Minimal f-divergence martingale measures and optimal portfolios for exponential Levy models with a change-point

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S. Cawston[1]and L. Vostrikova[2]

Abstract

We consider the exponential Levy models and we study the conditions under which f-minimal equivalent martingale measure preserves Levy property. Then we give a general formula for optimal strategy in a sense of utility maximization. Finally, we study change-point exponential Levy models, namely we give the conditions for the existence of f-minimal equivalent martingale measure and we obtain a general formula for optimal strategy from point of view of the utility maximization. We illustrate our results considering Black-Scholes model with change-point.

KEY WORDS AND PHRASES: f-divergence, exponential Levy models, change-point, optimal portfolio
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1 Introduction

The parameters of financial models are generally highly dependent on time : a number of events (for example the release of information in the press, changes in the price of raw materials or the first time a stock price hits some psychological level) can trigger a change in the behavior of stock prices. This time-dependency of the parameters can often be described using a piece-wise constant function : we will call this case a change-point situation. In this context, an important problem in financial mathematics will be option pricing and hedging. Of course, the time of change (change-point) for the parameters is not explicitly known, but it is often possible to make reasonable

[1,2] LAREMA, Département de Mathématiques, Université d'Angers, 2, Bd Lavoisier - 49045, ANGERS CEDEX 01.
[1]E-mail: suzanne.cawston@univ-angers.fr [2]E-mail: lioudmila.vostrikova@univ-angers.fr

assumptions about its nature and use statistical tests for its detection.

Change-point problems have a long history, probably beginning with the papers of Page [39],[40] in an a-posteriori setting, and of Shiryaev [46] in a quickest detection setting. The problem was later considered in many papers, see for instance [12], [43],[18],[42], [2],[50], [36], also the book [1] and references there. In the context of financial mathematics, the question was investigated in [27], [6], [26],[19],[13],[49], [51] and was often related to a quickest detection approach.

In this paper we study optimal portfolios from the point of view of utility maximization, for exponential Levy change-point models where the parameters of the model before and after the change are known and with a change-point which is independent from the observed processes. The case of a change-point at the first exit time for some functional of the price of the risky asset will be considered in a forthcoming paper.

Let us describe our change-point model more precisely. Let $L = (L_t)_{t\geq 0}$ be a Levy process with parameters (b, c, ν) where b is the drift parameter, c the diffusion parameter and ν the Levy measure which satisfies

$$\int_{\mathbb{R}^*} (x^2 \wedge 1)\nu(dx) < +\infty. \tag{1}$$

As is well known, the characteristic function of L_t for $t \in \mathbb{R}^+$ and $u \in \mathbb{R}$ is given by:

$$\phi_t(u) = \mathbb{E}e^{iuL_t} = e^{\psi(u)t}$$

where the characteristic exponent $\psi(u)$ is equal to:

$$\psi(u) = iub - \frac{1}{2}cu^2 + \int_{\mathbb{R}^*} (\exp(iux) - 1 - iul(x))\nu(dx),$$

where from now on, l is the truncation function.

Let $\tilde{L} = (\tilde{L}_t)_{t\geq 0}$ be a Levy process which is independent from L and with parameters $(\tilde{b}, \tilde{c}, \tilde{\nu})$ where as before $\tilde{b}$ is the drift parameter, $\tilde{c}$ the diffusion parameter and $\tilde{\nu}$ the Levy measure which satisfies (1) when replacing ν by $\tilde{\nu}$. Let τ be a random variable which is independent from L and $\tilde{L}$ and let $X = (X_t)_{t\geq 0}$ be the process given by:

$$X_t = L_t \mathbb{I}_{\{\tau > t\}} + (L_\tau + \tilde{L}_t - \tilde{L}_\tau)\mathbb{I}_{\{\tau \leq t\}} \tag{2}$$

where $\mathbb{I}(\cdot)$ is the indicator function. Let also

$$r(t) = r\mathbb{I}_{\{\tau > t\}} + \tilde{r}\mathbb{I}_{\{\tau \leq t\}} \tag{3}$$

be the interest rate changing from r to $\tilde{r}$ at the change-point τ. Our model for risky and non risky assets will then be given by

$$S_t = S_0 \exp(X_t), \quad B_t = B_0 \exp(\int_0^t r(s)ds) \tag{4}$$

respectively. The multidimensional case and multiple change-point situations can be defined in a similar way.

It is well known that exponential Levy models are in general incomplete so that for option pricing, one has to choose a "good" equivalent martingale measure. Many approaches have been developed and various criteria suggested for this choice of martingale measure, for example risk-minimization in an L^2-sense [16],[44], [45], Hellinger integrals minimization [9], [10], [25], entropy minimization [37], [17] ,[15], f^q-martingale measures [30] or Esscher measures[28]. In our change-point setting we will consider minimal quadratic variation measures, minimal entropy measures and f^q -minimal measures as special cases of minimal f-divergence martingale measures. The notion of f-divergence was introduced by Ciszar [11] and was investigated in a number of papers and books, see for instance [35] and references there. In a number of papers it was also noticed that utility maximization is closely related with the choice of a minimal martingale measure via the Fenchel-Legendre transform. The optimal portfolios for Levy models for some special cases of utility functions were considered in [24], [31], [32],[5].

The paper is organized in the following way. In 2. we consider minimal martingale measures and we give a general expression for minimal martingale measures in change-point situations. We restrict ourselves to f-divergences for which the minimal martingale measure preserves the Levy property and such that the multiplication of the argument of the f-divergence by a constant does not change the minimal martingale measure. The conditions under which the f-divergence minimal martingale measure preserves the Levy property are discussed in Theorem 1 and also in [4], and a number of important examples mentioned above satisfy both these properties. Then, we give the expression for f-divergence minimal martingale measure in the change-point case (see Theorem 2). We illustrate this result by an example considering the Black-Scholes model with a change-point.

In 3. we first recall some useful results about optimal strategies in general. Then, we give an expression for optimal strategies of exponential Levy model with no change-point (Theorem 4). Finally, we obtain the expression for optimal strategies for exponential Levy models in a change -point situation (Theorem 5). We illustrate this result by an example considering again the Black -Scholes model with a change-point.

2 Minimal martingale measures in change-point situations

We start by describing in more details our model for the risky asset. Let $(D, \mathcal{G}, \mathbb{G})$ be the canonical space of right-continuous functions with left-hand limits equipped with its natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ which satisfies standard conditions: it is right-continuous, $\mathcal{G}_0 = \{\emptyset, D\}$, $\bigvee_{t\geq 0} \mathcal{G}_t = \mathcal{G}$. On the product of such canonical spaces we define two independent Levy processes $L = (L_t)_{t\geq 0}$ and $\tilde{L} = (\tilde{L}_t)_{t\geq 0}$ with characteristics (b, c, ν) and $(\tilde{b}, \tilde{c}, \tilde{\nu})$ respectively and denote by P and $\tilde{P}$ their respective laws which are assumed to be locally equivalent: $P \overset{loc}{\sim} \tilde{P}$. We suppose that the Levy measures of these processes satisfy

$$\int_{x>1} (e^x - 1) d\nu < \infty, \quad \int_{x>1} (e^x - 1) d\tilde{\nu} < \infty.$$

As we will consider the market on a fixed finite time interval, we are really only interested in $P|_{\mathcal{G}_T}$ and $\tilde{P}|_{\mathcal{G}_T}$ for a fixed $T \geq 0$.
Our change-point will be represented by an independent random variable τ of law α taking values in $([0,T], \mathcal{B}([0,T])$. The set $\{\tau = T\}$ corresponds to the situation when the change-point does not take place, or at least not on the interval we are studying. On the probability space $(D \times D \times [0,T], \mathcal{G} \times \mathcal{G} \times \mathcal{B}([0,T], P \times \tilde{P} \times \alpha)$ we define a measurable map X by (2) and we denote by $\mathbb{P}$ its law. If we observe only the process X then the natural probability space to work is $(D, \mathcal{G}, \mathbb{P})$ equipped with the right-continuous version of the natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ where $\mathcal{G}_t = \sigma\{X_s,\ s \leq t\}$ for $t \geq 0$.
Now, if we observe not only the process X but also some complementary variables related with τ then we can take it in account by the enlargement of the filtration. First we consider the filtration $\mathbb{H}$ given by $\mathcal{H}_t = \sigma(1\!\!\mathrm{I}_{\{\tau \leq s\}}, s \leq t)$ and note that $\mathcal{H}_T = \sigma(\tau)$. Then we introduce two filtrations: the initially enlarged filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$

$$\mathcal{F}_0 = \mathcal{G}_0 \vee \mathcal{H}_T, \ \ \mathcal{F}_t = \bigcap_{s>t} (\mathcal{G}_s \vee \mathcal{H}_T) \tag{5}$$

and the progressively enlarged filtration $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \geq 0}$ which satisfies :

$$\hat{\mathcal{F}}_0 = \mathcal{G}_0 \vee \mathcal{H}_0, \ \ \hat{\mathcal{F}}_t = \bigcap_{s>t} (\mathcal{G}_s \vee \mathcal{H}_s) \tag{6}$$

In the case of additional information the most natural filtration from the point of view of observable events would be $\hat{\mathbb{F}}$. However, we will see that it is much easier to start by working with the initially enlarged filtration and then come back to the progressively enlarged filtration.

We notice that conditional law of X the conditionally to $\{\tau = T\}$ is P, the law of the first Levy process, and that $\mathbb{P} \overset{loc}{<<} P$. We set

$$\frac{d\mathbb{P}_t}{d\,P_t} = Y_t$$

where $\mathbb{P}_t$ and P_t denote the restrictions of the corresponding measures to the σ-algebra $\mathcal{F}_t$. We remark that

$$Y_t = 1\!\!\mathrm{I}_{[\![0,\tau]\!]}(t) + \frac{y_t}{y_\tau} 1\!\!\mathrm{I}_{]\!]\tau,+\infty[\![}(t) \tag{7}$$

with $y_t = \dfrac{d\tilde{P}_t}{dP_t}$. After all these precisions we define our risky asset by formula (4) and assume for simplicity that r and $\tilde{r}$ in (3) are equal to zero, and that $S_0 = 1$. In what folows we use $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$ as well as for the expectation with respect to $P \times \tilde{P} \times \alpha$.

2.1 Equivalent martingale measures in a change-point situation.

First of all, we need to describe the set of equivalent martingale measures (EMMs) for our model, in particular in relation to the sets of EMMs for the two associated Levy models L and $\tilde{L}$ which we denote by $\mathcal{M}(P)$ and $\mathcal{M}(\tilde{P})$ respectively. We assume these sets are non-empty. Let $Q \in \mathcal{M}(P)$ and $\tilde{Q} \in \mathcal{M}(\tilde{P})$. We introduce the Radon-Nikodym density processes $\zeta = (\zeta_t)_{t \geq 0}$ and $\tilde{\zeta} = (\tilde{\zeta}_t)_{t \geq 0}$ given by

$$\zeta_t = \frac{dQ_t}{dP_t}, \qquad \tilde{\zeta}_t = \frac{d\tilde{Q}_t}{d\tilde{P}_t}$$

where $Q_t, P_t, \tilde{Q}_t, \tilde{P}_t$ stand for the restrictions of the corresponding measures to the σ-algebra $\mathcal{F}_t$, and also the process $z = (z_t)_{t \geq 0}$ given by

$$z_t = \zeta_t \mathbb{I}_{[\![0,\tau]\!]}(t) + \zeta_\tau \frac{\tilde{\zeta}_t}{\tilde{\zeta}_\tau} \mathbb{I}_{]\!]\tau,+\infty[\![}(t) \tag{8}$$

We finally consider the measure $\mathbb{Q}$ defined by:

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = c(\tau) z_t \tag{9}$$

where $c(\cdot)$ is a measurable function $[0,T] \to \mathbb{R}^{+,*}$ such that $\mathbb{E}c(\tau) = 1$.

Lemma 1. *A measure $\mathbb{Q}$ is an equivalent martingale measure for the exponential model (4) related to the process X iff its density process has the form (9).*

Proof First we show that the process $Z = (Z_t)_{t \geq 0}$ given by

$$Z_t = c(\tau) z_t \tag{10}$$

is a density process with respect to $\mathbb{P}$ and that the process $S = (S_t)_{t \geq 0}$ such that

$$S_t = e^{L_t} \mathbb{I}_{[\![0,\tau]\!]}(t) + S_\tau e^{\tilde{L}_t - \tilde{L}_\tau} \mathbb{I}_{]\!]\tau,+\infty[\![}(t) \tag{11}$$

is a $(\mathbb{Q}, \mathbb{F})$ - martingale.
We begin by noticing that if $M, \tilde{M}$ are two strictly positive martingales on the same filtered probability space and τ is a stopping time independent of M and $\tilde{M}$ then $N = (N_t)_{t \geq 0}$ such that

$$N_t = c(\tau) \left[M_t \mathbb{I}_{[\![0,\tau]\!]}(t) + M_\tau \frac{\tilde{M}_t}{\tilde{M}_\tau} \mathbb{I}_{]\!]\tau,+\infty[\![}(t) \right]$$

is again a martingale. This can, for example, be seen by conditioning with respect to τ and using the facts that M and $\tilde{M}$ are martingales.
To show that Z is a $(\mathbb{P}, \mathbb{F})$-martingale, we prove the equivalent fact that $(Y_t Z_t)_{t \geq 0}$ is a $(P, \mathbb{F})$ - martingale. But this follows from the previous remark taking $M_t = \zeta_t$ and

$\tilde{M}_t = \tilde{\zeta}_t y_t$ and using (7), (8), (10).
Furthermore, taking conditional expectation with respect to $\mathcal{H}_T$ and using the fact that ζ and $\tilde{\zeta}$ are density processes independent from τ, we see that $\mathbb{E}Z_t = 1$.
To show that $S = (S_t)_{t\geq 0}$ is $(\mathbb{Q}, \mathbb{F})$-martingale we establish that $(Y_t\, Z_t\, S_t)_{t\geq 0}$ is a $(P, \mathbb{F})$ - martingale. For this we use the same remark with $M_t = e^{L_t}\zeta_t$ and $\tilde{M}_t = y_t\tilde{\zeta}_t e^{\tilde{L}_t}$.
Conversely, Z is the density of any equivalent martingale measure if and only if $(Z_t\, S_t)_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{F})$ - martingale. But the last fact is equivalent to the fact that for any bounded stopping time σ,

$$\mathbb{E}(Z_\sigma\, S_\sigma) = 1.$$

Replacing σ by $\sigma \wedge \tau$ in previous expression we deduce that $(Z_{t\wedge\tau})_{t\geq 0}$ is the density of a martingale measure for $(e^{L_{t\wedge\tau}})_{t\geq 0}$. In the same way, using the martingale properties of Z we get for any bounded stopping time σ

$$\mathbb{E}(\frac{Z_\sigma\, S_\sigma}{Z_{\sigma\wedge\tau}\, S_{\sigma\wedge\tau}}) = 1$$

and so $(\frac{Z_t}{Z_{t\wedge\tau}})_{t\geq\tau}$ is the density of an equivalent martingale measure for $(e^{\tilde{L}_t - \tilde{L}_{t\wedge\tau}})_{t\geq\tau}$. ■

2.2 The f-divergence minimal martingale measures.

We now turn to the problem of finding martingale measures which are minimal with respect to some f-divergence. We recall that if f is a convex function on $\mathbb{R}$, the f-divergence of Q with respect to P will be

$$f(Q|P) = \mathbb{E}_P[f(\frac{dQ}{dP})]$$

if the integral given above exists. By convention we set it equal to $+\infty$ otherwise. We recall that P here is the law of the Levy process L with the parameters (b, c, ν).

Definition 1. *We say that Q^* is an f-divergence minimal equivalent martingale measure if $f(Q^*|P) < +\infty$ and*

$$f(Q^*|P) = \inf_{Q\in\mathcal{M}(P)} f(Q|P)$$

where $\mathcal{M}(P)$ is the set of equivalent martingale measures supposed to be non-empty.

Definition 2. *We say that an f-divergence minimal martingale measure Q^* is invariant under scaling if for all $x \in \mathbb{R}^{+,*}$*

$$f(xQ^*|P) = \inf_{Q\in\mathcal{M}(P)} f(xQ|P)$$

Definition 3. *For a given exponential Levy model $S = e^L$, we say that an f-divergence minimal martingale measure Q^* preserves the Levy property if L remains a Levy process under Q^*.*

It was shown in [16],[15], [9], [30] that the properties mentioned above are satisfied for the most common f-divergence functions, in particular for $f(x) = x \ln x$, $f(x) = (1-\sqrt{x})^2$, $f(x) = (1-x)^2$, $f(x) = x^\gamma$ with $\gamma > 1$ or $\gamma < 0$, $f(x) = 1 - x^\gamma$ with $0 < \gamma < 1$. Note also that these functions satisfy: $f''(x) = ax^\gamma$ for some $a > 0$ and $\gamma \in \mathbb{R}$. Conversely, we show in the next theorem that under some conditions, the functions which preserve the Levy property are, up to a linear term, of these forms.

Let us denote

$$\zeta_t^* = \frac{dQ_t^*}{dP_t}$$

the Radon-Nikodym derivatives of Q_t^* with respect to P_t where t stands for the restrictions of the measures to the σ- algebra $\mathcal{G}_t$. Let (β^*, Y^*) be the Girsanov parameters, i.e. the parameters arising in Girsanov theorem and corresponding to the change of the measure P by Q^*. We exclude in advance from our consideration the trivial case when $Q^* = P$ corresponding to $\beta^* = 0, Y^* = 1$. Let also

$$\rho_\lambda(t,x) = \mathbb{E}_{Q^*}[f'(x\lambda\zeta_{T-t}^*)]$$

and

$$q_\lambda(t,x) = \lambda\mathbb{E}_{Q^*}[\zeta_{T-t}^* f''(x\lambda\zeta_{T-t}^*)]$$

when these expectations exist. For simplicity of notation in the case of $\lambda = 1$ we will omit the index 1.

Theorem 1. *Let f be a strictly convex function which belongs to $C^3(\mathbb{R}^{+,*})$ and such that an f-minimal equivalent martingale measure Q^* exists and preserves the Levy property. We suppose that $c \neq 0$ and that the Levy measure ν has a strictly positive density with respect to Lebesgue measure.*

We also suppose that the following integrability conditions are satisfied : for each $\lambda > 0$ and each compact set K belonging to $\mathbb{R}^{+,*}$

$$\mathbb{E}_P|f(\lambda\zeta_T^*)| < +\infty, \quad \mathbb{E}_P[\sup_{0\le t\le T} |\, \zeta_t^* \, f'(\lambda\zeta_t^*)\,|] < +\infty,$$

$$\mathbb{E}_P[\sup_{\lambda\in K,\, 0\le t\le T} (\zeta_t^*)^2 f''(\lambda\zeta_t^*)] < +\infty, \quad \mathbb{E}_P[\sup_{\lambda\in K,\, 0\le t\le T} (\zeta_t^*)^3 |f'''(\lambda\zeta_t^*)|] < +\infty$$

Then $f''(x) = ax^\gamma$ with $a > 0$ and $\gamma \in \mathbb{R}$, i.e. up to a multiplicative constant and a linear term, $f(x) = x\ln(x)$, or $f(x) = -\ln(x)$ or $f(x) = sign(p)sign(p-1)\, x^p$ with $p \neq 0, 1$. Moreover, Q^* is invariant under scaling.

Remark 1. *It should be noticed that for the functions $f''(x) = ax^\gamma$ with $a > 0$ and $\gamma \in \mathbb{R}$ due to the monotonicity of f', f'' and f''' we can omit $\sup_{\lambda\in K}$ in the integrability conditions and they become:*

$$\mathbb{E}_P|f(\lambda\zeta_T^*)| < +\infty, \quad \mathbb{E}_P[\sup_{0\le t\le T} |\, \zeta_t^* \, f'(\lambda\zeta_t^*)\,|] < +\infty.$$

Remark 2. *In the case when the continuous martingale part of L is zero and/or the Levy measure of L does not have a positive density with respect to the Lebesgue measure, the answer is given in [4].*

Proof of theorem 1 The proof of this theorem is highly related with the next section and the proof of Theorem 4. To simplify the notation we put $\lambda = 1$. Making the correspondence between f-divergence minimization and utility maximization we obtain that $Q^* \times \lambda \times \nu^{Q^*}$- a.s. :

$$\rho(t, \zeta^*_{t-} Y^*(x)) - \rho(t, \zeta^*_{t-}) - \beta^*(e^x - 1)\zeta^*_{t-}\, q(t, \zeta^*_{t-}) = 0.$$

In particular, as this equality is true λ-a.e, it must hold for some sequence $(t_n)_{n\geq 1}$ such that $t_n \to T$. Since $\rho(T, x) = f'(x)$ and $q(T, x) = f''(x)$, we get as n goes to infinity:

$$f'(\zeta^*_{T-} Y^*(x)) - f'(\zeta^*_{T-}) - \beta^*(e^x - 1)\zeta^*_{T-} f''(\zeta^*_{T-}) = 0$$

We note that $\zeta^*_{T-} = \zeta^*_T$ (P-a.s. and Q^*-a.s.) since the initial process has no fixed points of discontinuity.

Since the continuous martingale part of L is not zero, the law of ζ^*_T has a strictly positive density with respect to the Lebesgue measure. Hence, we have λ^2-a.e. :

$$f'(yY^*(x)) - f'(y) - \beta^*(e^x - 1)yf''(y) = 0$$

We observe that since f' is a strictly increasing function, Y^* is a strictly monotone function. Setting $Y^*(x) = v$ and writing $x = (Y^*)^{-1}(v)$ where $(Y^*)^{-1}$ is the inverse of Y^*, we obtain that λ^2-a.e. for all $v, y \in \mathbb{R}^{+,*}$

$$f'(yv) - f'(y) = g(v)yf''(y)$$

where g is some function. Since f' is differentiable and $f''(y) > 0$ for $y \in \mathbb{R}^{+,*}$, the function g is differentiable, too. Differentiating with respect to v gives us :

$$f''(yv) = g'(v)f''(y)$$

Taking $y = 1$ we get for all $y, v \in \mathbb{R}^{+,*}$

$$f''(yv) = af''(v)f''(y)$$

where $a = (f''(1))^{-1}$. Hence $f''(y) = ay^\gamma$ with $\gamma \in \mathbb{R}$.

2.3 f-divergence minimal martingale measures in a change-point situation

In the following theorem we give an expression for the density of the f-divergence minimal martingale measures in our change-point framework. We introduce the following hypotheses :

(H1): The f-divergence minimal equivalent martingale measures Q^* and $\tilde{Q}^*$ relative to L and $\tilde{L}$ exist.

(H2): The f-divergence minimal equivalent martingale measures Q^* and $\tilde{Q}^*$ preserve the Levy property and are invariant under scaling.

(H3): For all $c > 0$ and $t \in [0,T]$, we have: $\mathbb{E}_Q |\, f'(c\,\zeta_t^*)\,| < \infty$, $\mathbb{E}_{\tilde{Q}} |\, f'(c\,\tilde{\zeta}_t^*)\,| < \infty$ where ζ^* and $\tilde{\zeta}^*$ are the densities of the f-minimal equivalent martingale measures Q^* and $\tilde{Q}^*$ with respect to P and $\tilde{P}$ respectively.

We set for $t \in [0,T]$

$$z_T^*(t) = \zeta_t^* \, \frac{\tilde{\zeta}_T^*}{\tilde{\zeta}_t^*}$$

Theorem 2. *Assume that f is a strictly convex function, $f \in C^1(\mathbb{R}^{+,*})$, and that (H1), (H2), (H3) hold. Then if the f- minimal equivalent martingale measure $\mathbb{Q}^*$ for the change-point model (4) exists, it has the following structure:*

$$\frac{d\mathbb{Q}_T^*}{d\mathbb{P}_T} = c(\tau)\, z_T^*(\tau) \tag{12}$$

where $c(\cdot)$ is a measurable function $[0,T] \to \mathbb{R}^+$ such that $\mathbb{E}c(\tau) = 1$.
For $c > 0$, let

$$\lambda_t(c) = \mathbb{E}[f'(c\, z_T^*(t))\, z_T^*(t)]$$

where the expectation is taken with respect to $\mathbb{P}$ and let $c_t(\lambda)$ be its right-continuous inverse.
Then, if there exists λ^* such that

$$\int_0^T c_t(\lambda^*) d\alpha(t) = 1, \tag{13}$$

the f- minimal equivalent martingale measure for a change-point situation exists and is given by (12) with $c^*(t) = c_t(\lambda^*)$ for $t \in [0,T]$.
In particular , if $f'(x) = ax^\gamma$, for $a > 0$ and $\gamma \in \mathbb{R}^{+,*}$, then

$$c^*(t) = \frac{[\mathbb{E}(\, z_T^*(t)^{\gamma+1}\,)]^{-\frac{1}{\gamma}}}{\int_0^T [\mathbb{E}(\, z_T^*(t)^{\gamma+1}\,)]^{-\frac{1}{\gamma}}\, d\alpha(t)}$$

and for $f'(x) = \ln(x) + 1$,

$$c^*(t) = \frac{e^{-\mathbb{E}(\, z_T^*(t) \ln z_T^*(t)\,)}}{\int_0^T e^{-\mathbb{E}(\, z_T^*(t) \ln z_T^*(t)\,)}\, d\alpha(t)}.$$

Proof of Theorem 2 Since $\mathbb{Q}^*$ is an equivalent martingale measure we have from (9) that

$$f(\mathbb{Q}_T^* \,|\, \mathbb{P}_T) = \mathbb{E}[f(\, c(\tau)\, \zeta_\tau \, \frac{\tilde{\zeta}_T}{\tilde{\zeta}_\tau})]$$

It follows from the independence of $L, \tilde{L}$ and τ that

$$\mathbb{E}[f(\, c(\tau) \zeta_\tau \, \frac{\tilde{\zeta}_T}{\tilde{\zeta}_\tau}) | \tau = t] = \mathbb{E}[f(\, c(t)\, \zeta_t \, \frac{\tilde{\zeta}_T}{\tilde{\zeta}_t})]$$

Now, the independence of L and $\tilde{L}$ implies the conditional independence of ζ and $\tilde{\zeta}$ given $\sigma(L_s, s \leq T)$. Using the convexity of f and invariance of f under scaling, we see that in order to minimize f-divergence, the measure $\mathbb{Q}$ should be such that ζ is the density of an f-minimal martingale measure for $(e^{L_t})_{t\geq 0}$ and $\tilde{\zeta}$ the density of an f-minimal martingale measure for $(e^{\tilde{L}_t})_{t\geq 0}$. Hence (12) holds.
It follows from the previous formula that we have to minimize the function

$$F(c) = \int_0^T \mathbb{E}[f(\, c(t) z_T^*(t)\,)]\, d\alpha(t)$$

over all cadlag functions $c : [0, T] \to \mathbb{R}^{+,*}$ such that $\mathbb{E}c(\tau) = 1$. For that we consider the linear space $\mathcal{L}$ of such cadlag functions $c : [0; T] \to \mathbb{R}$ with the norm $||c|| = \sup_{t\in[0,T]} |c(t)|$ and the cone of such positive functions.
If f is strictly convex then the function F will be convex. Then according to the Kuhn-Tucker theorem (see [34]) one has to consider the function

$$F_\lambda(c) = F(c) - \lambda \int_0^T (c(t) - 1) d\alpha(t)$$

with Lagrangian factor $\lambda > 0$, and compute, if it exists, the Frechet derivative of $F_\lambda(c)$ denoted by $\frac{\partial F_\lambda}{\partial c}$. This derivative is a linear operator on the space $\mathcal{L}$ such that

$$\lim_{||\delta||\to 0} \frac{|F_\lambda(c+\delta) - F_\lambda(c) - \frac{\partial F_\lambda}{\partial c}\delta|}{||\delta||} = 0 \tag{14}$$

We show that

$$\frac{\partial F_\lambda}{\partial c}(\delta) = \int_0^T \left(\mathbb{E}[f'(c(t) z_T^*(t)) z_T^*(t)] - \lambda\right) \delta(t) d\alpha(t) \tag{15}$$

By the Taylor formula, we have for $\delta \in \mathcal{L}$:

$$F_\lambda(c+\delta) - F_\lambda(c) - \frac{\partial F_\lambda}{\partial c}\delta =$$

$$\int_0^T \mathbb{E}[(f'((c(t) + \theta(t)) z_T^*(t)) - f'(c(t) z_T^*(t))) z_T^*(t)] \delta(t) d\alpha(t)$$

where $\theta(t)$ is a function which takes values in $[0, \delta(t)]$. It is not difficult to see that the modulus of the right-hand side in the previous equality is bounded from above by:

$$\sup_{t\in[0,T]} \mathbb{E}[\,|\, f'((c(t) + \theta(t)) z_T^*(t)) - f'(c(t) z_T^*(t))\,|\, z_T^*(t)\,]\ ||\delta||$$

From hypothesis $(H3)$ it follows that for all $c > 0$

$$\mathbb{E}[f'(c z_T^*(t)) z_T^*(t)] < \infty$$

Since f' is continuous and increasing and the functions c and δ are bounded, we conclude by Lebesgue's dominated convergence theorem that (14) holds and then (15).

According to [34], we now have to find c such that $\frac{\partial F_\lambda}{\partial c}\delta = 0$ for all $\delta \in \mathcal{L}$. In order for the derivative of F_λ to be zero, it is necessary and sufficient to take c such that

$$\mathbb{E}[f'(c(t)z_T^*(t))z_T^*(t)] - \lambda = 0 \;\; \alpha\text{-a.s.}$$

For each $c > 0$ and $t \in [0, T]$ consider the function

$$\lambda_t(c) = \mathbb{E}[f'(cz_T^*(t))z_T^*(t)]$$

which is continuous and increasing in c. Its right-continuous inverse $c_t(\lambda)$ satisfies:

$$\lambda = \mathbb{E}[f'(c_t(\lambda)z_T^*(t))z_T^*(t)]$$

Now to obtain a minimizer c^*, it remains to find, if it exists, λ^* which satisfies (13). Let us now consider the special case $f'(x) = ax^\gamma$. Then we obtain up to a constant, that $\lambda_t(c) = ac^\gamma \mathbb{E}[z_T^*(t)^{\gamma+1}]$ and for $f'(x) = \ln(x)+1$ we get $\lambda_t(c) = \mathbb{E}[z_T^*(t)\ln z_T^*(t)]+1$. Writing now $c_t(\lambda)$ and integrating with respect to α we find λ^* and the expression of $c^*(t)$. ■

Example: *A change-point Black-Scholes model*
We now want to apply the results in the simplest of settings, namely when L and $\tilde{L}$ define Black-Scholes type models. Therefore, we assume that L and $\tilde{L}$ are continuous Levy processes with characteristics $(b, c, 0)$ and $(\tilde{b}, c, 0)$ respectively, $c > 0$. As is well known, the initial models will be complete, with a unique equivalent martingale measure which defines a unique price for options. However, in our change-point situation the martingale measure is not unique, and we have an infinite set of martingale measures of the form

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}(X) = c(\tau)\, \exp\Big(\int_0^t \beta_s dX_s^c - \frac{1}{2}\int_0^t \beta_s^2 c ds\Big)$$

where $c(\cdot)$ is a measurable function $[0, T] \to \mathbb{R}^{+,*}$ such that $\mathbb{E}[c(\tau)] = 1$ and

$$\beta_s = -\frac{1}{c}\Big[\,(b + \frac{c}{2})\, I_{[0,\tau]}(s) + (\tilde{b} + \frac{c}{2})\, I_{]\tau,+\infty[}(s)\,\Big]$$

If for example $f'(x) = ax^\gamma$, applying Theorem 2, we get

$$c^*(t) = \frac{e^{-\frac{\gamma+1}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]}}{\int_0^T e^{-\frac{\gamma+1}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]} d\alpha(t)}$$

and if $f'(x) = \ln(x) + 1$, then

$$c^*(t) = \frac{e^{-\frac{1}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]}}{\int_0^T e^{-\frac{1}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]} d\alpha(t)}$$

3 Optimal strategies for utility maximization

In this section, we are interested in finding optimal strategies for terminal wealth with respect to some utility functions. More precisely, we assume that our financial market consists of two assets : a non-risky asset B, with interest rate r, and a risky asset S, modeled using the change-point Levy model defined in (4). We denote by $\vec{S} = (B, S)$ the price process and by $\vec{\Phi} = (\phi^0, \phi)$ the amount of money invested in each asset. According to usual terminology, a predictable $\vec{S}$-integrable process $\vec{\Phi}$ is said to be a self-financing admissible strategy if for every $t \in [0, T]$ and x initial capital

$$\vec{\Phi}_t \cdot \vec{S}_t = x + \int_0^t \vec{\Phi}_u \cdot d\vec{S}_u \tag{16}$$

where the stochastic integral in the right-hand side is bounded from below. Here $\cdot$ denotes the scalar product. We will denote by $\mathcal{A}$ the set of all self-financing admissible strategies. In order to avoid unnecessary complications, we will assume that the interest rate r is 0, so that starting with an initial capital x, terminal wealth at time T is

$$V_T(\phi) = x + \int_0^T \phi_s dS_s$$

Let u denote a strictly increasing, strictly concave, continuously differentiable function on $dom(u) = \{x \in \mathbb{R} | u(x) > -\infty\}$ which satisfies

$$u'(+\infty) = \lim_{x \to +\infty} u'(x) = 0,$$

$$u'(\underline{x}) = \lim_{x \to \underline{x}} u'(x) = +\infty$$

where $\underline{x} = \inf\{u \in dom(u)\}$.

We will say that ϕ^* defines an optimal strategy with respect to u if

$$\mathbb{E}_P[u(x + \int_0^T \phi^*_s dS_s)] = \sup_{\phi \in \mathcal{A}} \mathbb{E}_P[u(x + \int_0^T \phi_s dS_s)]$$

As has been shown in [24], there is a strong link between this optimization problem and the previous problem of finding f-minimal martingale measures. Let f be the convex conjugate function of u :

$$f(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\} = u(I(y)) - yI(y) \tag{17}$$

where $I = (u')^{-1}$. We recall that in particular

$$\begin{aligned}
&\text{if } u(x) = \ln(x) \text{ then } f(x) = -\ln(x) - 1,\\
&\text{if } u(x) = \frac{x^p}{p}, p < 1 \text{ then } f(x) = -\frac{p-1}{p} x^{\frac{p}{p-1}},\\
&\text{if } u(x) = 1 - e^{-x} \text{ then } f(x) = 1 - x + x\ln(x).
\end{aligned}$$

The following result taken from [24] gives us the relation between portfolio optimization and f-minimal martingale measures. We assume for simplicity that $\underline{x} > -\infty$.

Theorem 3. *(see [24]) Let $x \in \mathbb{R}^+$ be fixed. Let Q^* be an equivalent martingale measure which satisfies*

$$\mathbb{E}_P[|f(\lambda\frac{dQ_T^*}{dP_T})|] < \infty, \;\; \mathbb{E}_{Q^*}|f'(\lambda\frac{dQ_T^*}{dP_T})| < \infty$$

for λ such that

$$-\mathbb{E}_{Q^*}f'(\lambda\frac{dQ_T^*}{dP_T}) = x.$$

Then Q^* is an f-minimal martingale measure if and only if

$$-f'(\lambda\frac{dQ_T^*}{dP_T}) = x + \int_0^T \phi_u^* dS_u \tag{18}$$

where ϕ^* is a predictable function such that $(\int_0^{\cdot} \phi_u^* dS_u)$ is a Q^*-martingale. If the last relation holds, then $\vec{\Phi}= (\phi^0, \phi)$ with $\phi_t^0 = x + \int_0^t \phi_u dS_u - \phi_t S_t$ is an admissible optimal portfolio strategy.

Remark 3. When $\underline{x} = -\infty$, (18) remains true but the process ϕ^* no longer necessarily defines an admissible strategy. As in [31], we will say that $\hat{\phi}$ is an asymptotically optimal strategy if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ which goes to infinity such that

$$\lim_{n\to+\infty} E[u(x + \int_0^{T\wedge\tau_n} \hat{\phi}_s dS_s)] = \sup_{\phi\in\mathcal{A}} E[u(x + \int_0^T \phi_s dS_s)]$$

Now if (18) holds and there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ which goes to infinity such that $(\phi_s^* 1\!\!1_{[\![0,T\wedge\tau_n]\!]})_{0\leq s\leq T}$ is a sequence of admissible strategies, then ϕ^* will indeed be asymptotically optimal.

It is obvious that

$$\lim_{n\to+\infty} \mathbb{E}_P[u(x + \int_0^{T\wedge\tau_n} \phi_s^* dS_s)] \leq \sup_{\phi\in\mathcal{A}} \mathbb{E}_P[u(x + \int_0^T \phi_s dS_s)] \tag{19}$$

Now, if ϕ is any admissible strategy, it follows from the concavity of u that

$$u(x + \int_0^T \phi_s dS_s) \leq u(x + \int_0^T \phi_s^* dS_s) + u'(x + \int_0^T \phi_s^* dS_s)[\int_0^T (\phi_s - \phi_s^*) dS_s]$$

Note that $u'(x + \int_0^T \phi_s^* dS_s) = \lambda\, \zeta_T^*$ where ζ_T^* is Radon-Nikodym density of the measure Q_T^* with respect to P_T. Note also that $(\int_0^t (\phi_s - \phi_s^*) dS_s)_{0\leq t\leq T}$ is the difference of a local martingale which is bounded from below and of a martingale with respect to Q^*, so that

$$\mathbb{E}_P[u(x + \int_0^T \phi_s dS_s)] \leq \mathbb{E}_P[u(x + \int_0^T \phi_s^* dS_s)]$$

and hence

$$\sup_{\phi\in\mathcal{A}} \mathbb{E}_P[u(x + \int_0^T \phi_s dS_s)] \leq \mathbb{E}_P[u(x + \int_0^T \phi_s^* dS_s)] \tag{20}$$

Next, we have in the same way from the concavity of u that

$$u(x+\int_0^{T\wedge\tau_n}\phi^*_s dS_s)\le u(x+\int_0^T\phi^*_s dS_s)+u'(x+\int_0^T\phi^*_s dS_s).\int_{T\wedge\tau_n}^T\phi^*_s dS_s$$

Since $(\int_0^{T\wedge\tau_n}\phi^*_s dS_s)_{n\geq 1}$ is a uniformly integrable Q^* martingale, the family $(\int_{T\wedge\tau_n}^T\phi^*_s dS_s)_{n\in\mathbb{N}}$ is uniformly integrable. Hence $(u(x+\int_0^{T\wedge\tau_n}\phi^*_s dS_s))_{n\in\mathbb{N}}$ is a uniformly integrable family and

$$\lim_{n\to+\infty}\mathbb{E}_P[u(x+\int_0^{T\wedge\tau_n}\phi^*_s dS_s)=\mathbb{E}_P[u(x+\int_0^T\phi^*_s dS_s)] \tag{21}$$

Finally, it follows from (19), (20) and (21) that ϕ^* defines an asymptotically optimal strategy.

3.1 Optimal strategies for exponential Levy models

In the following theorem we discuss the existence and we give, when it exists, the expression of an admissible optimal portfolio. Let Q be f-minimal equivalent martingale measure and P initial measure. To avoid unnecessary complications within this part we omit * in the notations concerning f-minimal martingale measure and related processes. Let $\zeta=(\zeta_t)_{t\geq 0}$ be the Radon-Nikodym density process of Q with respect to P:

$$\zeta_t=\frac{dQ_t}{dP_t}.$$

Let u be a utility function and f its convex conjugate. We set for fixed $\lambda>0$

$$\rho_\lambda(t,x)=\mathbb{E}_Q[f'(x\lambda\zeta_{T-t})] \tag{22}$$

and

$$q_\lambda(t,x)=\lambda\mathbb{E}_Q[\zeta_{T-t}f''(x\lambda\zeta_{T-t})]$$

when these expectations exist.

Theorem 4. *Let f be a strictly convex function which belongs to $C^3(\mathbb{R}^{+,*})$ such that an f-minimal equivalent martingale measure Q^* exists and preserves the Levy property. We suppose that $c\neq 0$. We also suppose that the following integrability conditions are satisfied : for each $\lambda>0$ and each compact set K belonging to $\mathbb{R}^{+,*}$*

$$\mathbb{E}_P|f(\lambda\zeta_T)|<+\infty,\quad \mathbb{E}_P[\sup_{0\leq t\leq T}|\,\zeta_t\,f'(\lambda\zeta_t)\,|]<+\infty,$$

$$(\mathcal{H}_4)\qquad \mathbb{E}_P[\sup_{\lambda\in K,\,0\leq t\leq T}(\zeta_t)^2f''(\lambda\zeta_t)]<+\infty,$$

$$\mathbb{E}_P[\sup_{\lambda\in K,\,0\leq t\leq T}(\zeta_t)^3f'''(\lambda\zeta_t)]<+\infty.$$

Then, for an initial capital $x = -\rho_\lambda(0,1)$ and a utility function u, an optimal strategy (an asymptotically optimal strategy) exists and is given by

$$\phi_t = -\frac{\beta\zeta_{t-}}{S_{t-}}\; q_\lambda(t, \zeta_{t-}) \tag{23}$$

If $f''(x) = ax^\gamma$ with $a > 0$, then

$$\phi_t = -\frac{a\beta\lambda^{\gamma+1}\zeta_{t-}^{\gamma+1}}{S_{t-}}\; \mathbb{E}_P[\,\zeta_{T-t}^{\gamma+2}\,]$$

Remark 4. *For classical utility functions: $u(x) = \ln(x)$, $u(x) = \frac{x^p}{p}, p < 1$, and $u(x) = 1 - e^{-x}$, the corresponding f satisfies $f''(x) = ax^\gamma$ with $a = 1, \gamma = -2$ in the first case, $a = (1-p)^{-1}, \gamma = \frac{2-p}{p-1}$ in the second case and $a = 1, \gamma = -1$ in the third case.*

Remark 5. *It should also be noticed that for the functions $f''(x) = ax^\gamma$ with $a > 0$ and $\gamma \in \mathbb{R}$ due to the monotonicity of f', f'' and f''' we can omit $\sup_{\lambda\in K}$ in the integrability conditions and they become:*

$$\mathbb{E}_P|f(\lambda\zeta_T^*)| < +\infty, \quad \mathbb{E}_P[\sup_{0\le t\le T} |\,\zeta_t^*\, f'(\lambda\zeta_t^*)\,|] < +\infty.$$

Lemma 2. *Suppose that the conditions of Theorem 4 are satisfied. Then the function ρ_λ is twice continuously differentiable in x and once continuously differentiable in t on the set $]0,T[\times\mathbb{R}^{+,*}$ and*

$$\frac{\partial\rho_\lambda}{\partial t}(t,x) = \beta^2\lambda\, x\mathbb{E}_Q[\,(\zeta_{T-t})\, f''(x\lambda\zeta_{T-t})\,]$$

$$+\frac{1}{2}\beta^2\lambda^2\, x^2\mathbb{E}_Q[\,(\zeta_{T-t})^2\, f'''(x\lambda\zeta_{T-t})\,]$$

$$+\int_{\mathbb{R}^*}\mathbb{E}_Q[Y(u)\, f'(x\lambda\zeta_{t-}Y(u)) - Y(u)\, f'(x\lambda\zeta_{t-}) - \lambda\, x\zeta_{t-}f''(x\lambda\zeta_{t-})(Y(u)-1)]\, d\nu(u),$$

$$\frac{\partial\rho_\lambda}{\partial x}(t,x) = \lambda\mathbb{E}_Q[\,\zeta_{T-t}\, f''(x\lambda\zeta_{T-t})\,],$$

$$\frac{\partial^2\rho_\lambda}{\partial x^2}(t,x) = \lambda^2\mathbb{E}_Q[\,(\zeta_{T-t})^2\, f'''(x\lambda\zeta_{T-t})\,]$$

Proof We remark that the regularity properties of $\rho_\lambda(T-t,x)$ are the same as $\rho_\lambda(t,x)$. We introduce for $a > 0$ the stopping times

$$s_n = \inf\{t \ge 0 | \zeta_t \ge n\}$$

We remark that $s_n \to \infty$ as $n \to \infty$ and that $\cup_{n>0}[\![0, s_n[\![= \Omega$. To obtain the formula for the partial derivative with respect to t we use Ito formula for $f'(x\lambda\zeta_{t\wedge s_n})$, then we take the expectation with respect to Q and we passe to the limit as $n \to \infty$. In the limit passage we use the fact that $Q_T << P_T$ expressed in terms of Hellinger process, namely P_T and Q_T-a.s.

$$h(\frac{1}{2}, P, Q)_T = \frac{1}{8}\beta^2 cT + \int_0^T \int_{\mathbb{R}^*} (\sqrt{Y} - 1)^2 d\nu_L^P) < \infty.$$

For the existence of partial derivatives with respect to x we use Taylor formula for $f'(x\lambda\zeta_t)$ with integral remainder. The continuity of the derivatives follows directly from continuity of the derivatives of f and the integrability condition $(\mathcal{H}_4)$. ■

Proof of Theorem 4 As L is a Levy process both under P and Q, ζ_t and ζ_T/ζ_t are independent and so

$$\mathbb{E}_Q(f'(\lambda\zeta_T)|\zeta_t = x) = \mathbb{E}_Q(f'(x\lambda\zeta_{T-t}))$$

implying

$$\mathbb{E}_Q(f'(\lambda\zeta_T)|\mathcal{G}_t) = \rho_\lambda(t, \zeta_t)$$

We recall that $\rho_\lambda(T, \zeta_T) = f'(\lambda\zeta_T)$. So, if we can write Ito formula for the process $\rho_\lambda(t, \zeta_t)_{t\geq 0}$ and obtain the integral representation as in Theorem 3, we will be able to identify the optimal strategy.

In order to ensure that ζ_{t^-} is bounded from above and below and that S_{t^-} is bounded away from zero, we introduce for $a > 0$ the stopping times

$$\tau_n = \inf\{t \geq 0 | \zeta_t \geq n, \zeta_t \leq 1/n, S_t \leq 1/n\}$$

We remark that $\tau_n \to \infty$ as $n \to \infty$ and that $\cup_{n>0}[\![0, \tau_n[\![= \Omega$.

We write the process ζ in the following form:

$$\zeta_t = \zeta_t^c + \zeta_t^d$$

and we separate the big jumps of $\Delta\zeta/\zeta_-$ using truncation function l:

$$\zeta_t^d = \zeta_t^{d,l} + \zeta_{t-}\left(\frac{x}{\zeta_{t-}} - l(\frac{x}{\zeta_{t-}})\right) \star (\mu_\zeta - \nu_\zeta)$$

Here and further on, μ_ζ and ν_ζ are respectively the jump measure of ζ with respect to P and its compensator. Then we put

$$\zeta_t^l = \zeta_t^c + \zeta_t^{d,l}$$

According to Lemma 2 the function ρ_λ is twice continuously differentiable with respect to x, once continuously differentiable with respect to t, and has bounded derivatives on compact sets of $]0, T[\times\mathbb{R}^{+,*}$. Therefore, we can write the Ito formula :

$$\rho_\lambda(t \wedge \tau_n, \zeta_{t\wedge\tau_n}) = \rho_\lambda(0, 1) +$$

$$\int_0^{t\wedge\tau_n} \frac{\partial\rho_\lambda}{\partial s}(s, \zeta_{s-})ds + \int_0^{t\wedge\tau_n} \frac{\partial\rho_\lambda}{\partial x}(s, \zeta_{s-})d\zeta_s^l + \frac{1}{2}\int_0^{t\wedge\tau_n} \frac{\partial^2\rho_\lambda}{\partial x^2}(s, \zeta_{s-})d\langle\zeta^c\rangle_s$$

$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}^*}[\rho_\lambda(s,\zeta_{s-}+x)-\rho_\lambda(s,\zeta_{s-})-\frac{\partial\rho_\lambda}{\partial x}(s,\zeta_{s-})\,\zeta_{s-}\,l(\frac{x}{\zeta_{s-}})]\,(d\mu_\zeta-d\nu_\zeta^Q)$$

$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}^*}[\rho_\lambda(s,\zeta_{s-}+x)-\rho_\lambda(s,\zeta_{s-})-\frac{\partial\rho_\lambda}{\partial x}(s,\zeta_{s-})\,\zeta_{s-}\,l(\frac{x}{\zeta_{s-}})]\,d\nu_\zeta^Q$$

Under the measure Q the process ζ^l is no more a local martingale, but a semi-martingale with the decomposition:

$$\zeta_t^l = B_t^l + m_t^l$$

where $m^l=(m_t^l)_{t\geq 0}$ is a Q-local martingale and $B^l=(B_t^l)_{t\geq 0}$ is a Q-drift of ζ^l. Let us calculate this drift.

We note that ζ^l can be written as $\mathcal{E}(M^l)$ where $\mathcal{E}(\cdot)$ is Dolean-Dade exponential and $M^l=(M_t^l)_{t\geq 0}$ is a local martingale with respect to P. Then $d\zeta_t^l=\zeta_{t-}^l dM_t^l$ and from [21],p.260 or from [29], p. 182 we get

$$M^l=\beta\cdot L^c + l(Y-1)\star(\mu_L-\nu_L^P)$$

Here and further on, μ_L and ν_L are respectively the jump measure of L with respect to P and its compensator. Using Girsanov Theorem (see [29],p.159) we find that the drift of M is given by:

$$dA_t^l=\beta^3 d<L^c>_t+[l(Y-1)]^2\,d\nu_L^P \tag{24}$$

and that

$$dB_t^l=\zeta_{t-}dA_t^l \tag{25}$$

Since $(\mathbb{E}_Q(f'(\lambda\zeta_T)|\mathcal{G}_t))_{t\geq 0}$ is a martingale with respect to Q, the process $\rho_\lambda(t,\zeta_t)_{t\geq 0}$ is Q-martingale, and hence, the stopped process $(\rho_\lambda(t\wedge\tau_n,\zeta_{t\wedge\tau_n})_{t\geq 0})$ is also a martingale with respect to Q. Using (25) and (24) and the fact that $d\zeta_t^l=\zeta_{t-}^l dM_t^l$, we see that we must have :

$$\int_0^{t\wedge\tau_n}\frac{\partial\rho_\lambda}{\partial s}(s,\zeta_{s-})ds+\int_0^{t\wedge\tau_n}\frac{\partial\rho_\lambda}{\partial s}(s,\zeta_{s-})dA_s^l+\frac{1}{2}\int_0^{t\wedge\tau_n}\frac{\partial^2\rho_\lambda}{\partial x^2}(s,\zeta_{s-})\beta^2\,\zeta_{s-}\,c\,ds \tag{26}$$

$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}^*}[\rho_\lambda(s,\zeta_{s-}Y)-\rho_\lambda(s,\zeta_{s-})-\frac{\partial\rho_\lambda}{\partial x}(s,\zeta_{s-})\,l(Y-1)]Y\,d\nu ds=0$$

Let us introduce now the process $\hat{L}=(\hat{L}_t)_{t\geq 0}$ such that

$$S_t=S_0\mathcal{E}(\hat{L})_t$$

where $\mathcal{E}(\cdot)$ is the Dolean's-Dade exponential. Hence, if $\hat{L}$ is a martingale with respect to Q, then S will be a local martingale with respect to Q. The parameters of the process $\hat{L}$ with respect to to P can easily be expressed using the parameters (b,c,ν) of the Levy process L :

$$\begin{cases}\hat{b}=b+\frac{1}{2}c+(e^x-1-x)\star\nu\\ \hat{c}=c\\ \hat{\nu}=(e^x-1)\cdot\nu\end{cases}$$

Since $L^c = \hat{L}^c$ and writing $\hat{L} = \hat{L}^c + \hat{L}^d$ we get:

$$\rho_\lambda(t \wedge \tau_n, \lambda\zeta_{t\wedge\tau_n}) = \rho_\lambda(0,1) + \int_0^{t\wedge\tau_n} \beta\zeta_{s-}\frac{\partial\rho_\lambda}{\partial x}(s,\zeta_{s-})d\hat{L}_s + N^d_{t\wedge\tau_n} \tag{27}$$

where $N^d = (N^d_t)_{t\geq 0}$ is the purely discontinuous martingale given by

$$N^d_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n}\int_{\mathbb{R}^*}[\,\rho_\lambda(s,\zeta_{s-}Y) - \rho_\lambda(s,\zeta_{s-}) - \zeta_{s-}\beta\,(e^x-1)\,\frac{\partial\rho_\lambda}{\partial x}(s,\zeta_{s-})\,]d(\mu_L - \nu_L^Q) \tag{28}$$

Since $c \neq 0$, in the case $\nu_L^Q \neq 0$ the integral with respect to $(\mu_L - \nu_L^Q)$ with a non-zero integrand can not be written as a stochastic integral with respect to $\hat{L}$. Then from Theorem 3 we deduce that $N^d_T = 0\,(Q-a.s.)$. But N^d is a Q-martingale, which implies $N^d_t = 0$ for all $t \in [0,T]\,(Q-a.s.)$ and we should necessarily have $Q\times\lambda\times\nu^Q$-a.s.:

$$\rho_\lambda(t,\zeta_{t-}Y) - \rho_\lambda(t,\zeta_{t-}) - \beta(e^x-1)\zeta_{t-}\frac{\partial\rho_\lambda}{\partial x}(t,\zeta_{t-}) = 0. \tag{29}$$

In the case ν^Q is zero measure, which implies that ν^P is zero measure, we also have $N^d_T = 0$.
Finally from (27) and (29) we deduce that on each stochastic interval $[\![0,\tau_n[\![$

$$\rho_\lambda(T,\lambda\zeta_T) = \rho_\lambda(0,1) + \int_0^T \beta\zeta_{s-}\frac{\partial\rho_\lambda}{\partial x}(s,\zeta_{s-})d\hat{L}_s$$

and, hence,

$$f'(\lambda\zeta_T) = \rho_\lambda(0,1) + \int_0^T \frac{\beta\zeta_{s-}}{S_{s-}}\,\frac{\partial\rho}{\partial x}(s,\zeta_{s-})dS_s \tag{30}$$

Note that for all $t \leq T$,

$$\rho_\lambda(t,\lambda\zeta_t) = \rho_\lambda(0,1) + \int_0^t \frac{\beta\zeta_{s^-}}{S_{s^-}}\frac{\partial\rho_\lambda}{\partial x}(x,\zeta_{s^-})dS_s$$

Now as $(\rho_\lambda(t,\lambda\zeta_t))_{0\leq t\leq T}$ defines a Q-martingale, the process on the right-hand side is also a Q-martingale.

Finally, in the decomposition (18), $x = -\rho_\lambda(0,1) = -\mathbb{E}_Q f'(\lambda\zeta_T)$ and ϕ is given by (23). Computations in the special cases presented above are straightforward. ■

3.2 Optimal strategies in a change-point situation

Let u be a utility function and f its convex conjugate. We denote by (β, Y) and $(\tilde{\beta},\tilde{Y})$ the Girsanov parameters corresponding to the changes of measure from P and $\tilde{P}$ to the f-divergence minimal measures Q^* and $\tilde{Q}^*$ respectively. Then the first Girsanov parameter for the change of the measure $\mathbb{P}$ to $\mathbb{Q}^*$ will be:

$$\beta_t = \beta I_{[0,\tau]}(t) + \tilde{\beta}\, I_{]\tau,+\infty[}(t)$$

For $0 \le t \le T$ we write

$$z_t^*(\tau) = \zeta_t^* \, I_{[0,\tau]}(t) + \zeta_\tau^* \, \frac{\tilde{\zeta}_t^*}{\tilde{\zeta}_\tau^*} \, I_{]\tau,+\infty[}(t)$$

From the Theorem 2 we know that the Radon-Nikodym derivative of $\mathbb{Q}^*$ with respect to $\mathbb{P}$ is

$$Z_T^*(\tau) = c^*(\tau) z_T^*(\tau)$$

where $c^*(\tau)$ is defined in Theorem 2 and ζ^*, $\tilde{\zeta}^*$ are the densities of Q^* and $\tilde{Q}^*$ with respect to P and $\tilde{P}$.

Taking the regular versions of conditional probabilities, we denote for $0 \le v \le T$

$$\rho^{(v)}(t,x) = \mathbb{E}[z_{T-t}^*((v-t)^+) f'(x\lambda z_{T-t}^*((v-t)^+)]$$

and

$$q^{(v)}(t,x) = \lambda \mathbb{E}[[z_{T-t}^*((v-t)^+)]^2 f''(x\lambda z_{T-t}^*((v-t)^+)]$$

when these expectations exist. Here $x^+ = \max(0,x)$. We can now state the following result:

Theorem 5. *Let f be a strictly convex function which belongs to $C^3(\mathbb{R}^{+,*})$ satisfying (H1), (H2), (H4) for Q and $\tilde{Q}$ and (13) . Then there exists an $\mathbb{F}$-optimal strategy ϕ^* for our change-point model. In addition, it is $\hat{\mathbb{F}}$-adapted and*

$$\phi_t^* = -\frac{\beta_t Z_{t-}^*(\tau)}{S_{t-}} q^{(\tau)}(t, Z_{t-}^*(\tau)) \tag{31}$$

In particular, when $f''(x) = ax^\gamma$ with $a > 0$, we have:

$$\phi_t^* = -\frac{a\lambda^{\gamma+1}\beta_t Z_{t-}^{*\gamma+1}(\tau)}{S_{t-}} \mathbb{E}([Z_{T-t}^*((\tau-t)^+)]^{\gamma+2} \,|\, \tau)$$

Proof From Theorem 2 it follows that there exists an f-minimal martingale measure $\mathbb{Q}^*$. Since the processes X and S are $\hat{\mathbb{F}}$-adapted, applying for example Theorem 3.1 in [24], we have the existence of an $\hat{\mathbb{F}}$-adapted optimal strategy ϕ^* such that

$$-f'(\lambda Z_T^*(\tau)) = x + \int_0^T \phi_u^* dS_u$$

and such that $\int_0^\cdot \phi_u^* dS_u$ defines a local martingale with respect to $(\mathbb{Q}, \hat{\mathbb{F}})$.

The proof of the formula for the optimal strategy is similar to that of formula (23) in Theorem 3. Namely, we write the Ito formula for conditional expectation replacing β by β_t, Y by Y_t and ζ_t by $Z_t^*(\tau)$. The corresponding formulas for N^d can be obtained with the same changes. Then using the relation

$$\frac{z_T^*(\tau)}{z_t^*(\tau)} \stackrel{\mathcal{L}}{=} z_{T-t}^*((\tau-t)^+)$$

and the fact that the left-hand side of this equality is independent from $z_t^*(\tau)$, we obtain

$$\mathbb{E}_{\mathbb{Q}^*}(f'(\lambda Z_T^*(\tau)) \,|\, \tau = v, Z_t^*(\tau) = x) = \rho^{(v)}(t, x).$$

Then for a regular version of conditional probabilities we have:

$$\mathbb{E}_{\mathbb{Q}^*}(f'(\lambda Z_T^*(\tau)) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}^*}(f'(\lambda Z_T^*(\tau)) \,|\, \mathcal{G}_t \vee \sigma(\tau)) = \rho^{(\tau)}(t, Z_t^*(\tau))$$

where the last function is obtained by replacing of v and x by τ and $Z_t^*(\tau)$ respectively. Then applying Ito's formula again, we find the continuous martingale part of the decomposition of $(\rho^{(v)}(t, Z_t^*(\tau))_{t\geq 0}$ and then we replace v by τ and it gives us a final result for optimal strategy.

■

Example: *A change-point Black-Scholes model*
As before, we now want to apply the results when L and $\tilde{L}$ define Black-Scholes type models. Therefore, we assume that L and $\tilde{L}$ are continuous Levy processes with characteristics $(b, c, 0)$ and $(\tilde{b}, c, 0)$ respectively. Let τ be a random variable bounded by T which is independent from L and $\tilde{L}$. Then the asymptotically optimal strategy from the point of view of maximization of exponential utility $u(x) = 1 - \exp(-x)$ will be :

$$\phi_t^* = -\frac{\beta_t}{S_{t-}} = \frac{(b + c/2)I_{[0,\tau]}(t) + (\tilde{b} + c/2)\, I_{]\tau,+\infty[}(t)}{cS_{t-}}$$

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