 2\tilde{\varepsilon} < 0$. Also, it follows from (2) that $h(X_{t-1})/H_t \rightarrow 0$. Therefore, (4.9) is negative for large t 's, implying that (3.5) holds true.

To check (R3) note that by (4.5),

$$(4.10) \quad E_\theta \{l_t^2(\theta + u) \mid \mathcal{F}_{t-1}\} = \ddot{\gamma}(\theta)h(X_{t-1}) + b_t^2(\theta, u)$$

and so,

$$H_t^\varepsilon (E_\theta \{H_t^{-2}\ddot{\gamma}^{-2}(\theta + \Delta_{t-1})l_t^2(\theta + \Delta_{t-1}) \mid \mathcal{F}_{t-1}\} - \mathcal{P}_t) = \frac{h(X_{t-1})}{H_t^{2-\varepsilon}}\frac{\ddot{\gamma}(\theta)}{\ddot{\gamma}^2(\theta + \Delta_{t-1})}.$$

Now, (R3) follows from (3) and Proposition A2 in Appendix A. \diamond

A particular example of conditional additive exponential family is the Gaussian autoregressive model defined by

$$X_t = \theta X_{t-1} + Z_t, \quad t = 1, 2, \dots,$$

where $\theta \in \mathbb{R}$, $X_0 = 0$ and Z_t 's are independent random variables with the standard normal distribution. In this model $m(y, x) = xy$ and $\beta(\theta, x) = \frac{1}{2}x^2\theta^2$ so that we can assume that $\gamma(\theta) = \theta^2/2$ and $h(x) = x^2$. Then

$$l_t(\theta) = X_t X_{t-1} - X_{t-1}^2 \theta, \quad I_t = I_t(\theta) = \sum_{s=1}^t X_{s-1}^2.$$

Therefore,

$$(4.11) \quad \begin{aligned} \hat{\theta}_t &= \hat{\theta}_{t-1} + \frac{1}{I_t} \left(X_t X_{t-1} - X_{t-1}^2 \hat{\theta}_{t-1} \right) \\ I_t &= I_{t-1} + X_{t-1}^2. \end{aligned}$$

Note that the rate of the conditional Fisher information I_t varies for the different values of θ . Suppose

$$(4.12) \quad \kappa_t(\theta) = \begin{cases} t(1 - \theta^2)^{-1} & \text{for } |\theta| < 1 \\ \frac{1}{2}t^2 & \text{for } |\theta| = 1 \\ \theta^{2t}(\theta^2 - 1)^{-2} & \text{for } |\theta| > 1. \end{cases}$$

For $|\theta| < 1$, $I_t/\kappa_t(\theta) \rightarrow 1$ in probability as $t \rightarrow \infty$, whereas $I_t/\kappa_t(\theta) \rightarrow W \sim \chi^2(1)$ almost surely in the case $|\theta| > 1$ (non-ergodic case). In the case $|\theta| = 1$, the ratio $I_t/\kappa_t(\theta)$ converges in distribution, but not in probability (for details, see White (1958) and Anderson (1959)). It is also well known that $I_t \rightarrow \infty$ almost surely for any $\theta \in \mathbb{R}$ (see, e.g, Shiriyayev (1984), Ch.VII, 5.5). Also, since $\dot{\gamma}(\theta)$ is linear and $H_t = I_t$, the conditions of Proposition 4.1 are trivially satisfied. Therefore, for any $\theta \in \mathbb{R}$, the recursive estimator $\hat{\theta}_t$ is strongly consistent for any choice of the initial $\hat{\theta}_0$.

To establish the rate of convergence we assume that the process is (strongly) stationary and ergodic. So, $|\theta| < 1$ and it follows from the ergodic theorem for stationary processes that the limit

$$(4.13) \quad \lim_{t \rightarrow \infty} \frac{1}{t} I_t$$

exist P^θ -a.s. and is finite (it can be proved this holds without assumption of strong stationarity.) Now, taking $H_t = I_t$, we obtain that

$$\frac{\Delta I_t}{I_{t-1}} = \frac{I_t}{I_{t-1}} - 1 = \frac{t}{t-1} d_t - 1 \rightarrow 0,$$

since $d_t = ((t-1)/I_{t-1})(I_t/t) \rightarrow 1$. This implies that (2) of Corollary 4.2 holds. (Note that for the non-ergodic case $|\theta| > 1$, we do not expect (2) to hold since in this case $\Delta\kappa_t/\kappa_{t-1} = \theta^2 - 1 \neq 0$.)

So, the conditions of Corollary 4.2 are satisfied implying that $t^\delta(\hat{\theta}_t - \theta) \rightarrow 0$ for any $0 < \delta < 1/2$.

APPENDIX A

Lemma A1 *Let $\mathcal{F}_0, \mathcal{F}_1, \dots$ be a non-decreasing sequence of σ -algebras and $X_n, \beta_n, \xi_n, \zeta_n \in \mathcal{F}_n$, $n \geq 0$, are nonnegative r.v.'s such that*

$$E(X_n | \mathcal{F}_{n-1}) \leq X_{n-1}(1 + \beta_{n-1}) + \xi_{n-1} - \zeta_{n-1}, \quad n \geq 1$$

eventually. Then

$$\left\{ \sum_{i=1}^{\infty} \xi_{i-1} < \infty \right\} \cap \left\{ \sum_{i=1}^{\infty} \beta_{i-1} < \infty \right\} \subseteq \{X \rightarrow\} \cap \left\{ \sum_{i=1}^{\infty} \zeta_{i-1} < \infty \right\} \quad (P\text{-a.s.}),$$

where $\{X \rightarrow\}$ denotes the set where $\lim_{n \rightarrow \infty} X_n$ exists and is finite.

Remark Proof can be found in Robbins and Siegmund (1971). Note also that this lemma is a special case of the theorem on the convergence sets non-negative semimartingales (see, e.g., Lazrieva, Sharia, and Toronjadze (1997)).

Proposition A2 *If d_n is a nondecreasing sequence of positive numbers such that $d_n \rightarrow +\infty$, then*

$$\sum_{n=1}^{\infty} \Delta d_n / d_n = +\infty$$

and

$$\sum_{n=1}^{\infty} \Delta d_n / d_n^{1+\varepsilon} < +\infty$$

for any $\varepsilon > 0$.

Proof The first claim is easily obtained by contradiction from the Kronecker lemma (see, e.g., Lemma 2, §3, Ch. IV in Shirayev (1984)). The second one is proved by the following argument

$$\begin{aligned} 0 &\leq \sum_{n=1}^N \frac{\Delta d_n}{d_n^{1+\varepsilon}} \leq \sum_{n=1}^N \int_0^1 \frac{\Delta d_n}{(d_{n-1} + t\Delta d_n)^{1+\varepsilon}} dt = \sum_{n=1}^N \frac{1}{\varepsilon} \left(\frac{1}{d_{n-1}^\varepsilon} - \frac{1}{d_n^\varepsilon} \right) \\ &= \frac{1}{\varepsilon} \left(\frac{1}{d_0^\varepsilon} - \frac{1}{d_N^\varepsilon} \right) \rightarrow \frac{1}{\varepsilon d_0^\varepsilon} < +\infty. \end{aligned}$$

◇

APPENDIX B

Theorem B1 (Sharia (2007), Theorem 3.2) *Suppose that for $\theta \in \mathbb{R}^m$ there exists a real valued nonnegative function $V_\theta(u) : \mathbb{R}^m \rightarrow \mathbb{R}$ having continuous and bounded partial second derivatives and*

(G1) $V_\theta(0) = 0$, and for each $\varepsilon \in (0, 1)$,

$$\inf_{\|u\| \geq \varepsilon} V_\theta(u) > 0;$$

(G2) *there exists a set $A \in \mathcal{F}$ with $P^\theta(A) > 0$ such that for each $\varepsilon \in (0, 1)$,*

$$\sum_{t=1}^{\infty} \inf_{\varepsilon \leq V_\theta(u) \leq 1/\varepsilon} [\mathcal{N}_t(u)]^- = \infty$$

on A , where

$$\begin{aligned} \mathcal{N}_t(u) &= \dot{V}_\theta(u)\Gamma_t^{-1}(\theta + u)E_\theta \{\psi_t(\theta + u) \mid \mathcal{F}_{t-1}\} \\ &\quad + \frac{1}{2} \sup_v \|\ddot{V}_\theta(v)\| E_\theta \{\|\Gamma_t^{-1}(\theta + u)\psi_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1}\}, \end{aligned}$$

(G3) *for $\Delta_t = \hat{\theta}_t - \theta$,*

$$\sum_{t=1}^{\infty} (1 + V_\theta(\Delta_{t-1}))^{-1} [\mathcal{N}_t(\Delta_{t-1})]^+ < \infty, \quad P^\theta\text{-a.s.}$$

Then $\hat{\theta}_t \rightarrow \theta$ (P^θ -a.s.) for any initial value $\hat{\theta}_0$, where $\hat{\theta}_t$ is defined by 3.1.

Proof of Proposition 4.2 As always (see the convention in Section 2), convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. Let us check that the conditions of Theorem B1 above are satisfied with $\psi_t(\theta) = l_t(\theta) = m(X_t, X_{t-1}) - \dot{\gamma}(\theta)h(X_{t-1})$, $\Gamma_t(\theta) = I_t(\theta) = H_t\ddot{\gamma}(\theta)$, and $V_t = u^2$. Using (4.8) and (4.10), we have

$$\begin{aligned} \mathcal{N}_t(u) &= 2u \frac{1}{H_t\ddot{\gamma}(\theta + u)} b_t(\theta, u) + \frac{1}{H_t^2\ddot{\gamma}^2(\theta + u)} E_\theta \{l_t^2(\theta + u) \mid \mathcal{F}_{t-1}\} \\ &= \frac{h(X_{t-1})}{H_t} \frac{\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u)}{\ddot{\gamma}(\theta + u)} u \left(2 + \frac{h(X_{t-1})}{H_t} \frac{\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u)}{u\ddot{\gamma}(\theta + u)} \right) + \frac{h(X_{t-1})}{H_t^2} \frac{\ddot{\gamma}(\theta)}{\ddot{\gamma}^2(\theta + u)} \\ &\quad (4.14) \hspace{25em} =: \mathcal{N}_{1t}(u) + \mathcal{N}_{2t}(u), \end{aligned}$$

with the convention that $0/0 = 0$. Let us show that for large t 's,

$$(4.15) \quad 2 + \frac{h(X_{t-1})}{H_t} \frac{\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u)}{u\ddot{\gamma}(\theta + u)} \geq 1.$$

If $\dot{\gamma}$ is linear, the above inequality trivially holds since $h(X_{t-1})/H_t = \Delta H_t/H_t \leq 1$. For a non-linear case we have (assuming that $u \neq 0$),

$$(4.16) \quad |(\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u))/u\ddot{\gamma}(\theta + u)| = \ddot{\gamma}(\theta + \tilde{u})/\ddot{\gamma}(\theta + u)$$

where $|\tilde{u}| \leq |u|$. Suppose now that $|u| \leq M$ where $0 < M < \infty$. Then it follows from (M2) that the left hand side of (4.16) is bounded by some positive constant. Also, using the obvious inequality $(a - b)^2 \leq 2a^2 + 2b^2$ and (M3), we obtain that $(\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u))^2/\ddot{\gamma}^2(\theta + u) \leq \tilde{B}(1 + u^2)$ for any u (where \tilde{B} may depend on θ). So, the left hand side of (4.16) is less than or equal to $\sqrt{\tilde{B}(1 + u^2)}/u^2 = \sqrt{\tilde{B}(1/u^2 + 1)}$ which is bounded by a positive constant if $|u| \geq M$. So, the left hand side of (4.16) is bounded by a constant (which may depend on θ) for any u . So, because of (M1) it follows that (4.15) holds for large t 's. This implies that $\mathcal{N}_{1t}(u) \leq 0$ for large t 's (recall that $\ddot{\gamma}(\cdot)$ is positive). So, using (M3) we obtain that for large t 's,

$$\frac{1}{(1 + u^2)} [\mathcal{N}_t(u)]^+ \leq \frac{1}{(1 + u^2)} [\mathcal{N}_{2t}(u)]^+ \leq \frac{h(X_{t-1})}{H_t^2} B_1,$$

for some constant B_1 which may depend on θ . Now, since $\sum_{t=1}^{\infty} h(X_{t-1})/H_t^2 < \infty$ (see Proposition A2 in Appendix A), condition (G3) of Theorem B1 is satisfied. To check condition (G2), note that $(\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u))u \leq 0$, use the obvious inequality $[x]^- \geq -x$, and (4.15) to obtain that for large t 's

$$\begin{aligned} [\mathcal{N}_t(u)]^- &\geq -\mathcal{N}_{1t}(u) - \mathcal{N}_{2t}(u) \geq -\frac{h(X_{t-1})}{H_t} \frac{\dot{\gamma}(\theta) - \dot{\gamma}(\theta + u)}{\ddot{\gamma}(\theta + u)} u - \mathcal{N}_{2t}(u) \\ &= \frac{h(X_{t-1})}{H_t} \frac{\ddot{\gamma}(\theta + \tilde{u})}{\ddot{\gamma}(\theta + u)} u^2 - \frac{h(X_{t-1})}{H_t^2} \frac{\ddot{\gamma}(\theta)}{\ddot{\gamma}^2(\theta + u)} \end{aligned}$$

where $|\tilde{u}| \leq |u|$. Then, it follows from (M2) that $\sup_{\varepsilon \leq |u| \leq 1/\varepsilon} \ddot{\gamma}(\theta)/\ddot{\gamma}^2(\theta + u) < R$ and $\inf_{\varepsilon \leq |u| \leq 1/\varepsilon} \ddot{\gamma}(\theta + \tilde{u})u^2/\ddot{\gamma}(\theta + u) > r > 0$ (where the positive constants R and r may depend on θ). Note also that these inequalities trivially hold for the linear case. Therefore, using once more Proposition A2 in Appendix A we obtain that $\sum_{t=1}^{\infty} \inf_{\varepsilon \leq |u| \leq 1/\varepsilon} [\mathcal{N}_t(u)]^- = \infty$ which completes the proof. \diamond

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