

Recursive Parameter Estimation: Convergence

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Abstract

We consider estimation procedures which are recursive in the sense that each successive estimator is obtained from the previous one by a simple adjustment. We propose a wide class of recursive estimation procedures for the general statistical model and study convergence.

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1 Introduction

Let X_1, \dots, X_n be independent identically distributed (i.i.d.) random variables (r.v.'s) with a common distribution function F_θ with a real unknown parameter θ . An M -estimator of θ is defined as a statistic $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$, which is a solution w.r.t. v of the estimating equation

$$(1.1) \quad \sum_{i=1}^n \psi(X_i; v) = 0,$$

where ψ is a suitably chosen function. For example, if θ is a location parameter in the normal family of distribution functions, the choice $\psi(x, v) = x - v$ gives the MLE (maximum likelihood estimator). For the same problem, if $\psi(x, v) = \text{sign}(x - v)$, the solution of (1.1) reduces to the median of X_1, \dots, X_n . In general, if $f(x, \theta)$ is the probability density function (or probability function) of $F_\theta(x)$ (w.r.t. a σ -finite measure μ) then the choice $\psi(x, v) = f'(x, v)/f(x, v)$ yields the MLE.

Suppose now that X_1, \dots, X_n are not necessarily independent or identically distributed r.v's, with a joint distribution depending on a real parameter θ . Then an M -estimator of θ is defined as a solution of the estimating equation

$$(1.2) \quad \sum_{i=1}^n \psi_i(v) = 0,$$

where $\psi_i(v) = \psi_i(X_{i-k}^i; v)$ with $X_{i-k}^i = (X_{i-k}, \dots, X_i)$. So, the ψ -functions may now depend on the past observations as well. For instance, if X_i 's are observations from a discrete time Markov process, then one can assume that $k = 1$. In general, if no restrictions are placed on the dependence structure of the process X_i , one may need to consider ψ -functions depending on the vector of all past and present observations of the process (that is, $k = i - 1$). If the conditional probability density function (or probability function) of the observation X_i , given X_{i-k}, \dots, X_{i-1} , is $f_i(x, \theta) = f_i(x, \theta | X_{i-k}, \dots, X_{i-1})$, then one can obtain the MLE on choosing $\psi_i(v) = f'_i(X_i, v) / f_i(X_i, v)$. Besides MLEs, the class of M -estimators includes estimators with special properties such as robustness. Under certain regularity and ergodicity conditions it can be proved that there exists a consistent sequence of solutions of (1.2) which has the property of local asymptotic linearity. (See e.g., Serfling [24], Huber [9], Lehman [16]. A comprehensive bibliography can be found in Launer and Wilkinson [12], Hampel et al [7], Rieder [21], and Jurečková and Sen [10].)

If ψ -functions are nonlinear, it is rather difficult to work with the corresponding estimating equations, especially if for every sample size n (when new data are acquired), an estimator has to be computed afresh. In this paper we consider estimation procedures which are recursive in the sense that each successive estimator is obtained from the previous one by a simple adjustment. Note that for a linear estimator, e.g., for the sample mean, $\hat{\theta}_n = \bar{X}_n$ we have $\bar{X}_n = (n-1)\bar{X}_{n-1}/n + X_n/n$, that is $\hat{\theta}_n = \hat{\theta}_{n-1}(n-1)/n + X_n/n$, indicating that the estimator $\hat{\theta}_n$ at each step n can be obtained recursively using the estimator at the previous step $\hat{\theta}_{n-1}$ and the new information X_n . Such an exact recursive relation may not hold for nonlinear estimators (see, e.g., the case of the median).

In general, the following heuristic argument can be used to establish a possible form of an approximate recursive relation (see also Jurečková and Sen [10], Khas'minskii and Nevelson [11], Lazrieva and Toronjadze [15]). Since $\hat{\theta}_n$ is defined as a root of the estimating equation (1.2), denoting the left hand side of (1.2) by $M_n(v)$ we have $M_n(\hat{\theta}_n) = 0$ and $M_{n-1}(\hat{\theta}_{n-1}) = 0$. Assuming that the difference $\hat{\theta}_n - \hat{\theta}_{n-1}$ is "small" we can write

$$0 = M_n(\hat{\theta}_n) - M_{n-1}(\hat{\theta}_{n-1}) = M_n\left(\hat{\theta}_{n-1} + (\hat{\theta}_n - \hat{\theta}_{n-1})\right) - M_{n-1}(\hat{\theta}_{n-1})$$

$$\begin{aligned} &\approx M_n(\hat{\theta}_{n-1}) + M'_n(\hat{\theta}_{n-1})(\hat{\theta}_n - \hat{\theta}_{n-1}) - M_{n-1}(\hat{\theta}_{n-1}) \\ &= M'_n(\hat{\theta}_{n-1})(\hat{\theta}_n - \hat{\theta}_{n-1}) + \psi_n(\hat{\theta}_{n-1}). \end{aligned}$$

Therefore,

$$\hat{\theta}_n \approx \hat{\theta}_{n-1} - \frac{\psi_n(\hat{\theta}_{n-1})}{M'_n(\hat{\theta}_{n-1})},$$

where $M'_n(\theta) = \sum_{i=1}^n \psi'_i(\theta)$. Now, depending on the nature of the underlying model, $M'_n(\theta)$ can be replaced by a simpler expression. For instance, in i.i.d. models with $\psi(x, v) = f'(x, v)/f(x, v)$ (the MLE case), by the strong law of large numbers,

$$\frac{M'_n(\theta)}{n} = \frac{1}{n} \sum_{i=1}^n (f'(X_i, \theta)/f(X_i, \theta))' \approx E_\theta [(f'(X_1, \theta)/f(X_1, \theta))'] = -i(\theta)$$

for large n 's, where $i(\theta)$ is the one-step Fisher information. So, in this case, one can use the recursion

$$(1.3) \quad \hat{\theta}_n = \hat{\theta}_{n-1} + \frac{1}{n} \frac{f'(X_n, \hat{\theta}_{n-1})}{i(\hat{\theta}_{n-1}) f(X_n, \hat{\theta}_{n-1})}, \quad n \geq 1,$$

to construct an estimator which is ‘‘asymptotically equivalent’’ to the MLE.

Motivated by the above argument, we consider a class of estimators

$$(1.4) \quad \hat{\theta}_n = \hat{\theta}_{n-1} + \Gamma_n^{-1}(\hat{\theta}_{n-1})\psi_n(\hat{\theta}_{n-1}), \quad n \geq 1,$$

where ψ_n is a suitably chosen vector process, Γ_n is a (possibly random) normalizing matrix process and $\hat{\theta}_0 \in \mathbb{R}^m$ is some initial value. Note that while the main goal is to study recursive procedures with non-linear ψ_n functions, it is worth mentioning that any linear estimator can be written in the form (1.4) with linear, w.r.t. θ , ψ_n functions. Indeed, if $\hat{\theta}_n = \Gamma_n^{-1} \sum_{k=1}^n h_k(X_k)$, where Γ_k and $h_k(X_k)$ are matrix and vector processes of suitable dimensions, then (see Section 4.2 for details)

$$\hat{\theta}_n = \hat{\theta}_{n-1} + \Gamma_n^{-1} \left(h_n(X_n) - (\Gamma_n - \Gamma_{n-1})\hat{\theta}_{n-1} \right),$$

which is obviously of the form (1.4) with $\psi_n(\theta) = h_n(X_n) - (\Gamma_n - \Gamma_{n-1})\theta$.

It should be noted that at first glance, recursions (1.3) and (1.4) resemble the Newton-Raphson iterative procedure of numerical optimisation. In the i.i.d. case, the Newton-Raphson iteration for the likelihood equation is

$$(1.5) \quad \vartheta_k = \vartheta_{k-1} + J^{-1}(\vartheta_{k-1}) \sum_{i=1}^n \frac{f'(X_i, \vartheta_{k-1})}{f(X_i, \vartheta_{k-1})}, \quad k \geq 1,$$

where $J(v)$ is minus the second logarithmic derivative of the log-likelihood function, that is, $-\sum_{i=1}^n \frac{\partial}{\partial v} (f'(X_i, v)/f(X_i, v))$ or its expectation, that is, the information matrix $ni(v)$. In the latter case, the iterative scheme is often called the method of scoring, see e.g., Harvey [8]. (We do not consider the so called one-step Newton-Raphson method since it requires an auxiliary consistent estimator). The main feature of the scheme (1.5) is that ϑ_k , at each step $k = 1, 2, \dots$, is $\sigma(X_1, \dots, X_n)$ -measurable (where $\sigma(X_1, \dots, X_n)$ is the σ -field generated by the random variables X_1, \dots, X_n). In other words, (1.5) is a deterministic procedure to find a root, say $\tilde{\theta}_n$, of the likelihood equation

$\sum_{i=1}^n (f'(X_i, v)/f(X_i, v)) = 0$. On the other hand the random variable $\hat{\theta}_n$ derived from (1.3) is an estimator of θ for each $n=1, 2, \dots$ (is $\sigma(X_1, \dots, X_n)$ -measurable at each n). Note also that in the iid case, (1.3) can be regarded as a stochastic iterative scheme, i.e., a classical stochastic approximation procedure, to detect the root of an unknown function when the latter can only be observed with random errors (see Remark 3.1). A theoretical implication of this is that by studying the procedures (1.3), or in general (1.4), we study asymptotic behaviour of the estimator of the unknown parameter. As far as applications are concerned, there are several advantages in using (1.4). Firstly, these procedures are easy to use since each successive estimator is obtained from the previous one by a simple adjustment and without storing all the data unnecessarily. This is especially convenient when the data come sequentially. Another potential benefit of using (1.4) is that it allows one to monitor and detect certain changes in probabilistic characteristics of the underlying process such as change of the value of the unknown parameter. So, there may be a benefit in using these procedures in linear cases as well.

In i.i.d. models, estimating procedures similar to (1.4) have been studied by a number of authors using methods of stochastic approximation theory (see, e.g., Khas'minskii and Nevelson [11], Fabian [4], Ljung and Soderstrom [19], Ljung et al [18], and references therein). Some work has been done for non i.i.d. models as well. In particular, Englund et al [3] give an asymptotic representation results for certain type of X_n processes. In Sharia [25] theoretical results on convergence, rate of convergence and the asymptotic representation are given under certain regularity and ergodicity assumptions on the model, in the one-dimensional case with $\psi_n(x, \theta) = \frac{\partial}{\partial \theta} \log f_n(x, \theta)$ (see also Campbell [2], Sharia [26] and Lazrieva et al [13]).

In the present paper, we study multidimensional estimation procedures of type (1.4) for the general statistical model. Section 2 introduces the basic model, objects and notation. In Section 3, imposing "global" restrictions on the processes ψ and Γ , we study "global" convergence of the recursive estimators, that is the convergence for an arbitrary starting point $\hat{\theta}_0$. In

Section 4, we demonstrate the use of these results on some examples. (Results on rate of convergence, asymptotic linearity and efficiency, and numerical simulations will appear in subsequent publications, see Sharia [27], [28].)

2 Basic model, notation and preliminaries

Let X_t , $t = 1, 2, \dots$, be observations taking values in a measurable space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ equipped with a σ -finite measure μ . Suppose that the distribution of the process X_t depends on an unknown parameter $\theta \in \Theta$, where Θ is an open subset of the m -dimensional Euclidean space \mathbb{R}^m . Suppose also that for each $t = 1, 2, \dots$, there exists a regular conditional probability density of X_t given values of past observations of X_{t-1}, \dots, X_2, X_1 , which will be denoted by

$$f_t(\theta, x_t | x_1^{t-1}) = f_t(\theta, x_t | x_{t-1}, \dots, x_1),$$

where $f_1(\theta, x_1 | x_1^0) = f_1(\theta, x_1)$ is the probability density of the random variable X_1 . Without loss of generality we assume that all random variables are defined on a probability space (Ω, \mathcal{F}) and denote by $\{P^\theta, \theta \in \Theta\}$ the family of the corresponding distributions on (Ω, \mathcal{F}) .

Let $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$ be the σ -field generated by the random variables X_1, \dots, X_t . By $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ we denote the m -dimensional Euclidean space with the Borel σ -algebra $\mathcal{B}(\mathbb{R}^m)$. Transposition of matrices and vectors is denoted by T . By (u, v) we denote the standard scalar product of $u, v \in \mathbb{R}^m$, that is, $(u, v) = u^T v$.

Suppose that h is a real valued function defined on $\Theta \subset \mathbb{R}^m$. We denote by $\dot{h}(\theta)$ the row-vector of partial derivatives of $h(\theta)$ with respect to the components of θ , that is,

$$\dot{h}(\theta) = \left(\frac{\partial}{\partial \theta^1} h(\theta), \dots, \frac{\partial}{\partial \theta^m} h(\theta) \right).$$

Also we denote by $\ddot{h}(\theta)$ the matrix of second partial derivatives. The $m \times m$ identity matrix is denoted by $\mathbf{1}$.

If for each $t = 1, 2, \dots$, the derivative $\dot{f}_t(\theta, x_t | x_1^{t-1})$ w.r.t. θ exists, then we can define the function

$$l_t(\theta, x_t | x_1^{t-1}) = \frac{1}{f_t(\theta, x_t | x_1^{t-1})} \dot{f}_t^T(\theta, x_t | x_1^{t-1})$$

with the convention $0/0 = 0$.

The *one step conditional Fisher information matrix* for $t = 1, 2, \dots$ is defined as

$$i_t(\theta | x_1^{t-1}) = \int l_t(\theta, z | x_1^{t-1}) l_t^T(\theta, z | x_1^{t-1}) f_t(\theta, z | x_1^{t-1}) \mu(dz).$$

We shall use the notation

$$\begin{aligned} f_t(\theta) &= f_t(\theta, X_t | X_1^{t-1}), & l_t(\theta) &= l_t(\theta, X_t | X_1^{t-1}), \\ i_t(\theta) &= i_t(\theta | X_1^{t-1}). \end{aligned}$$

Note that the process $i_t(\theta)$ is “predictable”, that is, the random variable $i_t(\theta)$, is \mathcal{F}_{t-1} measurable for each $t \geq 1$.

Note also that by definition, $i_t(\theta)$ is a version of the conditional expectation w.r.t. \mathcal{F}_{t-1} , that is,

$$i_t(\theta) = E_\theta \{l_t(\theta)l_t^T(\theta) | \mathcal{F}_{t-1}\}.$$

Everywhere in the present work conditional expectations are meant to be calculated as integrals w.r.t. the conditional probability densities.

The *conditional Fisher information* at time t is

$$I_t(\theta) = \sum_{s=1}^t i_s(\theta), \quad t = 1, 2, \dots$$

If the X_t 's are independent random variables, $I_t(\theta)$ reduces to the standard Fisher information matrix. Sometimes $I_t(\theta)$ is referred as the incremental expected Fisher information. Detailed discussion of this concept and related work appears in Barndorff-Nielsen and Sorensen [1], Prakasa-Rao [20] Ch.3, and Hall and Heyde [6].

We say that $\psi = \{\psi_t(\theta, x_t, x_{t-1}, \dots, x_1)\}_{t \geq 1}$ is a sequence of estimating functions and write $\psi \in \Psi$, if for each $t \geq 1$, $\psi_t(\theta, x_t, x_{t-1}, \dots, x_1) : \Theta \times \mathbf{X}^t \rightarrow \mathbb{R}^m$ is a Borel function.

Let $\psi \in \Psi$ and denote $\psi_t(\theta) = \psi_t(\theta, X_t, X_{t-1}, \dots, X_1)$. We write $\psi \in \Psi^M$ if $\psi_t(\theta)$ is a martingale-difference process for each $\theta \in \Theta$, i.e., if $E_\theta \{\psi_t(\theta) | \mathcal{F}_{t-1}\} = 0$ for each $t = 1, 2, \dots$ (we assume that the conditional expectations above are well-defined and \mathcal{F}_0 is the trivial σ -algebra).

Note that if differentiation of the equation

$$1 = \int f_t(\theta, z | x_1^{t-1})\mu(dz)$$

is allowed under the integral sign, then $\{l_t(\theta, X_t | X_1^{t-1})\}_{t \geq 1} \in \Psi^M$.

Convention *Everywhere in the present work $\theta \in \mathbb{R}^m$ is an arbitrary but fixed value of the parameter. Convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. A sequence of random variables $(\xi_t)_{t \geq 1}$ has some property eventually if for every ω in a set Ω^θ of P^θ probability 1, ξ_t has this property for all t greater than some $t_0(\omega) < \infty$.*

3 Main results

Suppose that $\psi \in \Psi$ and $\Gamma_t(\theta)$, for each $\theta \in \mathbb{R}^m$, is a predictable $m \times m$ matrix process with $\det \Gamma_t(\theta) \neq 0$, $t \geq 1$. Consider the estimator $\hat{\theta}_t$ defined by

$$(3.1) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + \Gamma_t^{-1}(\hat{\theta}_{t-1})\psi_t(\hat{\theta}_{t-1}), \quad t \geq 1,$$

where $\hat{\theta}_0 \in \mathbb{R}^m$ is an arbitrary initial point.

Let $\theta \in \mathbb{R}^m$ be an arbitrary but fixed value of the parameter and for any $u \in \mathbb{R}^m$ define

$$b_t(\theta, u) = E_\theta \{ \psi_t(\theta + u) \mid \mathcal{F}_{t-1} \}, \quad t \geq 1.$$

Theorem 3.1 *Suppose that*

$$(C1) \quad u^T \Gamma_t^{-1}(\theta + u) b_t(\theta, u) < 0 \quad \text{for each } u \neq 0, \quad P^\theta\text{-a.s.}^1;$$

$$(C2) \quad \text{for each } \varepsilon \in (0, 1),$$

$$\sum_{t=1}^{\infty} \inf_{\varepsilon \leq \|u\| \leq 1/\varepsilon} |u^T \Gamma_t^{-1}(\theta + u) b_t(\theta, u)| = \infty, \quad P^\theta\text{-a.s.};$$

(C3) *there exists a predictable scalar process $(B_t^\theta)_{t \geq 1}$ such that*

$$E_\theta \{ \|\Gamma_t^{-1}(\theta + u) \psi_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1} \} \leq B_t^\theta (1 + \|u\|^2)$$

for each $u \in \mathbb{R}^m$, P^θ -a.s., and

$$\sum_{t=1}^{\infty} B_t^\theta < \infty, \quad P^\theta\text{-a.s..}$$

Then $\hat{\theta}_t$ is strongly consistent (i.e., $\hat{\theta}_t \rightarrow \theta$ P^θ -a.s.) for any initial value $\hat{\theta}_0$.

We will derive this theorem from a more general result (see the end of the section). Let us first comment on the conditions used here.

¹Note that the set of P^θ probability 0 where the inequalities in (C1) and (C3) are not valid should not depend on u .

Remark 3.1 Conditions (C1), (C2), and (C3) are natural analogues of the corresponding assumptions in theory of stochastic approximation. Indeed, let us consider the i.i.d. case with

$$f_t(\theta, z \mid x_1^{t-1}) = f(\theta, z), \quad \psi_t(\theta) = \psi(\theta, z)|_{z=X_t},$$

where $\int \psi(\theta, z)f(\theta, z)\mu(dz) = 0$ and $\Gamma_t(\theta) = t\gamma(\theta)$ for some invertible non-random matrix $\gamma(\theta)$. Then

$$b_t(\theta, u) = b(\theta, u) = \int \psi(\theta + u, z)f(\theta, z)\mu(dz),$$

implying that $b(\theta, 0) = 0$. Denote $\Delta_t = \hat{\theta}_t - \theta$ and rewrite (3.1) in the form

$$(3.2) \quad \Delta_t = \Delta_{t-1} + \frac{1}{t} (\gamma^{-1}(\theta + \Delta_{t-1})b(\theta, \Delta_{t-1}) + \varepsilon_t^\theta),$$

where

$$\varepsilon_t^\theta = \gamma^{-1}(\theta + \Delta_{t-1}) \{ \psi(\theta + \Delta_{t-1}, X_t) - b(\theta, \Delta_{t-1}) \}.$$

Equation (3.2) defines a Robbins-Monro stochastic approximation procedure that converges to the solution of the equation

$$R^\theta(u) := \gamma^{-1}(\theta + u)b(\theta, u) = 0,$$

when the values of the function $R^\theta(u)$ can only be observed with zero expectation errors ε_t^θ . Note that in general, recursion (3.1) cannot be considered in the framework of classical stochastic approximation theory (see Lazrieva et al [13], [14] for the generalized Robbins-Monro stochastic approximations procedures). For the i.i.d. case, conditions (C1), (C2) and (C3) can be written as **(I)** and **(II)** in Section 4, which are standard assumptions for stochastic approximation procedures of type (3.2) (see, e.g., Robbins and Monro [22], Gladyshev [5], Khas'minskii and Nevelson [11], Ljung and Soderstrom [19], Ljung et al [18]).

Remark 3.2 To understand how the procedure works, consider the one-dimensional case, denote $\Delta_t = \hat{\theta}_t - \theta$ and rewrite (3.1) in the form

$$\Delta_t = \Delta_{t-1} + \Gamma_t^{-1}(\theta + \Delta_{t-1})\psi_t(\theta + \Delta_{t-1}).$$

Then,

$$E_\theta \left\{ \hat{\theta}_t - \hat{\theta}_{t-1} \mid \mathcal{F}_{t-1} \right\} = E_\theta \left\{ \Delta_t - \Delta_{t-1} \mid \mathcal{F}_{t-1} \right\} = \Gamma_t^{-1}(\theta + \Delta_{t-1})b_t(\theta, \Delta_{t-1}).$$

Suppose now that at time $t-1$, $\hat{\theta}_{t-1} < \theta$, that is, $\Delta_{t-1} < 0$. Then, by (C1), $\Gamma_t^{-1}(\theta + \Delta_{t-1})b_t(\theta, \Delta_{t-1}) > 0$ implying that $E_\theta \left\{ \hat{\theta}_t - \hat{\theta}_{t-1} \mid \mathcal{F}_{t-1} \right\} > 0$. So, the next step $\hat{\theta}_t$ will be in the direction of θ . If at time $t-1$, $\hat{\theta}_{t-1} > \theta$, by the same reason, $E_\theta \left\{ \hat{\theta}_t - \hat{\theta}_{t-1} \mid \mathcal{F}_{t-1} \right\} < 0$. So, the condition (C1) ensures that, on average, at each step the procedure moves towards θ . However, the magnitude of the jumps $\hat{\theta}_t - \hat{\theta}_{t-1}$ should decrease, for otherwise, $\hat{\theta}_t$ may oscillate around θ without approaching it. This is guaranteed by (C3). On the other hand, (C2) ensures that the jumps do not decrease too rapidly to avoid failure of $\hat{\theta}_t$ to reach θ .

Now, let us consider a maximum likelihood type recursive estimator

$$\hat{\theta}_t = \hat{\theta}_{t-1} + I_t^{-1}(\hat{\theta}_{t-1})l_t(\hat{\theta}_{t-1}), \quad t \geq 1,$$

where $l_t(\theta) = \dot{f}_t^T(\theta, X_t \mid X_1^{t-1})/f_t(\theta, X_t \mid X_1^{t-1})$ and $I_t(\theta)$ is the conditional Fisher information with $\det I_t(\theta) \neq 0$ (see also (1.3) for the i.i.d. case). By Theorem 3.1, $\hat{\theta}_t$ is strongly consistent if conditions (C1), (C2) and (C3) are satisfied with $l_t(\theta)$ and $I_t(\theta)$ replacing $\psi_t(\theta)$ and $\Gamma_t(\theta)$ respectively. On the other hand, if e.g., in the one-dimensional case, $b_t(\theta, u)$ is differentiable at $u = 0$ and the differentiation is allowed under the integral sign, then

$$\frac{\partial}{\partial u} b_t(\theta, u) \Big|_{u=0} = E_\theta \left\{ \dot{l}_t(\theta) \mid \mathcal{F}_{t-1} \right\}.$$

So, if the differentiation w.r.t. θ of $E_\theta \left\{ l_t(\theta) \mid \mathcal{F}_{t-1} \right\} = 0$ is allowed under the integral sign, $\frac{\partial}{\partial u} b_t(\theta, u) \Big|_{u=0} = -i_t(\theta)$ implying that (C1) always holds for small values of $u \neq 0$.

Condition (C2) in the i.i.d. case is a requirement that the function $\gamma^{-1}(\theta + u)b(\theta, u)$ is separated from zero on each finite interval that does not contain 0. For the i.i.d. case with continuous w.r.t u functions $b(\theta, u)$ and $i(\theta + u)$, condition (C2) is an easy consequence of (C1).

Condition (C3) is a boundedness type assumption which restricts the growth of $\psi_t(\theta)$ w.r.t. θ with certain uniformity w.r.t. t .

We denote by η^+ (respectively η^-) the positive (respectively negative) part of η .

Theorem 3.2 *Suppose that for $\theta \in \mathbb{R}^m$ there exists a real valued nonnegative function $V_\theta(u) : \mathbb{R}^m \rightarrow \mathbb{R}$ having continuous and bounded partial second derivatives and*

(G1) $V_\theta(0) = 0$, and for each $\varepsilon \in (0, 1)$,

$$\inf_{\|u\| \geq \varepsilon} V_\theta(u) > 0;$$

(G2) there exists a set $A \in \mathcal{F}$ with $P^\theta(A) > 0$ such that for each $\varepsilon \in (0, 1)$,

$$\sum_{t=1}^{\infty} \inf_{\varepsilon \leq V_\theta(u) \leq 1/\varepsilon} [\mathcal{N}_t(u)]^- = \infty$$

on A , where

$$\begin{aligned} \mathcal{N}_t(u) &= \dot{V}_\theta(u) \Gamma_t^{-1}(\theta + u) b_t(\theta, u) \\ &\quad + \frac{1}{2} \sup_v \|\ddot{V}_\theta(v)\| E_\theta \{ \|\Gamma_t^{-1}(\theta + u) \psi_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1} \}, \end{aligned}$$

(G3) for $\Delta_t = \hat{\theta}_t - \theta$,

$$\sum_{t=1}^{\infty} (1 + V_\theta(\Delta_{t-1}))^{-1} [\mathcal{N}_t(\Delta_{t-1})]^+ < \infty, \quad P^\theta\text{-a.s.}$$

Then $\hat{\theta}_t \rightarrow \theta$ (P^θ -a.s.) for any initial value $\hat{\theta}_0$.

Proof. As always (see the convention in Section 2), convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. Rewrite (3.1) in the form

$$\Delta_t = \Delta_{t-1} + \Gamma_t^{-1}(\theta + \Delta_{t-1}) \psi_t(\theta + \Delta_{t-1}).$$

By the Taylor expansion,

$$\begin{aligned} V_\theta(\Delta_t) &= V_\theta(\Delta_{t-1}) + \dot{V}_\theta(\Delta_{t-1}) \Gamma_t^{-1}(\theta + \Delta_{t-1}) \psi_t(\theta + \Delta_{t-1}) \\ &\quad + \frac{1}{2} [\Gamma_t^{-1}(\theta + \Delta_{t-1}) \psi_t(\theta + \Delta_{t-1})]^T \ddot{V}_\theta(\tilde{\Delta}_t) \Gamma_t^{-1}(\theta + \Delta_{t-1}) \psi_t(\theta + \Delta_{t-1}), \end{aligned}$$

where $\tilde{\Delta}_t \in \mathbb{R}^m$. Taking the conditional expectation w.r.t. \mathcal{F}_{t-1} yields

$$E_\theta \{ V_\theta(\Delta_t) \mid \mathcal{F}_{t-1} \} \leq V_\theta(\Delta_{t-1}) + \mathcal{N}_t(\Delta_{t-1}).$$

Using the obvious decomposition $\mathcal{N}_t(\Delta_{t-1}) = [\mathcal{N}_t(\Delta_{t-1})]^+ - [\mathcal{N}_t(\Delta_{t-1})]^-$, the previous inequality can be rewritten as

$$(3.3) \quad E_\theta \{ V_\theta(\Delta_t) \mid \mathcal{F}_{t-1} \} \leq V_\theta(\Delta_{t-1})(1 + B_t) + B_t - [\mathcal{N}_t(\Delta_{t-1})]^-,$$

where

$$B_t = (1 + V_\theta(\Delta_{t-1}))^{-1} [\mathcal{N}_t(\Delta_{t-1})]^+.$$

By condition (G3),

$$(3.4) \quad \sum_{t=1}^{\infty} B_t < \infty.$$

According to Lemma A1 in Appendix A (with $X_n = V_\theta(\Delta_n)$, $\beta_{n-1} = \xi_{n-1} = B_n$ and $\zeta_{n-1} = [\mathcal{N}_n(\Delta_{n-1})]^-$), inequalities (3.3) and (3.4) imply that the processes $V_\theta(\Delta_t)$ and

$$Y_t = \sum_{s=1}^t [\mathcal{N}_s(\Delta_{s-1})]^-$$

converge to some finite limits. It therefore follows that $V_\theta(\Delta_t) \rightarrow r \geq 0$. Suppose that $\{r > 0\}$. Then there exists $\varepsilon > 0$ such that $\varepsilon \leq V_\theta(\Delta_t) \leq 1/\varepsilon$ eventually. Because of (G2), this implies that for some (possibly random) t_0 ,

$$\sum_{s=t_0}^{\infty} [\mathcal{N}_s(\Delta_{s-1})]^- \geq \sum_{s=t_0}^{\infty} \inf_{\varepsilon \leq V_\theta(u) \leq 1/\varepsilon} [\mathcal{N}_s(u)]^- = \infty$$

on the set A with $P^\theta(A) > 0$, which contradicts the existence of a finite limit of Y_t . Hence, $r = 0$ and so, $V_\theta(\Delta_t) \rightarrow 0$. Now, $\Delta_t \rightarrow 0$ follows from (G1) (otherwise there would exist a sequence $t_k \rightarrow \infty$ such that $\|\Delta_{t_k}\| \geq \varepsilon$ for some $\varepsilon > 0$, and (G1) would imply that $\inf_k V_\theta(\Delta_{t_k}) > 0$). \diamond

Proof of Theorem 3.1. As always (see the convention in Section 2), convergence and all relations between random variables are meant with probability one w.r.t. the measure P^θ unless specified otherwise. Let us show that the conditions of Theorem 3.1 imply those in Theorem 3.2 with $V_\theta(u) = (u, u) = u^T u = \|u\|^2$. Condition (G1) trivially holds. Since $\dot{V}_\theta(u) = 2u^T$ and $\ddot{V}_\theta(u) = 2 \times \mathbf{1}$, it follows that

$$(3.5) \quad \mathcal{N}_t(u) = 2u^T \Gamma_t^{-1}(\theta + u) b_t(\theta, u) + E_\theta \left\{ \|\Gamma_t^{-1}(\theta + u) \psi_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1} \right\}.$$

Then, by (C1) and (C3),

$$(3.6) \quad \begin{aligned} & \sum_{t=1}^{\infty} (1 + \|\Delta_{t-1}\|^2)^{-1} [\mathcal{N}_t(\Delta_{t-1})]^+ \\ & \leq \sum_{t=1}^{\infty} (1 + \|\Delta_{t-1}\|^2)^{-1} E_\theta \left\{ \|\Gamma_t^{-1}(\theta + \Delta_{t-1}) \psi_t(\theta + \Delta_{t-1})\|^2 \mid \mathcal{F}_{t-1} \right\} \\ & \leq \sum_{t=1}^{\infty} B_t < \infty. \end{aligned}$$

So, (G3) holds. To derive (G2), using the obvious inequality $[a]^- \geq -a$ and (C1), we write

$$\begin{aligned} \inf [\mathcal{N}_t(u)]^- &\geq \inf [-2u^T \Gamma_t^{-1}(\theta + u)b_t(\theta, u) \\ &\quad - E_\theta \{ \|\Gamma_t^{-1}(\theta + u)\psi_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1} \}] \\ &\geq \inf |2u^T \Gamma_t^{-1}(\theta + u)b_t(\theta, u)| \\ &\quad - \sup [E_\theta \{ \|\Gamma_t^{-1}(\theta + u)\psi_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1} \}], \end{aligned}$$

where inf's and sup's are taken over $\{u : \varepsilon \leq \|u\|^2 \leq 1/\varepsilon\}$. From (C3),

$$\sup [E_\theta \{ \|\Gamma_t^{-1}(\theta + u)\psi_t(\theta + u)\|^2 \mid \mathcal{F}_{t-1} \}] \leq B_t(1 + 1/\varepsilon^2)$$

and $\sum_{t=1}^{\infty} B_t < \infty$. Now, using (C2), we finally obtain

$$\sum_{t=1}^{\infty} \inf [\mathcal{N}_t(u)]^- \geq \sum_{t=1}^{\infty} \inf |2u^T \Gamma_t^{-1}(\theta + u)b_t(\theta, u)| - (1 + 1/\varepsilon^2) \sum_{t=1}^{\infty} B_t = \infty,$$

which implies (G2). So, Theorem 3.1 follows on application of Theorem 3.2. \diamond

Remark 3.3 It follows from the proof of Theorem 3.2 that if conditions (G1) and (G3) are satisfied then $(\hat{\theta}_t - \theta)^2$ converges (P^θ -a.s.) to a finite limit, for any initial value $\hat{\theta}_0$. In particular, to guarantee this convergence, it suffices to require conditions (C1) and (C3) of Theorem 3.1 (this can be seen by taking $V_\theta(u) = (u, u) = u^T u = \|u\|^2$ and (3.6)).

4 SPECIAL MODELS AND EXAMPLES

4.1 The i.i.d. scheme.

Consider the classical scheme of i.i.d. observations X_1, X_2, \dots , with a common probability density/mass function $f(\theta, x)$, $\theta \in \mathbb{R}^m$. Suppose that $\psi(\theta, z)$ is an estimating function with

$$\int \psi(\theta, z) f(\theta, z) \mu(dz) = 0.$$

Let us define the recursive estimator $\hat{\theta}_t$ by

$$(4.1) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t} \gamma^{-1}(\hat{\theta}_{t-1}) \psi(\hat{\theta}_{t-1}, X_t), \quad t \geq 1,$$

where $\gamma(\theta)$ is a non-random matrix such that $\gamma^{-1}(\theta)$ exists for any $\theta \in \mathbb{R}^m$ and $\hat{\theta}_0 \in \mathbb{R}^m$ is any initial value.

Corollary 4.1 *Suppose that for any $\theta \in \mathbb{R}^m$, the following conditions hold.*

(I) *For any $0 < \varepsilon < 1$,*

$$\sup_{\varepsilon \leq \|u\| \leq \frac{1}{\varepsilon}} u^T \gamma^{-1}(\theta + u) \int \psi(\theta + u, x) f(\theta, x) \mu(dx) < 0.$$

(II) *For each $u \in \mathbb{R}^m$,*

$$\int \|\gamma^{-1}(\theta + u) \psi(\theta + u, x)\|^2 f(\theta, x) \mu(dx) \leq K_\theta (1 + \|u\|^2)$$

for some constant K_θ .

Then the estimator $\hat{\theta}_t$ is strongly consistent for any initial value $\hat{\theta}_0$.

Proof Since $b_t(\theta, u) = b(\theta, u) = \int \psi(\theta + u, z) f(\theta, z) \mu(dz)$ and $\Gamma_t(\theta) = t\gamma(\theta)$, it is easy to see that (I) and (II) imply (C1), (C2) and (C3) from Theorem 3.1 which yields $(\hat{\theta}_t - \theta) \rightarrow 0$ (P^θ -a.s.).

Similar results (for i.i.d. schemes) were obtained by Khas'minskii and Nevelson [11] Ch.8, §4, and Fabian [4]. Note that conditions (I) and (II) are derived from Theorem 3.1 and are sufficient conditions for the convergence of (4.1). Applying Theorem 3.2 to (4.1), one can obtain various alternative sufficient conditions analogous to those given in Fabian (1978). Note also that, in (4.1), the normalising sequence is $\Gamma_t(\theta) = t\gamma(\theta)$, but Theorems 3.1 and 4.1 allow to consider procedures with arbitrary predictable $\Gamma_t(\theta)$.

4.2 Linear procedures.

Consider the recursion

$$(4.2) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + \Gamma_t^{-1} \left(h_t - \gamma_t \hat{\theta}_{t-1} \right), \quad t \geq 1,$$

where the Γ_t and γ_t are predictable processes, h_t is an adapted process (i.e., h_t is \mathcal{F}_t -measurable for $t \geq 1$) and all three are independent of θ . The following result gives a sets of sufficient conditions for the convergence of (4.2) in the case when the linear $\psi_t(\theta) = h_t - \gamma_t \theta$ is a martingale-difference.

Corollary 4.2 *Suppose that for any $\theta \in \mathbb{R}$,*

$$(a) \quad E_\theta \{h_t \mid \mathcal{F}_{t-1}\} = \gamma_t \theta, \quad \text{for } t \geq 1, \quad P^\theta\text{-a.s.},$$

(b) $0 \leq \gamma_t/\Gamma_t \leq 2 - \delta$ eventually for some $\delta > 0$, and

$$\sum_{t=1}^{\infty} \gamma_t/\Gamma_t = \infty,$$

on a set A of positive probability P^θ .

(c)

$$\sum_{t=1}^{\infty} \frac{E_\theta \{(h_t - \theta\gamma_t)^2 \mid \mathcal{F}_{t-1}\}}{\Gamma_t^2} < \infty, \quad P^\theta\text{-a.s.}$$

Then $\hat{\theta}_t \rightarrow \theta$ (P^θ -a.s.) for any initial value $\hat{\theta}_0 \in \mathbb{R}$.

Proof. We need to check that the conditions of Theorem 3.2 hold for for $V_\theta(u) = u^2$. Using (a) we obtain

$$b_t(\theta, u) = E_\theta \{(h_t - (\theta + u)\gamma_t) \mid \mathcal{F}_{t-1}\} = -u\gamma_t$$

and

$$\begin{aligned} E_\theta \{(\psi_t(\theta + u))^2 \mid \mathcal{F}_{t-1}\} &= E_\theta \{(h_t - (\theta + u)\gamma_t)^2 \mid \mathcal{F}_{t-1}\} \\ &= E_\theta \{(h_t - \theta\gamma_t)^2 \mid \mathcal{F}_{t-1}\} + u^2\gamma_t^2 = \mathcal{P}_t^\theta + u^2\gamma_t^2, \end{aligned}$$

where $\mathcal{P}_t^\theta = E_\theta \{(h_t - \theta\gamma_t)^2 \mid \mathcal{F}_{t-1}\}$. Now, using (3.5),

$$\begin{aligned} N_t(u) &= -2u^2\gamma_t\Gamma_t^{-1} + \Gamma_t^{-2}\mathcal{P}_t^\theta + u^2\gamma_t^2\Gamma_t^{-2} \\ &= -\delta u^2\gamma_t\Gamma_t^{-1} - u^2\gamma_t\Gamma_t^{-1}((2 - \delta) - \gamma_t\Gamma_t^{-1}) + \Gamma_t^{-2}\mathcal{P}_t^\theta. \end{aligned}$$

To derive (G2), we use the obvious inequality $[a]^- \geq -a$ (for any a), conditions (b) and (c), and write

$$\sum_{t=1}^{\infty} \inf_{\varepsilon \leq u^2 \leq 1/\varepsilon} [\mathcal{N}_t(u)]^- \geq \sum_{t=1}^{\infty} \inf_{\varepsilon \leq u^2 \leq 1/\varepsilon} (\delta u^2\gamma_t\Gamma_t^{-1} - \Gamma_t^{-2}\mathcal{P}_t^\theta) = \infty$$

on A . To check (G3) we write

$$\sum_{t=1}^{\infty} (1 + \Delta_{t-1}^2)^{-1} [\mathcal{N}_t(\Delta_{t-1})]^+ \leq \sum_{t=1}^{\infty} [\mathcal{N}_t(\Delta_{t-1})]^+ \leq \sum_{t=1}^{\infty} \Gamma_t^{-2}\mathcal{P}_t^\theta < \infty$$

(P^θ -a.s.), which completes the proof. \diamond

Remark 4.1 Suppose that $\Delta\Gamma_t = \gamma_t$. Then

$$(4.3) \quad \hat{\theta}_t = \Gamma_t^{-1} \left(\hat{\theta}_0 + \sum_{s=1}^t h_s(X_s) \right).$$

This can be easily seen by inspecting the difference $\hat{\theta}_t - \hat{\theta}_{t-1}$ for the sequence (4.3), to check that (4.2) holds. It is also interesting to observe that since in this case, $\Gamma_t = \sum_{s=1}^t \gamma_s$,

$$\hat{\theta}_t = \Gamma_t^{-1} \hat{\theta}_0 + \Gamma_t^{-1} \sum_{s=1}^t (h_s(X_s) - \gamma_s \theta) + \theta = \Gamma_t^{-1} \hat{\theta}_0 + \Gamma_t^{-1} M_t^\theta + \theta$$

where, $M_t^\theta = \sum_{s=1}^t (h_s(X_s) - \gamma_s \theta)$ is a P^θ martingale. Now, if $\Gamma_t \rightarrow \infty$, a necessary and sufficient condition for the convergence to θ is the convergence to zero of the sequence $\Gamma_t^{-1} M_t^\theta$. Condition (c) in Corollary 4.2 is a standard sufficient condition in martingale theory to guarantee $\Gamma_t^{-1} M_t^\theta \rightarrow 0$ (see e.g., Shiriyayev [29], Ch.VII, §5 Theorem 4). The first part of (b) will trivially hold if $\gamma_t = \Delta\Gamma_t \geq 0$. Also, in this case, $\Gamma_t \rightarrow \infty$ implies $\sum_{t=1}^\infty \Delta\Gamma_t / \Gamma_t = \infty$ (see Proposition A3 in Appendix A).

Remark 4.2 As a particular example, consider the process

$$X_t = \theta X_{t-1} + \xi_t, \quad t \geq 1,$$

where, ξ_t is a P^θ martingale-difference with $D_t = E_\theta \{\xi_t^2 \mid \mathcal{F}_{t-1}\} > 0$. The choice $h_t = D_t^{-1} X_{t-1} X_t$ and $\Delta\Gamma_t = \gamma_t = D_t^{-1} X_{t-1}^2$, in (4.2) yields the least square estimator of θ . It is easy to verify that (a) holds. Also, since

$$E_\theta \{(h_t - \gamma_t \theta)^2 \mid \mathcal{F}_{t-1}\} = D_t^{-2} X_{t-1}^2 E_\theta \{\xi_t^2 \mid \mathcal{F}_{t-1}\} = D_t^{-1} X_{t-1}^2 = \Delta\Gamma_t,$$

it follows that (c) in Corollary 4.2 is equivalent to $\sum_{t=1}^\infty \Delta\Gamma_t / \Gamma_t^2 < \infty$. This, as well as (b) hold if $\Gamma_t \rightarrow \infty$ (see Proposition A3 in Appendix A). So, if $\Gamma_t \rightarrow \infty$ the least square procedure is strongly consistent. If, e.g., ξ_t are i.i.d. r.v.'s, then $\Gamma_t \rightarrow \infty$ for all values of $\theta \in \mathbb{R}$ (see, e.g, Shiriyayev [29], Ch.VII, 5.5).

4.3 AR(m) process

Consider an AR(m) process

$$X_i = \theta_1 X_{i-1} + \cdots + \theta_m X_{i-m} + \xi_i = \theta^T X_{i-m}^{i-1} + \xi_i,$$

where $X_{i-m}^{i-1} = (X_{i-1}, \dots, X_{i-m})^T$, $\theta = (\theta_1, \dots, \theta_m)^T$ and ξ_i is a sequence of i.i.d. random variables.

A reasonable class of procedures in this model should have a form

$$(4.4) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + \Gamma_t^{-1}(\hat{\theta}_{t-1})\psi_t(X_t - \hat{\theta}_{t-1}^T X_{t-m}^{t-1}),$$

where $\psi_t(z)$ and $\Gamma_t^{-1}(z)$ ($z \in \mathbb{R}^m$) are respectively vector and matrix processes meeting conditions of the previous section. Suppose that the probability density function of ξ_t w.r.t. Lebesgue's measure is $g(x)$. Then the conditional probability density function is $f_t(\theta, x_t | x_1^{t-1}) = g(x_t - \theta^T x_{t-m}^{t-1})$. So, denoting

$$(4.5) \quad \psi_t(z) = -\frac{g'(z)}{g(z)} X_{t-m}^{t-1},$$

it is easy to see that

$$\psi_t(X_t - \theta^T X_{t-m}^{t-1}) = \frac{\dot{f}_t^T(\theta, X_t | X_1^{t-1})}{f_t(\theta, X_t | X_1^{t-1})}$$

and (4.4) becomes a likelihood recursive procedure. A possible choice of $\Gamma_t(z)$ in this case would be the conditional Fisher information matrix

$$I_t = i^g \sum_{s=1}^t X_{t-m}^{s-1} (X_{t-m}^{s-1})^T$$

where

$$i^g = \int \left(\frac{g'(z)}{g(z)} \right)^2 g(z) dz.$$

An interesting class of recursive estimators for strongly stationary AR(m) processes is studied in Campbell [2]. These estimators are recursive versions of robust modifications of the least squares method and are defined as

$$(4.6) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + a_t \gamma(X_{t-m}^{t-1}) \phi(X_t - \hat{\theta}_{t-1}^T X_{t-m}^{t-1}),$$

where a_t is a sequence of a positive numbers with $a_t \rightarrow 0$, ϕ is a bounded scalar function and $\gamma(u)$ is a vector function of the form $uh(u)$ for some non-negative function h of u (See also Leonov [17]). The class of procedures of type (4.6) is clearly a subclass of that defined by (4.4) and therefore can be studied using the results of the previous section.

Suppose that ξ_i are i.i.d. random variables with a bell-shaped, symmetric about zero probability density function $g(z)$ (that is, $g(-z) = g(z)$, and $g \downarrow 0$ on \mathbb{R}_+). Suppose also that $\phi(x)$ is an odd, continuous in zero function. Let us write conditions of Theorem 3.1 for

$$(4.7) \quad \Gamma(\theta) = a_t^{-1} \mathbf{1} \quad \text{and} \quad \psi_t(\theta) = X_{t-m}^{t-1} h(X_{t-m}^{t-1}) \phi(X_t - \theta^T X_{t-m}^{t-1}).$$

We have

$$\begin{aligned} E_\theta \left\{ \phi \left(X_t - (\theta + u)^T X_{t-m}^{t-1} \right) \mid \mathcal{F}_{s-1} \right\} &= E_\theta \left\{ \phi \left(\xi_t - u^T X_{t-m}^{t-1} \right) \mid \mathcal{F}_{s-1} \right\} \\ &= \int \phi \left(z - u^T X_{t-m}^{t-1} \right) g(z) dz. \end{aligned}$$

It follows from Lemma A2 in Appendix A that if $w \neq 0$,

$$G(w) = -w \int_{-\infty}^{\infty} \phi(z-w) g(z) dz > 0.$$

Therefore,

$$\begin{aligned} u^T \Gamma_t^{-1}(\theta + u) b_t(\theta, u) &= a_t u^T X_{t-m}^{t-1} h(X_{t-m}^{t-1}) E_\theta \left\{ \phi(\xi_t - u^T X_{t-m}^{t-1}) \mid \mathcal{F}_{s-1} \right\} \\ (4.8) \quad &= -a_t h(X_{t-m}^{t-1}) G(u^T X_{t-m}^{t-1}) \leq 0. \end{aligned}$$

Also, since ϕ is a bounded function,

$$E_\theta \left\{ \left\| \Gamma_t^{-1}(\theta + u) \psi_t(\theta + u) \right\|^2 \mid \mathcal{F}_{t-1} \right\} \leq C^\theta a_t^2 \|X_{t-m}^{t-1}\|^2 h^2(X_{t-m}^{t-1})$$

for some positive constant C^θ . Therefore, conditions of Theorem 3.1 hold if (P^θ -a.s.),

$$(4.9) \quad \sum_{t=1}^{\infty} a_t h(X_{t-m}^{t-1}) \inf_{\varepsilon \leq \|u\| \leq 1/\varepsilon} G(u^T X_{t-m}^{t-1}) = \infty$$

and

$$(4.10) \quad \sum_{t=1}^{\infty} a_t^2 \|X_{t-m}^{t-1}\|^2 h^2(X_{t-m}^{t-1}) < \infty.$$

If X_t is a stationary process, these conditions can be verified using limit theorems for stationary processes. Suppose, e.g., that $a_t = 1/t$, $h(\mathbf{x}) \neq 0$ for any $\mathbf{x} \neq 0$, and $g(z)$ is continuous. Then $h(\mathbf{x}) \inf_{\varepsilon \leq \|u\| \leq 1/\varepsilon} G(u^T \mathbf{x}) > 0$ for any $\mathbf{x} \neq 0$ (see Appendix A, Lemma A2). Therefore, it follows from an ergodic theorem for stationary processes that in probability P^θ ,

$$(4.11) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=1}^t h(X_{s-m}^{s-1}) \inf_{\varepsilon \leq \|u\| \leq 1/\varepsilon} G(u^T X_{s-m}^{s-1}) > 0.$$

Now, (4.9) follows from Proposition A4, in Appendix A.

Examples of the procedures of type (4.6) as well as some simulation results are presented in Campbell [2].

4.4 An explicit example

As a particular example of (4.4), consider the process

$$X_t = \theta X_{t-1} + \xi_t, \quad t \geq 1,$$

where, ξ_t , $t \geq 1$, are independent Student random variables with degrees of freedom α . So, the probability density functions of ξ_t is

$$g(x) = C_\alpha \left(1 + \frac{x^2}{\alpha}\right)^{-\frac{\alpha+1}{2}}$$

where $C_\alpha = \Gamma((\alpha + 1)/2)/(\sqrt{\pi\alpha} \Gamma(\alpha/2))$.

Since

$$\frac{g'(z)}{g(z)} = -(\alpha + 1) \frac{z}{\alpha + z^2}$$

(see also (4.5)),

$$\frac{\dot{f}_t(\theta, X_t | X_{t-1})}{f_t(\theta, X_t | X_{t-1})} = -X_{t-1} \frac{g'}{g}(X_t - \theta X_{t-1}) = (\alpha + 1) X_{t-1} \frac{X_t - \theta X_{t-1}}{\alpha + (X_t - \theta X_{t-1})^2}$$

and the conditional Fisher information is

$$I_t = i^g \sum_{s=1}^t X_{s-1}^2$$

where

$$\begin{aligned} i^g &= \int \left(\frac{g'(z)}{g(z)} \right)^2 g(z) dz = C_\alpha (\alpha + 1)^2 \int \frac{z^2 dz}{(\alpha + z^2)^2 (1 + \frac{z^2}{\alpha})^{\frac{\alpha+1}{2}}} \\ &= C_\alpha \frac{(\alpha + 1)^2}{\sqrt{\alpha}} \int \frac{z^2 dz}{(1 + z^2)^{\frac{\alpha+5}{2}}} \\ &= C_\alpha \frac{(\alpha + 1)^2}{\sqrt{\alpha}} \frac{\sqrt{\pi} \Gamma((\alpha + 5)/2 - 3/2)}{2\Gamma((\alpha + 5)/2)} \\ &= \frac{2(\alpha + 1)}{\alpha + 3}. \end{aligned}$$

Therefore, a likelihood recursive procedure is

$$(4.12) \quad \hat{\theta}_t = \hat{\theta}_{t-1} + I_t^{-1}(\hat{\theta}_{t-1})(\alpha + 1) X_{t-1} \frac{X_t - \hat{\theta}_{t-1} X_{t-1}}{\alpha + (X_t - \hat{\theta}_{t-1} X_{t-1})^2}, \quad t \geq 1,$$

where $\hat{\theta}_0$ is any starting point. Note that I_t can also be derived recursively by

$$I_t = I_{t-1} + i^g X_{t-1}^2.$$

Clearly, (4.12) is a recursive procedure of type (4.6) but with a stochastic normalizing sequence $a_t = I_t^{-1}$. Now, ψ_t is of a form of (4.7) with $h(u) = 1$ and $\phi(z) = (\alpha + 1)z/(\alpha + z^2)$, and $g(z)$ is a bell-shaped and symmetric about zero. Therefore, to show convergence to θ , it suffices to check conditions (4.9) and (4.10), which, in this case can be written as

$$(4.13) \quad \sum_{t=1}^{\infty} \frac{1}{I_t} \inf_{\varepsilon \leq |u| \leq 1/\varepsilon} G(uX_{t-1}) = \infty$$

and

$$(4.14) \quad \sum_{t=1}^{\infty} \frac{X_{t-1}^2}{I_t^2} < \infty,$$

(P^θ -a.s.). We have, $I_t \rightarrow \infty$ for any $\theta \in \mathbb{R}$ (see, e.g, Shiriyayev [29], Ch.VII, 5.5). Since $\Delta I_t = i^g (X_{t-1})^2$, we obtain that (4.14) follows from Proposition A3 in Appendix A. Let us assume now that $|\theta| < 1$. By Lemma A2 in Appendix A, $\inf_{\varepsilon \leq |u| \leq 1/\varepsilon} G(ux) > 0$ for any $x \neq 0$. Then if we assume that the process is strongly stationary, it follows from the ergodic theorem that in probability P^θ ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} I_t > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=1}^t \inf_{\varepsilon \leq |u| \leq 1/\varepsilon} G(uX_{s-1}) > 0.$$

(It can be proved that these hold without assumption of strong stationarity.) Therefore, in probability P^θ , $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=1}^t \inf_{\varepsilon \leq |u| \leq 1/\varepsilon} G(uX_{s-1}) > 0$ and (4.13) now follows on application of Proposition A4 in Appendix A.

Remark 4.3 We have shown above that the recursive estimator (4.12) is strongly consistent, i.e., converges to θ a.s., if $|\theta| < 1$. It is worth mentioning that (4.14), and therefore, (4.10) holds for any $\theta \in \mathbb{R}$, which guarantees (C3) of Theorem 3.1. Also, (4.8) implies that (C1) of Theorem 3.1 holds as well. Therefore, according to Remark 3.3, we obtain that $|\hat{\theta}_t - \theta|$ converges (P^θ -a.s.) to a finite limit for any $\theta \in \mathbb{R}$.

Remark 4.4 Note that conditions (4.13) and (4.14) will still hold if we replace I_t by $c_t I_t$ where c_t is a sequence of non-negative r.v.'s such that

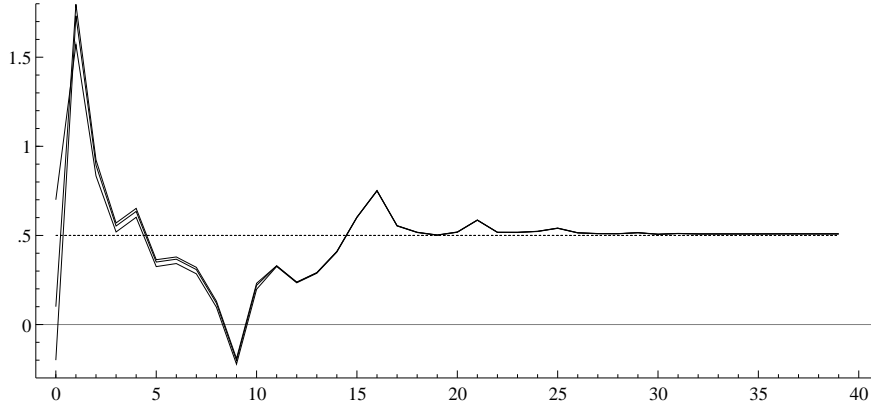


Figure 1: Realisations of (4.12) for $\alpha = 3$ and $\theta = 0.5$ for three different starting values $\theta_0 = -0.2, 0, 1$ and 0.7 . The number of observations is 40.

$c_t = 1$ eventually. So, the procedure (4.12) will remain consistent if I_t is replaced by $c_t I_t$, i.e., if tuning constants are introduced. We have shown that the procedure is consistent, i.e., the recursive estimator is close to the value of the unknown parameter for the large t 's. But in practice, the tuning constants may be useful to control the behaviour of a recursion at the “beginning” of the procedure. Fig.1 shows realisations of (4.12) for $\alpha = 3$ and $\theta = 0.5$ for three different starting values. The number of observations is 40. As we can see from these graphs, the recursive procedure, at each step moves in the direction of the parameter (see also Remark 3.2), but oscillates quite violently for the first ten steps and then settles down nicely after another ten steps. This oscillation is due to the small values of the normalising sequence for the first several steps and can be dealt with by introducing tuning constants. On other occasions, it may be desirable to lower the value of the normalising sequence for the first several steps. This happens when a procedure settles down too quickly without any, or little oscillation (before reaching the actual value of the parameter). The detailed discussion of these and related topics will appear elsewhere.

APPENDIX A

Lemma A1 *Let $\mathcal{F}_0, \mathcal{F}_1, \dots$ be a non-decreasing sequence of σ -algebras and $X_n, \beta_n, \xi_n, \zeta_n \in \mathcal{F}_n$, $n \geq 0$, are nonnegative r.v.'s such that*

$$E(X_n | \mathcal{F}_{n-1}) \leq X_{n-1}(1 + \beta_{n-1}) + \xi_{n-1} - \zeta_{n-1}, \quad n \geq 1$$

eventually. Then

$$\left\{ \sum_{i=1}^{\infty} \xi_{i-1} < \infty \right\} \cap \left\{ \sum_{i=1}^{\infty} \beta_{i-1} < \infty \right\} \subseteq \{X \rightarrow\} \cap \left\{ \sum_{i=1}^{\infty} \zeta_{i-1} < \infty \right\} \quad (P\text{-a.s.}),$$

where $\{X \rightarrow\}$ denotes the set where $\lim_{n \rightarrow \infty} X_n$ exists and is finite.

Remark Proof can be found in Robbins and Siegmund [23]. Note also that this lemma is a special case of the theorem on the convergence sets nonnegative semimartingales (see, e.g., Lazrieva et al [13]).

Lemma A2 *Suppose that $g \not\equiv 0$ is a nonnegative even function on \mathbb{R} and $g \downarrow 0$ on \mathbb{R}_+ . Suppose also that ϕ is a measurable odd function on \mathbb{R} such that $\phi(z) > 0$ for $z > 0$ and $\int_{\mathbb{R}} |\phi(z-w)|g(z)dz < \infty$ for all $w \in \mathbb{R}$. Then*

$$(A1) \quad w \int_{-\infty}^{\infty} \phi(z-w)g(z)dz < 0$$

for any $w \neq 0$. Furthermore, if $g(z)$ is continuous, then for any $\varepsilon \in (0, 1)$

$$(A2) \quad \sup_{\varepsilon \leq |w| \leq 1/\varepsilon} w \int_{-\infty}^{\infty} \phi(z-w)g(z)dz < 0.$$

Proof Denote

$$(A3) \quad \Phi(w) = \int_{-\infty}^{\infty} \phi(z-w)g(z)dz = \int_{-\infty}^{\infty} \phi(z)g(z+w)dz.$$

Using the change of variable $z \longleftrightarrow -z$ in the integral over $(-\infty, 0)$ and the equalities $\phi(-z) = -\phi(z)$ and $g(-z+w) = g(z-w)$, we obtain

$$\begin{aligned} \Phi(w) &= \int_{-\infty}^0 \phi(z)g(z+w)dz + \int_0^{\infty} \phi(z)g(z+w)dz \\ &= \int_0^{\infty} \phi(z)(g(z+w) - g(-z+w))dz \\ &= \int_0^{\infty} \phi(z)(g(z+w) - g(z-w))dz. \end{aligned}$$

Suppose now that $w > 0$. Then $z-w$ is closer to 0 than $z+w$, and the properties of g imply that $g(z+w) - g(z-w) \leq 0$. Since $\phi(z) > 0$ for $z > 0$, $\Phi(w) \leq 0$. The equality $\Phi(w) = 0$ would imply that $g(z+w) - g(z-w) = 0$ for all $z \in (0, +\infty)$ since, being monotone, g has right and left limits at each point of $(0, +\infty)$. The last equality, however, contradicts the restrictions on g . Therefore, (A1) holds. Similarly, if $w < 0$, then $z+w$ is closer to 0 than

$z - w$, and $g(z + w) - g(z - w) \geq 0$. Hence $w(g(z + w) - g(z - w)) \leq 0$, which yields (A1) as before.

To prove (A2) note that the continuity of g implies that $g(z + w) - g(z - w)$ is a continuous functions of w and (A2) will follow from (A1) if one proves that $\Phi(w)$ is also continuous in w . So, it is sufficient to show that the integral in (A3) is uniformly convergent for $\varepsilon \leq |w| \leq 1/\varepsilon$. It follows from the restrictions we have placed on g that there exists $\delta > 0$ such that $g \geq \delta$ in a neighbourhood of 0. Then the condition

$$\int_0^\infty \phi(z) (g(z + w) + g(z - w)) dz = \int_{-\infty}^\infty |\phi(z - w)|g(z)dz < \infty, \quad \forall w \in \mathbb{R}$$

implies that ϕ is locally integrable on \mathbb{R} . It is easy to see that, for any $\varepsilon \in (0, 1)$,

$$g(z \pm w) \leq g(0)\chi_\varepsilon(z) + g(z - 1/\varepsilon), \quad z \geq 0, \quad \varepsilon \leq |w| \leq 1/\varepsilon,$$

where χ_ε is the indicator function of the interval $[0, 1/\varepsilon]$. Since the function $\phi(\cdot)(g(0)\chi_\varepsilon + g(\cdot - 1/\varepsilon))$ is integrable on $(0, +\infty)$ and does not depend on w , we conclude that the integral in (A3) is indeed uniformly convergent for $\varepsilon \leq |w| \leq 1/\varepsilon$. \diamond

Proposition A3 *If d_n is a nondecreasing sequence of positive numbers such that $d_n \rightarrow +\infty$, then*

$$\sum_{n=1}^{\infty} \Delta d_n / d_n = +\infty$$

and

$$\sum_{n=1}^{\infty} \Delta d_n / d_n^2 < +\infty.$$

Proof The first claim is easily obtained by contradiction from the Kronecker lemma (see, e.g., Lemma 2, §3, Ch. IV in Shirayev [29]). The second one is proved by the following argument

$$0 \leq \sum_{n=1}^N \frac{\Delta d_n}{d_n^2} \leq \sum_{n=1}^N \frac{\Delta d_n}{d_{n-1}d_n} = \sum_{n=1}^N \left(\frac{1}{d_{n-1}} - \frac{1}{d_n} \right) = \frac{1}{d_0} - \frac{1}{d_N} \rightarrow \frac{1}{d_0} < +\infty.$$

\diamond

Proposition A4 *Suppose that d_n , c_n , and c are random variables, such that, with probability 1, $d_n > 0$, $c_n \geq 0$, $c > 0$ and $d_n \rightarrow +\infty$ as $n \rightarrow \infty$. Then*

$$\frac{1}{d_n} \sum_{i=1}^n c_i \rightarrow c \quad \text{in probability}$$

implies

$$\sum_{n=1}^{\infty} \frac{c_n}{d_n} = \infty \quad \text{with probability 1.}$$

Proof Denote $\xi_n = \frac{1}{d_n} \sum_{i=1}^n c_i$. Since $\xi_n \rightarrow c$ in probability, it follows that there exists a subsequence ξ_{i_n} of ξ_n with the property that $\xi_{i_n} \rightarrow c$ with probability 1. Now, assume that $\sum_{n=1}^{\infty} c_n/d_n < \infty$ on a set A of positive probability. Then, it follows from the Kronecker lemma, (see, e.g., Lemma 2, §3, Ch. IV in Shirayev [29]) that $\xi_n \rightarrow 0$ on A . Then it follows that $\xi_{i_n} \rightarrow 0$ on A as well, implying that $c = 0$ on A which contradicts the assumptions that $c > 0$ with probability 1. \diamond

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