DIFFUSION COVARIATION AND CO-JUMPS IN BIDIMENSIONAL ASSET PRICE PROCESSES WITH STOCHASTIC VOLATILITY AND INFINITE ACTIVITY LÉVY JUMPS

Fabio Gobbi* and Cecilia Mancini†

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Abstract

In this paper we consider two processes driven by diffusions and jumps. The jump components are Lévy processes and they can both have finite activity and infinite activity. Given discrete observations we estimate the covariation between the two diffusion parts and the co-jumps. The detection of the co-jumps allows to gain insight in the dependence structure of the jump components and has important applications in finance.

Our estimators are based on a threshold principle allowing to isolate the jumps. This work follows Gobbi and Mancini (2006) where the asymptotic normality for the estimator of the covariation, with convergence speed $\sqrt{h}$, was obtained when the jump components have finite activity. Here we show that the speed is $\sqrt{h}$ only when the activity of the jump components is moderate.[1]

Keywords: co-jumps, diffusion correlation coefficient, stable Lévy jumps, threshold estimator.

1 Introduction

We consider two state variables evolving as follows

$$dX_t^{(1)} = a_t^{(1)}dt + \sigma_t^{(1)}dW_t^{(1)} + dJ_t^{(1)},$$

$$dX_t^{(2)} = a_t^{(2)}dt + \sigma_t^{(2)}dW_t^{(2)} + dJ_t^{(2)},$$

for $t \in [0,T]$, T fixed, where $W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1-\rho_t^2}W_t^{(3)}$; $W^{(1)} = (W_t^{(1)})_{t\in[0,T]}$ and $W^{(3)} = (W_t^{(3)})_{t\in[0,T]}$ are independent Wiener processes. $J^{(1)}$ and $J^{(2)}$ are possibly correlated pure jump processes. We are interested in the separate identification of the dependence elements of the processes $X^{(q)}$, i.e. both of the covariation $\int_0^T \rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt$ between the two diffusion parts and of the *co-jumps* $\Delta J_t^{(1)}\Delta J_t^{(2)}$, the simultaneous jumps of $X^{(1)}$ and $X^{(2)}$.

*Dipartimento di Matematica per le Decisioni, Università degli Studi di Firenze

†Dipartimento di Matematica per le Decisioni, Università degli Studi di Firenze

[1] **Corresponding author** Fabio Gobbi, Dipartimento di Matematica per le Decisioni, Università di Firenze, via C. Lombroso 6/17, 50134 Firenze,

fgobbi@ds.unifi.it, phone number +39 (0)55 4796809, fax number +39 (0)55 4796800

Given discrete equally spaced observations $X^{(1)}_{t_j}, X^{(2)}_{t_j},\ j = 1..n$, in the interval $[0,T]$ (with $t_j = j\frac{T}{n}$), a commonly used approach to estimate $\int_0^T \rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt$ is to take the sum of cross products $\sum_{j=1}^n (X^{(1)}_{t_j} - X^{(1)}_{t_{j-1}})(X^{(2)}_{t_j} - X^{(2)}_{t_{j-1}})$; however, this estimate can be highly biased when the processes $X^{(q)}$ contain jumps; in fact, such a sum approaches the global quadratic covariation $[X^{(1)}, X^{(2)}]_T = \int_0^T \rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt + \sum_{0\le t\le T}\Delta J_t^{(1)}\Delta J_t^{(2)}$ containing also the co-jumps. It is crucial to single out the time intervals where the jumps have not occurred. Our estimator is based on a threshold criterion ([6]) allowing to isolate the jump part. In particular, we asymptotically identify when jumps larger than a given *thershold* occurred in a given time interval $]t_{j-1}, t_j]$, depending on whether the increment $|X_{t_j} - X_{t_{j-1}}|$ is too big with respect to the threshold. In Gobbi and Mancini (2006) we derived an asymptotically unbiased estimator of the continuous part of the covariation process as well as of the co-jumps. More precisely, the following threshold estimator

$$\tilde{v}_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T = \sum_{j=1}^n \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2\le r(h)\}}\Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2\le r(h)\}},$$

is a truncated version of the realized quadratic covariation and it is shown to be consistent to $\int_0^T \rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt$, as the number n of observations tends to infinity. Moreover, in the case where each $J^{(q)}$ is a *finite activity* jump process (i.e. only a finite number of jumps can occur, along each path, in each finite time interval) we show that our estimator is asymptotically Gaussian and converges with speed $\sqrt{h}$. Here we find the speed of convergence of the estimator of the covariation even in the case of infinite activity jumps, which turns out to be $\sqrt{h}$ only for moderate activity of the jump processes.
For the literature on non parametric inference for stochastic processes driven by diffusions plus jumps, see Gobbi and Mancini (2006).

Applications of the theory we present here is of strong interest in finance, in particular in financial econometrics (see e.g. [1]), in the framework of portfolio risk ([3]) and for hedge funds management.

An outline of the paper is as follows. In section 2 we illustrate the framework; in section 3 we present some preliminary results in the case where each component $J^{(q)}$ of $X^{(q)}$ has finite activity of jump. In section 4 we deal with the more complex case where each $J^{(q)}$ can have an *infinite activity* jump component $\tilde{J}_2^{(q)}$ (which makes an infinite number of jumps in each finite time interval). We assume that such component $\tilde{J}_2^{(q)}$ is a Lévy process and we show that our estimator is consistent and we develop some preliminaries for the asymptotic normality in the case where $\tilde{J}_2^{(q)}$ have stable-like laws and the joint law is characterized by a Copula ranging in a given class.

2 The framework

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, let $X^{(1)} = (X_t^{(2)})_{t\in[0,T]}$ and $X^{(2)} = (X_t^{(2)})_{t\in[0,T]}$ be two real processes defined by

$$\begin{aligned} X_t^{(1)} &= \int_0^t a_s^{(1)}ds + \int_0^t \sigma_s^{(1)}dW_s^{(1)} + J_t^{(1)}, \quad t\in[0,T], \\ X_t^{(2)} &= \int_0^t a_s^{(2)}ds + \int_0^t \sigma_s^{(2)}dW_s^{(2)} + J_t^{(2)}, \quad t\in[0,T], \end{aligned} \tag{1}$$

where

A1. $W^{(1)} = (W_t^{(1)})_{t\in[0,T]}$ and $W^{(2)} = (W_t^{(2)})_{t\in[0,T]}$ are two correlated Wiener processes, with $\rho_t = Corr(W_t^{(1)}, W_t^{(2)})$, $t \in [0,T]$; we can write

$$W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1-\rho_t^2}\, W_t^{(3)},$$

where $W^{(1)}$ and $W^{(3)}$ are independent Wiener processes.

A2. The diffusion stochastic coefficients $\sigma^{(q)} = (\sigma_t^{(q)})_{t\in[0,T]}$, $a^{(q)} = (a_t^{(q)})_{t\in[0,T]}$, $q = 1,2$, and $\rho = (\rho_t)_{t\in[0,T]}$ are adapted càdlàg.

A3. For $q = 1,2$

$$J^{(q)} = J_1^{(q)} + \tilde{J}_2^{(q)},$$

where $J_1^{(q)}$ are finite activity jump processes

$$J_{1t}^{(q)} = \int_0^t \gamma_s^{(q)} dN_s^{(q)} = \sum_{k=1}^{N_t^{(q)}} \gamma_{\tau_k^{(q)}}, \quad q = 1,2,$$

where $N^{(q)} = (N_t^{(q)})_{t\in[0,T]}$ are counting processes with $E[N_T^{(q)}] < \infty$; $\{\tau_k^{(q)},\ k = 1, ..., N_T^{(q)}\}$ denote the instants of jump of $J_1^{(q)}$ and $\gamma_{\tau_k^{(q)}}$ denote the sizes of the jumps occurred at $\tau_k^{(q)}$. We assume

$$P(\gamma_{\tau_k^{(q)}} = 0) = 0, \quad \forall\, k = 1, ..., N_T^{(q)},\ q = 1,2. \tag{2}$$

Denote, for each $q = 1,2$, $\underline{\gamma}^{(q)} = \min_{k=1,\dots,N_T^{(q)}} |\gamma_{\tau_k^{(q)}}|$. By condition (2), a.s. we have $\underline{\gamma}^{(q)} > 0$.

A4. $\tilde{J}_2^{(q)}$ are infinite activity Lévy pure jump processes of small jumps,

$$\tilde{J}_{2t}^{(q)} = \int_0^t \int_{|x|\le 1} x\ \tilde{\mu}^{(q)}(dx, ds), \tag{3}$$

where $\mu^{(q)}$ is the Poisson random measure of the jumps of $\tilde{J}_2^{(q)}$, $\tilde{\mu}^{(q)}(dx, ds) = \mu^{(q)}(dx, ds) - \nu^{(q)}(dx)ds$ is its compensated measure, where $\nu^{(q)}$ is the Lévy measure of $\tilde{J}_2^{(q)}$ (see [3]).

Each $\nu^{(q)}$ has the property that $\nu^{(q)}(\mathbb{R} - \{0\}) = \infty$, which characterizes the fact that the path of $\tilde{J}_2^{(q)}$ jumps infinitely many times on each compact time interval. $\tilde{J}_2^{(q)}$ is a compensated sum of jumps, each of which is bounded in absolute value by 1, so that substantially $J_1^{(q)}$ accounts for the "big" (bigger in absolute value than $\underline{\gamma}^{(q)}$) and rare jumps of $X^{(q)}$, while $\tilde{J}_2^{(q)}$ accounts for the very frequent and small jumps.

Remark 2.1. If $J^{(q)}$ is a pure jump Lévy process, it is always possible to decompose it as

$$J^{(q)} = J_1^{(q)} + \tilde{J}_2^{(q)},$$

(see [3]) where J_1 is a compound Poisson process accounting for the jumps bigger in absolute value than 1, J_1 satisfies assumption **A3** and $\tilde{J}_2$ is as in (3).

Notation. c denotes any constant.

A5. Let α_q be the Blumenthal Getoor index of each $J^{(q)}$, $q = 1, 2$ (see [3]). Let each $\nu^{(q)}$ satisfy:

A5.1 $\int_{|x|\le\varepsilon} x^2 \nu^{(q)}(dx) = O(\varepsilon^{2-\alpha_q})$

A5.2 $\int_{\varepsilon<|x|\le 1} |x|\nu^{(q)}(dx) = O(c - c\varepsilon^{1-\alpha_q})$.

Assumption **A5** is satisfied if for instance each $\nu^{(q)}$ has a density $f^{(q)}(x)$ behaving as $\frac{K^{(q)}(|x|)}{|x|^{1+\alpha_q}}$ when $x \to 0$, where $K^{(q)}$ is a real function with $\lim_{x\to 0} K^{(q)}(x) \in \mathbb{R} - \{0\}$, and α_q is the Blumenthal-Getoor index of $J^{(q)}$.
In particular **A5** is true for anyone of the commonly used models (e.g. NIG, VG, CGMY, α-stable, GHL).

Let, for each n, $\pi_n^{[0,T]} = \{0 = t_{0,n} < t_{1.n} < \cdots < t_{n,n} = T\}$ be a partition of $[0, T]$. We assume equally spaced subdivisions, i.e. $h_n := t_{j,n} - t_{j-1,n} = \frac{T}{n}$ for every $n = 1, 2, ...$ Hence $h_n \to 0$ as $n \to \infty$. Let $\Delta_{j,n}X$ be the increment $X_{t_{j,n}} - X_{t_{j-1,n}}$. To simplify notations we write h in place of h_n and $\Delta_j X$ in place of $\Delta_{j,n}X$.

A6. We choose a deterministic function, $h \mapsto r(h)$, satisfying the following properties

$$\lim_{h\to 0} r(h) = 0, \quad \lim_{h\to 0} \frac{hlog\frac{1}{h}}{r(h)} = 0.$$

We denote $r(h)$ by r_h. Denote also, for each $q = 1, 2$,

$$D_t^{(q)} = \int_0^t a_s^{(q)} ds + \int_0^t \sigma_s^{(q)} dW_s^{(q)},$$

the diffusion part of $X^{(q)}$, and

$$Y_t^{(q)} = D_t^{(q)} + J_{1t}^{(q)}.$$

3 Preliminary results

By the Paul Lévy law of the modulus of continuity of the Brownian motion paths (see [14]), we know that the increments of the diffusion part of each $\Delta_j X^{(q)}$ tend to zero at speed $\sqrt{h \ln \frac{1}{h}}$. This is the key point to understand when an increment $\Delta_j X^{(q)}$ is likely to contain some jumps. In fact if, for small h, $|\Delta_j X^{(q)}| > r_h > \sqrt{h \ln \frac{1}{h}}$, then or some jumps of $J_1^{(q)}$ occurred, or some jumps of $\tilde{J}_2^{(q)}$ larger than $2\sqrt{r_h}$ occurred (Mancini, 2005). In Gobbi and Mancini (2006) we obtain the following consequences.

Remark 3.1. *(Mancini, 2005) Under **A2** we have a.s.*

$$\sup_{1\le j\le n} \frac{|\Delta_j D^{(q)}|}{\sqrt{2hlog\frac{1}{h}}} \le K_q(\omega) < \infty, \quad q = 1, 2,$$

where K_q are finite random variables.

Theorem 3.2. *(Estimation of the correlation between the continuous parts) Let* $(X_t^{(1)})_{t\in[0,T]}$ *and* $(X_t^{(2)})_{t\in[0,T]}$ *two processes of the form (1). Assume **A1-A4** and* **A6** *are satisfied. Then*

$$\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} \int_0^T \rho_t \sigma_t^{(1)}\sigma_t^{(2)}dt,$$

as $n\to\infty$*, where for* r *and* $l\in\mathbb{N}$

$$\tilde{v}_{r,l}^{(n)}(X^{(1)},X^{(2)})_T = h^{1-\frac{r+l}{2}}\sum_{j=1}^{n}(\Delta_j X^{(1)})^r 1_{\{(\Delta_j X^{(1)})^2\le r_h\}}(\Delta_j X^{(2)})^l 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}.$$

□

$v_{r,l}^{(n)}(X^{(1)},X^{(2)})_T = h^{1-\frac{r+l}{2}}\sum_{j=1}^n(\Delta_j X^{(1)})^r(\Delta_j X^{(2)})^l$, was used in [2] to estimate the covariation in the case of diffusion processes. $\tilde{v}_{r,l}^{(n)}(X^{(1)},X^{(2)})_T$ is a *threshold* modified version for the case of jump diffusion processes where we exclude from the sums the terms containing some jumps.

Remark 3.3. An estimate of the sum of the co-jumps is obtained simply subtracting the diffusion covariation estimator from the quadratic covariation estimator. In fact

$$\sum_{j=1}^{n}\Delta_j X^{(1)}\Delta_j X^{(2)} - \tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} \sum_{0\le s\le T}\Delta J_s^{(1)}\Delta J_s^{(2)},$$

as $n\to\infty$. Therefore an estimate of each $\Delta J_s^{(1)}\Delta J_s^{(2)}$ is obtained using

$$\Delta_j X^{(1)}\Delta_j X^{(2)} - \Delta_j X^{(1)}1_{\{(\Delta_j X^{(1)})^2\le r_h\}}\Delta_j X^{(2)}1_{\{(\Delta_j X^{(2)})^2\le r_h\}},$$

with j such that $s\in]t_{j-1},t_j]$, whose limit for $h\to 0$ coincides with the limit of

$$\Delta_j X^{(1)}1_{\{(\Delta_j X^{(1)})^2> r_h\}}\Delta_j X^{(2)}1_{\{(\Delta_j X^{(2)})^2> r_h\}}.$$

Theorem 3.4. *If* $\tilde{J}_2^{(q)}\equiv 0$*, under the assumptions **A1-A3**, and choosing* r_h *as in **A6**, we have*

$$\mathcal{NB}(h) := \frac{\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T\rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt}{\sqrt{h}\sqrt{\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}} \xrightarrow{d} Z,$$

where Z *has law* $\mathcal{N}(0,1)$ *and*

$$\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T = h^{-1}\sum_{j=1}^{n-1}\prod_{i=0}^{1}\Delta_{j+i}X^{(1)}1_{\{(\Delta_{j+i}X^{(1)})^2\le r_h\}}\prod_{i=0}^{1}\Delta_{j+i}X^{(2)}1_{\{(\Delta_{j+i}X^{(2)})^2\le r_h\}}.$$

4 Main results

In this paper we study the behavior of the normalized bias $\mathcal{NB}(h)$ when infinite activity jump components $\tilde{J}_2^{(q)}$ are included in the models $X^{(q)}$. First we show that the standard error

$$\sqrt{h}\sqrt{\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}$$

converges even in the present framework. We need the following notations and remarks.

Remark 4.1. *[Remark 4.3 in [4]] Under assumptions* **A2** *and* **A5.2**

1. *If processes a and σ are càdlàg then, under* **A5**, *a.s., for small h, $1_{\{(\Delta_j D^{(q)})^2 > r_h\}} = 0$, uniformly in j;*

2. *Let us consider the sequence $\tilde{v}_{1,1}^{(n)}, n \in \mathbb{N}$. As long as $\tilde{J}_2^{(q)}$ is a semimartingale, we can find a subsequence n_k for which a.s., for large k, for all $j = 1..n_k$, on $\{\Delta_j X^{(q)} \le 4r(h_k)\}$ we have that*

$$(\Delta \tilde{J}_{2,s})^2 \le 4r(h_k), \quad \forall s \in]t_{j-1}, t_j].$$

3. *If $\tilde{J}_2^{(q)}$ is Lévy and independent of $N^{(q)}$, and if $P\{\Delta_j N \neq 0\} = O(h)$ as $h \to 0$, then for any $j = 1..n$, $nP\{\Delta_j N \neq 0, (\Delta_i \tilde{J}_2)^2 > r(h)\} \to 0$ as $h \to 0$.*

Notations. For each $q = 1, 2$ we denote

$$\Delta_j \tilde{J}_{2m}^{(q)} := \int_{t_{j-1}}^{t_j} \int_{|x| \le 2\sqrt{r_h}} x\ \tilde{\mu}^{(q)}(dx, dt), \quad \Delta_j \tilde{J}_{2c}^{(q)} := \int_{t_{j-1}}^{t_j} \int_{2\sqrt{r_h} < |x| \le 1} x\ \nu^{(q)}(dx) dt$$

so that

$$\Delta_j \tilde{J}_2^{(q)} 1_{\{|\Delta_j \tilde{J}_2^{(q)}| \le 2\sqrt{r_h}\}} = \Delta_j \tilde{J}_{2m}^{(q)} - \Delta_j \tilde{J}_{2c}^{(q)}. \tag{4}$$

We also set

$$\Delta_{j\star} H^{(q)} := \Delta_j H^{(q)} 1_{\{(\Delta_j X^{(q)})^2 \le r_h\}}$$

for any process $H^{(q)}$ (e.g. $H^{(q)} = Y^{(q)}$, or $H^{(q)} = \tilde{J}_2^{(q)}$ and so on).

Note that for each $q = 1, 2$

$$E[(\Delta_j \tilde{J}_{2m}^{(q)})^2] = h \int_{|x| \le 2\sqrt{r_h}} x^2 \nu^{(q)}(dx) := h\eta_q^2(2\sqrt{r_h}) \to 0$$

as $h \to 0$, and under assumption **A5** we have

$$\Delta_j \tilde{J}_{2c}^{(q)} = O\big(h(c - c r_h^{\frac{1-\alpha_q}{2}})\big). \tag{5}$$

Theorem 4.2 (standard error)**.** *Under the assumptions* **A1**-**A6**, *if $\frac{h \log^2 \frac{1}{h}}{r_h} \to 0$, and $r_h = h^\beta$, $\beta \in]0, 1[$, then*

$$\tilde{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T (1 + \rho_t^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt$$

as $n \to \infty$.

Proof. We prove that

$$\tilde{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T (2\rho_t^2 + 1)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt$$

and

$$\tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T \rho_t^2 (\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt.$$

Note that a.s. for small h that

$$1_{\{(\Delta_j X^{(q)})^2 \le r_h\}} = 1_{\{(\Delta_j X^{(q)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(q)})^2 \le 4r_h\}} + 1_{\{(\Delta_j X^{(q)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\}}, \tag{6}$$

and, trivially, we also have that

$$1_{\{|\Delta_j X^{(q)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(q)}| \le 2\sqrt{r_h}\}} = 1_{\{|\Delta_j X^{(q)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(q)}| \le 2\sqrt{r_h}, \Delta_j N^{(q)} = 0\}}. \tag{7}$$

Let us now deal with $\tilde{v}_{22}^{(n)}$. As in the proof of proposition 3.5 in [4] we can write

$$\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T (2\rho_t^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt =$$

$$\Big[h^{-1}\sum_{j=1}^n (\Delta_{j\star}Y^{(1)})^2(\Delta_{j\star}Y^{(2)})^2 - \int_0^T (2\rho_t^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt\Big]+$$

$$h^{-1}\sum_{j=1}^n \big[(\Delta_{j\star}Y^{(1)})^2(\Delta_{j\star}\tilde{J}_2^{(2)})^2 + 2(\Delta_{j\star}Y^{(1)})^2(\Delta_{j\star}Y^{(2)})(\Delta_j\tilde{J}_2^{(2)}) + (\Delta_{j\star}\tilde{J}_2^{(1)})^2(\Delta_{j\star}Y^{(2)})^2 + \tag{8}$$

$$+(\Delta_{j\star}\tilde{J}_2^{(1)})^2(\Delta_{j\star}\tilde{J}_2^{(2)})^2 + 2(\Delta_j\tilde{J}_2^{(1)})^2(\Delta_{j\star}Y^{(2)})(\Delta_{j\star}\tilde{J}_2^{(2)}) + 2(\Delta_{j\star}Y^{(1)})(\Delta_{j\star}\tilde{J}_2^{(1)})(\Delta_j Y^{(2)})^2 +$$

$$2(\Delta_{j\star}Y^{(1)})(\Delta_j\tilde{J}_2^{(1)})(\Delta_{j\star}\tilde{J}_2^{(2)})^2 + 4(\Delta_{j\star}Y^{(1)})(\Delta_j\tilde{J}_2^{(1)})(\Delta_{j\star}Y^{(2)})(\Delta_j\tilde{J}_2^{(2)})\big] := \sum_{k=1}^9 I_k(h).$$

The terms of the right hand side within brackets are denoted by $I_1(h)$ and can be split into two parts by adding and subtracting the quantity $h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2 1_{\{(\Delta_j Y^{(1)})^2\le 4r_h\}}(\Delta_j Y^{(2)})^2 1_{\{(\Delta_j Y^{(2)})^2\le 4r_h\}}$ in the following way

$$|I_1(h)| = \big|h^{-1}\sum_{j=1}^n (\Delta_{j\star}Y^{(1)})^2(\Delta_{j\star}Y^{(2)})^2 - \int_0^T (2\rho_t^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt\big| \le$$

$$\big|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2 1_{\{(\Delta_j Y^{(1)})^2\le 4r_h\}}(\Delta_j Y^{(2)})^2 1_{\{(\Delta_j Y^{(2)})^2\le 4r_h\}} - \int_0^T (2\rho_t^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt\big| + \tag{9}$$

$$\big|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})^2(1_{\{(\Delta_j X^{(1)})^2\le r_h\}}1_{\{(\Delta_j X^{(2)})^2\le r_h\}} - 1_{\{(\Delta_j Y^{(1)})^2\le 4r_h\}}1_{\{(\Delta_j Y^{(2)})^2\le 4r_h\}})\big|$$

The first term of the right hand side of (9) tends to zero in probability by proposition 5.1. Developing the second one we find that it is the sum of terms which a.s. for small h are zero because by remark 4.1 point 1 we have

$$1_{\{(\Delta_j X^{(q)})^2\le r_h, (\Delta_j Y^{(q)})^2>4r_h\}} \le 1_{\{|\Delta_j\tilde{J}_2^{(q)}|>\sqrt{r_h}\}} \tag{10}$$

and

$$1_{\{(\Delta_j X^{(q)})^2> r_h, (\Delta_j Y^{(q)})^2\le 4r_h\}} \le 1_{\{|\Delta_j D^{(q)}|>\frac{\sqrt{r_h}}{2}\}} + 1_{\{|\Delta_j\tilde{J}_2^{(q)}|>\frac{\sqrt{r_h}}{2}\}} = 1_{\{|\Delta_j\tilde{J}_2^{(q)}|>\frac{\sqrt{r_h}}{2}\}}, \tag{11}$$

uniformly in j, so that the terms containing $\Delta_j J_1^{(q)}$ tends to zero by remark 4.1 point 3, whereas

$$h^{-1}\sum_{j=1}^n (\Delta_j D^{(1)})^2(\Delta_j D^{(2)})^2 1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}} \le$$

$$K_1^2(\omega)K_2^2(\omega)h log^2\frac{1}{h}\sum_{j=1}^n 1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}},$$

which converges to zero in L^1

$$E\big|h log^2\frac{1}{h}\sum_{j=1}^n 1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}\big| \le nh log^2\frac{1}{h}E\big[1_{\{|\Delta_1\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}\big] = T\frac{h log^2\frac{1}{h}}{r_h}\eta_2^2(1)\to 0$$

The other terms in the right hand side of (8) tend to zero in probability. We only deal with I_2, I_3, I_5, I_8 and I_9, the other ones being analogue. Note that for each $q=1,2$

$$E\left[\sup_{1\le j\le n}\frac{(\Delta_j\tilde{J}_2^{(q)})^2 1_{\{|\Delta_j\tilde{J}_2^{(q)}|\le 2\sqrt{r_h}\}}}{h}\right] \le 2\sup_{1\le j\le n}\frac{E(\Delta_j\tilde{J}_{2m}^{(q)})^2}{h} + 2\sup_{1\le j\le n}\frac{E(\Delta_j\tilde{J}_{2c}^{(q)})^2}{h}$$

$$= 2\eta_q^2\Big(2\sqrt{r_h}\Big) + O\Big(2h(c - cr_h^{\frac{1-\alpha_2}{2}})^2\Big) = O\big(h^{1+\beta(1-\alpha_q)}\big)$$

and $h^{1+\beta(1-\alpha_q)}$ tends to zero as $h \to 0$. That is trivial if $\alpha_q \le 1$; however even if α_q belongs to $]1,2[$ it is ensured that $1+\beta(1-\alpha_q) > 0$, i.e. $\beta < \frac{1}{\alpha_q - 1}$, since $\beta < 1$ while $\frac{1}{\alpha_q-1} > 1$. We have then that as $h \to 0$

$$\sup_{1\le j\le n} \frac{(\Delta_j \tilde{J}_2^{(q)})^2 1_{\{|\Delta_j \tilde{J}_2^{(q)}| \le 2\sqrt{r_h}\}}}{h} \xrightarrow{P} 0. \tag{12}$$

Now, by (6) and (7) a.s. for small h

$$|I_2 + I_3 + I_5 + I_8 + I_9| \le h^{-1} \sum_{j=1}^n \Big| (\Delta_j D^{(1)})^2 (\Delta_j \tilde{J}_2^{(2)})^2 + 2(\Delta_j D^{(1)})^2 (\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(2)})$$

$$+(\Delta_j \tilde{J}_2^{(1)})^2 (\Delta_j \tilde{J}_2^{(2)})^2 + 2(\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j \tilde{J}_2^{(2)})^2$$

$$+4(\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(2)})\Big| 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}},$$

and since $1_{\{(\Delta_j X^{(q)})^2 \le r_h\}} = 1_{\{(\Delta_j X^{(1)})^2 \le r_h, \Delta_j \tilde{J}_2^{(q)})^2 \le 2\sqrt{r_h}\}} + 1_{\{(\Delta_j X^{(1)})^2 \le r_h, \Delta_j \tilde{J}_2^{(q)})^2 > 2\sqrt{r_h}\}}$ by (7) the terms containing the indicator of the set $\{\Delta_j \tilde{J}_2^{(q)})^2 \le 2\sqrt{r_h}\}$ are dominated by

$$\sup_{1\le j\le n} \frac{(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}}}{h} \Big[\sum_{j=1}^n (\Delta_j D^{(1)})^2 + \sum_{j=1}^n (\Delta_j \tilde{J}_2^{(1)})^2 + \sum_{j=1}^n (\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)}) \Big]$$

$$+2\bar{K}^2 h \ln\frac{1}{h} \sum_{j=1}^n (\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(2)})$$

$$+4 \sup_{1\le j\le n} \frac{|\Delta_j \tilde{J}_2^{(1)}| 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}}}{\sqrt{h}} \sup_{1\le j\le n} \frac{|\Delta_j \tilde{J}_2^{(2)}| 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}}}{\sqrt{h}} \sum_{j=1}^n (\Delta_j D^{(1)})(\Delta_j D^{(2)}),$$

where $\bar{K} := \sqrt{2}(K_1 \vee K_2)$. Each term tends to zero in probability by (12) and using that

$$\sum_{j=1}^n (\Delta_j D^{(q)})^2 \xrightarrow{P} \int_0^T (\sigma_t^{(q)})^2 dt < \infty \text{ a.s. }, \tag{13}$$

$\sum_{j=1}^n (\Delta_j \tilde{J}_2^{(1)})^2 \xrightarrow{P} T \int_{|x|\le 1} x^2 \nu^{(1)}(dx) < \infty$ a.s., $\sum_{j=1}^n (\Delta_j D^{(q)})(\Delta_j \tilde{J}_2^{(q)}) \xrightarrow{P} [D^{(1)}, \tilde{J}_2^{(1)}]_T = 0$ and $\sum_{j=1}^n (\Delta_j D^{(1)})(\Delta_j D^{(2)}) \xrightarrow{P} \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt < \infty$ a.s., where by $[M,N]$ we denote the quadratic covariation process associated to two semimartingales M and N (see [3]).

It remains to consider

$$h^{-1} \sum_{j=1}^n \Big| (\Delta_j D^{(1)})^2 (\Delta_j \tilde{J}_2^{(2)})^2 + 2(\Delta_j D^{(1)})^2 (\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(2)})$$

$$+(\Delta_j \tilde{J}_2^{(1)})^2 (\Delta_j \tilde{J}_2^{(2)})^2 + 2(\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j \tilde{J}_2^{(2)})^2$$

$$+4(\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(2)})\Big| 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 2\sqrt{r_h}\}} 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(2)})^2 > 2\sqrt{r_h}\}}.$$

Now observing that on $\{(\Delta_j X^{(q)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(q)})^2 > 2\sqrt{r_h}\}$, $q = 1,2$, we have $\{(\Delta_j Y^{(q)})^2 > r_h\}$, so that, a.s. for small h

$$1_{\{(\Delta_j X^{(q)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(q)})^2 > 2\sqrt{r_h}\}} \le 1_{\{|\Delta_j J_1^{(q)}| > \frac{\sqrt{r_h}}{2}\}} + 1_{\{|\Delta_j D^{(q)}| > \frac{\sqrt{r_h}}{2}\}} \le 1_{\{\Delta_j N^{(q)} \ne 0\}}$$

by remark 4.1 point 3 we note that all terms tend to zero.
We can conclude that $I_2+I_3+I_5+I_8+I_9 \xrightarrow{P} 0$ as $h\to 0$, and this concludes the proof of the convergence of $\tilde v_{22}$.

Now, we show that $\tilde w^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} \int_0^T \rho_t^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt$. Note that

$$\left| h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}\Delta_{j\star}X^{(q)}\prod_{q=1}^{2}\Delta_{j+1,\star}X^{(q)} - \int_0^T \rho_t^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt\right|$$

is the sum of

$$\left| h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}\Delta_{j\star}Y^{(q)}\prod_{q=1}^{2}\Delta_{j+1,\star}Y^{(q)} - \int_0^T \rho_t^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt\right| \tag{14}$$

and of other 15 terms of type $h^{-1}\sum_{j=1}^n \Delta_{j\star}M^{(1)}\Delta_{j+1,\star}H^{(1)}\Delta_{j\star}M^{(2)}\Delta_{j+1,\star}H^{(2)}$ where (since $\Delta_jX^{(q)} = \Delta_jY^{(q)}+\Delta_j\tilde J_2$ for each $q=1,2$) both M and H can be Y or $\tilde J_2$ and at least one factor is the increment of one of the two $\tilde J_2^{(q)}$, $q=1,2$. Each one of the 15 terms tends to zero in probability as $h\to 0$. In fact the terms where only one factor is the increment of one of the $\tilde J_2^{(q)}$s are bounded by

$$\sqrt{h^{-1}\sum_{j=1}^n(\Delta_{j+s}\tilde J_2^{(q)})^2 1_{\{|\Delta_{j+s}\tilde J_2^{(q)}|\le 2\sqrt{r(h)}\}}(\Delta_{j+s}D^{(r)})^2}\sqrt{h^{-1}\sum_{j=1}^n(\Delta_{j+\bar s}D^{(1)})^2(\Delta_{j+\bar s}D^{(2)})^2}, \tag{15}$$

where $s=0$ or 1, $\bar s$ is 1 iff s is 0 and $q,r\in\{1,2\}$. Using (12), (13) and using that $h^{-1}\sum_{j=1}^n(\Delta_{j+\bar s}D^{(1)})^2\cdot(\Delta_{j+\bar s}D^{(2)})^2 = v_{22}(D^{(1)},D^{(2)})_T$ converges to the a.s. finite correlation term $\int_0^T(1+2\rho_t^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt$ ([2], and cfr proposition 5.1), we reach that (15) tends to zero in probability.
The terms containing two increments of kind $\tilde J_{j+s}^{(q)}$ are dominated in probability, thanks to (12), by

$$o(1)\sum_{j=1}^n \Delta_jD^{(r)}\Delta_{j+s}D^{(q)} \le o(1)\sqrt{\sum_{j=1}^n(\Delta_jD^{(r)})^2}\sqrt{\sum_{j=1}^n(\Delta_{j+s}D^{(q)})^2}\xrightarrow{P} 0.$$

The terms containing three increments of kind $\tilde J_{j+s}^{(q)}$ are dominated by

$$o(1)\sum_{j=1}^n \Delta_{j+u}\tilde J_2^{(r)}\Delta_{j+s}D^{(q)} \le o(1)\sqrt{\sum_{j=1}^n(\Delta_{j+u}\tilde J_2^{(r)})^2}\sqrt{\sum_{j=1}^n(\Delta_{j+s}D^{(q)})^2}\xrightarrow{P} 0,$$

where $u,s\in\{0,1\}$. The unique term of type (**??**) containing four increments of kind $\tilde J_{j+s}^{(q)}$ is simply dominated, thanks to (12), by $o(1)nh\to 0$.

As for (14), adding and subtracting

$$h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jY^{(q)})^2\le 4r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}Y^{(q)})^2\le 4r_h\}},$$

we obtain

$$\left|h^{-1}\sum_{j=1}^{n-1}\Big[\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jX^{(q)})^2\le r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2\le r_h\}}\Big] - \int_0^T \rho_t^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt\right|$$

$$\le \left|h^{-1}\sum_{j=1}^{n-1}\Big[\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jY^{(q)})^2\le 4r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}Y^{(q)})^2\le 4r_h\}}\Big] - \int_0^T \rho_t^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt\right|$$

$$+\Big|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}\Delta_{j+1}Y^{(1)}\Delta_j Y^{(1)}\Delta_j \tilde{J}_2^{(2)}\times$$

$$\times\big(1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_{j+1}X^{(1)})^2\le r_h,(\Delta_j X^{(2)})^2\le r_h,(\Delta_{j+1}X^{(2)})^2\le r_h\}}+$$

$$-1_{\{(\Delta_j Y^{(1)})^2\le 4r_h,(\Delta_{j+1}Y^{(1)})^2\le 4r_h,(\Delta_j Y^{(2)})^2\le 4r_h,(\Delta_{j+1}Y^{(2)})^2\le 4r_h\}}\big)\Big|.$$

The first term tends to zero in probability by theorem 5.1, whereas for the second one we note that developing the difference of the two indicators we obtain a sum of terms which are dominated by indicators as in (10) and (11) and thus they vanish a.s. for small h (analogously as in (9)). □

Next we check the speed of convergence to zero of the estimation error $\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T-\int_0^T\rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt$. Within $\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T-\int_0^T\rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt$ it is the co-jumps term $\sum_{s\le t}\Delta\tilde{J}_{2s}^{(1)}\Delta\tilde{J}_{2s}^{(2)}$ to determine such a speed. However the speed of convergence of such term depends both on the amount of jump activity of each $\tilde{J}_2^{(q)}$ and on the dependence structure giving the joint law $\big(\tilde{J}_2^{(1)},\tilde{J}_2^{(2)}\big)(P)$. We specialize our analysis to the case where $\tilde{J}_2^{(q)}$ have stable-like laws and the joint law is characterized by a copula C ranging in a given class.

A7 Assume $\alpha_q\in]0,2[$ for each $q=1,2$. Consider (w.l.g.) $\alpha_1\le\alpha_2$.
Each marginal law $\big(\tilde{J}_2^{(q)}\big)(P)$ has a Stable-like density of the form

$$\nu^{(q)}=c_q x^{-1-\alpha_q}1_{\{x>0\}}+d_q\,|x|^{-1-\alpha_q}1_{\{x<0\}}.$$

For simplicity, but w.l.g., we develop our proofs for the case where each $\tilde{J}_2^{(q)}$ has only positive jump sizes, i.e.

$$\nu^{(q)}=c_q x^{-1-\alpha_q}1_{\{x>0\}},$$

which have support $\mathbb{R}_+$.
We denote for each $q=1,2$ by

$$U_q(x):=\nu^{(q)}\big([x_q,+\infty[\big)=c_q\frac{x_q^{-\alpha_q}}{\alpha_q}\qquad(16)$$

the tail integral of the marginal law of $\tilde{J}_2^{(q)}$.

A8 The joint law $\big(\tilde{J}_2^{(1)},\tilde{J}_2^{(2)}\big)(P)$ has tail integrals given by

$$U(x,y)=C_\gamma(U_1(x),U_2(y))$$

where $C_\gamma(u,v)$ is a Lévy copula (see [3]) of the form

$$C_\gamma(u,v)=\gamma C_\perp(u,v)+(1-\gamma)C_\parallel(u,v),$$

where $C_\perp(u,v)=u1_{\{v=\infty\}}+v1_{\{u=\infty\}}$ is the independence copula, $C_\parallel(u,v)=u\wedge v$ is the total dependence copula and γ ranges in $[0,1]$.

Such choices are quite representative since in fact many commonly used models in finance (Variance Gamma model, CGMY model, NIG model, etc.) have $\nu^{(q)}$ related to the ones in assumption **A7** in the sense that they are tempered stable processes where the order of magnitude of the tail integrals as $x_q\to 0$

is as for (16). Moreover C allows to range from a framework of independent components to a framework where the components are completely positively monotonic.

Remark 4.3. *We need assumption* **A8** *in order to control the speed of convergence to zero of integrals like* $\int_{0\le x,y\le\varepsilon} xyd\nu(x,y)$, $\int_{0\le x,y\le\varepsilon} x^2y^2d\nu(x,y)$, *where* ν *is the bivariate Lévy measure of* $(\tilde J_2^{(1)}, \tilde J_2^{(2)})$. *Note that when the copula within* ν *is the independence copula then both integrals are zero so that under assumption* **A8**

$$\int_{0\le x,y\le\varepsilon} x^k y^k \nu(dx,dy) = (1-\gamma)\int_{0\le x,y\le\varepsilon} x^k y^k dC_{\|}(U_1(x),U_2(y))$$

for $k=1,2$, *and the speed is given only by the complete dependence component.*

Now we compute the speed of convergence to zero of the small co-increments of the two $\tilde J_2^{(q)}$.

Theorem 4.4. *Choose* $r_h = h^\beta, \beta\in]0,1[$ *and* $C_\gamma(u,v)\equiv C_{\|}(u,v)$ *(i.e.* $\gamma=0$*). Assume* **A1**-**A8**. *Then*

$$\frac{\sum_{j=1}^n \Delta_j\tilde J_2^{(1)} 1_{\{(\Delta_j\tilde J_2^{(1)})^2\le 4r_h\}}\Delta_j\tilde J_2^{(2)} 1_{\{(\Delta_j\tilde J_2^{(2)})^2\le 4r_h\}} - nE[H'_{n1}]}{\sqrt{nVar(H'_{n1})}} \xrightarrow{d} \mathcal{N}(0,1),$$

as $h\to 0$, *where for* $j=1..n$

$$H'_{nj} := \Delta_j\tilde J_2^{(1)} 1_{\{(\Delta_j\tilde J_2^{(1)})^2\le 4r_h\}}\Delta_j\tilde J_2^{(2)} 1_{\{(\Delta_j\tilde J_2^{(2)})^2\le 4r_h\}}$$

is such that as $h\to 0$

$$E[H'_{nj}] = O(h^{1+\beta\frac{\alpha_1+\alpha_2-\alpha_1\alpha_2}{2\alpha_1}}) + h^2 O\Big((c-ch^{\beta\frac{1-\alpha_1}{2}})(c-ch^{\beta\frac{1-\alpha_2}{2}})\Big)$$

and

$$Var(H'_{nj}) = O(h^{2+\frac{\beta}{2}(4-\alpha_1-\alpha_2)}) + O(h^{1+\beta\frac{2\alpha_1+2\alpha_2-\alpha_1\alpha_2}{2\alpha_1}}).$$

Proof. We use the Lindeberg-Feller theorem. Using **A7** and (4) we have

$$E[H'_{nj}] = h\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} xy\nu(dx,dy) + \Delta_j\tilde J_{2c}^{(1)}\Delta_j\tilde J_{2c}^{(2)}$$

$$= h\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} xydC_{\|}(U_1(x),U_2(y)) + \Delta_j\tilde J_{2c}^{(1)}\Delta_j\tilde J_{2c}^{(2)}$$

$$= h\int_{\frac{(h^{\frac{\beta}{2}})^{-\alpha_1}}{\alpha_1}\vee\frac{(h^{\frac{\beta}{2}})^{-\alpha_2}}{\alpha_2}}^{+\infty} U_1^{-1}(u)U_2^{-1}(u)du + \Delta_j\tilde J_{2c}^{(1)}\Delta_j\tilde J_{2c}^{(2)}$$

$$= O(h^{1+\beta\frac{\alpha_1+\alpha_2-\alpha_1\alpha_2}{2\alpha_1}}) + O(h(c-ch^{\beta\frac{1-\alpha_1}{2}}))O(h(c-ch^{\beta\frac{1-\alpha_2}{2}})).$$

Moreover, since

$$E[\Delta_j\tilde J_{2m}^{(1)}\Delta_j\tilde J_{2m}^{(2)}]^2 = h\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} x^2y^2\nu(dx,dy)$$

$$+h^2\Big(\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} x^2\nu(dx,dy)\Big)\Big(\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} y^2\nu(dx,dy)\Big)$$

$$+2h^2\Big(\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} xy\nu(dx,dy)\Big)^2,$$

$$E[(\Delta_j\tilde J_{2m}^{(1)})^2\ \Delta_j\tilde J_{2m}^{(2)}] = h\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} x^2y\nu(dx,dy)$$

and

$$E[\Delta_j \tilde{J}_{2m}^{(1)}\ (\Delta_j \tilde{J}_{2m}^{(2)})^2] = h\int_{]0,h^{\frac{\beta}{2}}]}\int_{]0,h^{\frac{\beta}{2}}]} xy^2\nu(dx,dy),$$

we get

$$Var(H'_{nj}) = O(h^{2+\frac{\beta}{2}(4-\alpha_1-\alpha_2)}) + O(+h^{1+\beta\frac{2\alpha_1+2\alpha_2-\alpha_1\alpha_2}{2\alpha_1}}).$$

Notice that $\forall\alpha_q \in]0,2[$ we have $\frac{\alpha_1+\alpha_2-\alpha_1\alpha_2}{2\alpha_1} > 0$. Denote

$$H_{nj} = \frac{H'_{nj} - E[H'_{nj}]}{\sqrt{nVar(H'_{nj})}}$$

the normalized versions of H'_{nj}. In order to verify the Lindeberg condition we consider the following sets

$$\{|H_{nj}| > \eta\} = \left\{\frac{H'_{nj} - E[H'_{nj}]}{\sqrt{nVar(H'_{nj})}} > \eta\right\} = \left\{|H'_{nj} - E[H'_{nj}]| > \eta\sqrt{nVar(H'_{nj})}\right\}.$$

We show that in fact, for small h, $H'_{nj} \le E[H'_{nj}] + \sqrt{nVar(H'_{nj})}$ $\forall j$, thus $\{|H_{nj}| > \eta\} = \emptyset$. Actually, after boring computations[2] we reach that

$$E[H'_{nj}] + \eta\sqrt{n\ Var(H'_{nj})} = O(h^{1+\beta\frac{\alpha_1+\alpha_2-\alpha_1\alpha_2}{2\alpha_1}}) + O\Big(\sqrt{h^{1+\frac{\beta}{2}(4-\alpha_1-\alpha_2)} + h^{\beta\frac{2\alpha_1+2\alpha_2-\alpha_1\alpha_2}{2\alpha_1}}}\ \Big)$$

as $h \to 0$. Note that, using (4) and (5),

$$H'_{nj} = \Delta_j\tilde{J}^{(1)}_{2m}\Delta_j\tilde{J}^{(2)}_{2m} - \Delta_j\tilde{J}^{(1)}_{2m}\Delta_j\tilde{J}^{(2)}_{2c} - \Delta_j\tilde{J}^{(1)}_{2c}\Delta_j\tilde{J}^{(2)}_{2m} + \Delta_j\tilde{J}^{(1)}_{2c}\Delta_j\tilde{J}^{(2)}_{2c}$$

$$= \Delta_j\tilde{J}^{(1)}_{2m}\Delta_j\tilde{J}^{(2)}_{2m} - \Delta_j\tilde{J}^{(1)}_{2m}O(h(c - ch^{\beta\frac{1-\alpha_2}{2}}))+$$

$$-\Delta_j\tilde{J}^{(2)}_{2m}O(h(c - ch^{\beta\frac{1-\alpha_1}{2}})) + O(h(c - ch^{\beta\frac{1-\alpha_1}{2}}))O(h(c - ch^{\beta\frac{1-\alpha_2}{2}})),$$

therefore

$$H'_{nj} = o\Big(E[H'_{nj}] + \eta\sqrt{nVar(H'_{nj})}\Big)$$

as $h \to 0$. Since $\frac{h^2(c-ch^{\beta\frac{1-\alpha_1}{2}})(c-ch^{\beta\frac{1-\alpha_2}{2}})}{\sqrt{h^{1+\frac{\beta}{2}(4-\alpha_1-\alpha_2)}}} \to 0$ it follows that $h^2(c - ch^{\beta\frac{1-\alpha_1}{2}})(c - ch^{\beta\frac{1-\alpha_2}{2}}) = o\Big(E[H'_{nj}] + \eta\sqrt{nVar(H'_{nj})}\Big)$. Moreover for each $q = 1, 2$

$$\Delta_j\tilde{J}^{(q)}_{2m}O(h(c - ch^{\beta\frac{1-\alpha_q}{2}}))$$

$$= \Big(\Delta_j\tilde{J}^{(q)}_2 1_{\{|\Delta_j\tilde{J}^{(q)}_2| \le 2\sqrt{r_h}\}} + \int_{t_{j-1}}^{t_j}\int_{2\sqrt{r_h}\le|x|<1} x\nu^{(q)}(dx)dt\Big)O(h(c - ch^{\beta\frac{1-\alpha_q}{2}}))$$

$$\le \big(2\sqrt{r_h} + O(h(c - ch^{\beta\frac{1-\alpha_q}{2}}))\big)O(h(c - ch^{\beta\frac{1-\alpha_q}{2}}))$$

$$= \big(O(h^{\frac{\beta}{2}}) + O(h(c - ch^{\beta\frac{1-\alpha_q}{2}}))\big)O(h(c - ch^{\beta\frac{1-\alpha_q}{2}}))$$

$$= o(\sqrt{h^{1+\frac{\beta}{2}(4-\alpha_1-\alpha_2)}}),$$

so that as $h \to 0$

$$\Delta_j\tilde{J}^{(q)}_{2m}O(h(c - ch^{\beta\frac{1-\alpha_q}{2}})) = o\Big(E[H'_{nj}] + \eta\sqrt{nVar(H'_{nj})}\Big). \tag{17}$$

[2] These are available if requested.

Now using (4) we can write

$$\Delta_j \tilde{J}_{2m}^{(1)} \Delta_j \tilde{J}_{2m}^{(2)} \le \Delta_j \tilde{J}_{2m}^{(2)} O(h^{\frac{\beta}{2}}) + \Delta_j \tilde{J}_{2m}^{(2)} O(h(c - ch^{\beta \frac{1-\alpha_1}{2}})).$$

But

$$E\left|\frac{(\Delta_j \tilde{J}_{2m}^{(2)})h^{\frac{\beta}{2}}}{\sqrt{h^{1+\frac{\beta}{2}(4-\alpha_1-\alpha_2)}}}\right| \le \frac{h^{\frac{\beta}{2}}}{h^{\frac{1}{2}+\frac{\beta}{4}(4-\alpha_1-\alpha_2)}} \sqrt{E[(\Delta_j \tilde{J}_{2m}^{(2)})^2]} = \frac{h^{\beta - \frac{\beta\alpha_2}{4}}}{h^{\frac{\beta}{2}(4-\alpha_1-\alpha_2)}} \to 0,$$

as $h \to 0$. It follows, using also (17), that $\Delta_j \tilde{J}_{2m}^{(1)} \Delta_j \tilde{J}_{2m}^{(2)} = o\Big(E[H'_{nj}] + \eta\sqrt{nVar(H'_{nj})}\Big)$. Therefore for small h, uniformly on j, we have $\{|H_{nj}| < \eta\} = \emptyset$ and the Lindeberg condition is satisfied and the proof of theorem is complete. □

5 Appendix

Proposition 5.1. *(Proposition 3.5 in [4]) If $\tilde{J}_2^{(q)} \equiv 0$, under the assumptions **A1-A3**, and choosing r_h as in **A5**, we have*

$$\tilde{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T (2\rho_t^2 + 1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt,$$

and

$$\tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T \rho_t^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt.$$

Theorem 5.2 (Lindeberg-Feller)**.** *Let $\{H_{nj},\ j = 1, ..., j_n,\ n = 1, 2, ...\}$ be a double array of r.v.s independent in each row such that $EH_{nj} = 0$ and $EH_{nj}^2 = \sigma_{nj}^2 < \infty$ for each n and j and moreover $\sum_{j=1}^{j_n} \sigma_{nj}^2 = 1$. Let F_{nj} be the distribution function of H_{nj}. In order that*

1. $\max_{1 \le j \le j_n} P(|H_{nj}| > \epsilon) \to 0,\ \forall \epsilon > 0,$
2. $\sum_{j=1}^{j_n} H_{nj} \xrightarrow{d} \mathcal{N}(0, 1),$

it is necessary and sufficient that for each $\eta > 0$ that the Lindeberg condition holds, i.e.

$$\sum_{j=1}^{j_n} \int_{|x|>\eta} x^2 F_{nj}(dx) = \sum_{j=1}^{j_n} EH_{nj}^2 1_{\{|H_{nj}|>\eta\}} \to 0.$$

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