Portfolio Optimization in a Defaults Model under Full/Partial Information

Thomas LIM
Laboratoire de Probabilités et Modèles Aléatoires
CNRS, UMR 7599
Universités Paris 7
e-mail: tlim@math.jussieu.fr

Marie-Claire QUENEZ
Laboratoire de Probabilités et Modèles Aléatoires
CNRS, UMR 7599
Universités Paris 7
e-mail: quenez@math.jussieu.fr

March 16, 2010

Abstract

In this paper, we consider a financial market with assets exposed to some risks inducing jumps in the asset prices, and which can still be traded after default times. We use a default-intensity modeling approach, and address in this incomplete market context the problem of maximization of expected utility from terminal wealth for logarithmic, power and exponential utility functions. We study this problem as a stochastic control problem both under full and partial information. Our contribution consists in showing that the optimal strategy can be obtained by a direct approach for the logarithmic utility function, and the value function for the power utility function can be determined as the minimal solution of a backward stochastic differential equation. For the partial information case, we show how the problem can be divided into two problems: a filtering problem and an optimization problem. We also study the indifference pricing approach to evaluate the price of a contingent claim in an incomplete market and the information price for an agent with insider information.

Keywords Optimal investment, default time, default intensity, filtering, dynamic programming principle, backward stochastic differential equation, indifference price, information pricing, logarithmic utility, power utility, exponential utility.

1 Introduction

One of the important problems in mathematical finance is the portfolio optimization problem when the investor wants to maximize the expected utility from terminal wealth. In this paper, we study this problem by considering a small investor on an incomplete financial market who can trade in a finite time interval $[0, T]$ by investing in risky stocks and a riskless bond. We assume that there exist some default times on the market, and each default time generates a jump of stock prices. The underlying traded assets are assumed to be some local martingales driven by a Brownian motion and a default indicating process. In such a context, we solve the portfolio optimization problem when the investors want to maximize the expected utility from terminal wealth. We assume that in the market there are two kinds of agents: the insider agents (the agents with insider information) and the classical agents (they only observe the asset prices and the default times). These situations are referred as full information and partial information. We will be interested not only in describing the investor's optimal utility, but also the strategies which he may follow to reach this goal.

The utility maximization problem with full information has been largely studied in the literature. In the framework of a continuous-time model the problem was studied for the first time by Merton (1971). Using the methods of stochastic optimal control, the author derives a nonlinear partial equation for the value function of the optimization problem. Some papers study this problem by using the dual problem, we can quote, for instance, Karatzas, Lehoczky and Shreve (1987) for the case of complete financial models, and Karatzas *et al.* (1991) and Kramkov and Schachermayer (1999) for the case of incomplete financial models, they find the solution of the original problem by convex duality. These papers are useful to prove the existence of an optimal strategy in the general case, but in practice it is difficult to find the optimal strategy with the dual method. Some others study the problem by using the dynamic programming principle, we can quote Jeanblanc and Pontier (1990) for a complete model with discontinuous prices, Bellamy (2001) in the case of a filtration generated by a Brownian motion and a Poisson measure, Hu, Imkeller and Muller (2005) for an incomplete model in the case of a Brownian filtration, and Jiao and Pham (2009) in the case with a default, in which the authors study the case before the default and the case after the default.

Models with partial observation are essentially studied in the literature in a complete market framework. Detemple (1986), Dothan and Feldman (1986), Gennotte (1986) use dynamic programming methods in a linear gaussian filtering. Lakner (1995, 1998) solves the optimization problem via a martingale approach and works out the special case of linear gaussian model. We mention that Frey and Runggaldier (1999) and Lasry and Lions (1999) study hedging problems in finance under restricted information. Pham and Quenez (2001) treat the case of an incomplete stochastic volatility model. Callegaro *et al.* (2006) and Roland (2007) study the case of a market model with jumps.

We first study the case of full information. For the logarithmic utility function, we use a direct approach, which allows to give an expression of the optimal strategy depending uniquely on the coefficients of the model satisfied by the stocks. For the power utility function, we look for a necessary condition characterizing the value function which is solution of the maximization problem. We show that this value function is the smallest solution of a BSDE. We also give an approximation of the value function by a sequence of solutions of BSDEs. These solutions are the value functions of the maximization problem restricted to some bounded subsets of strategies. For the exponential utility function, we refer to the companion paper Lim and Quenez (2009).

In order to solve the partial information problem, the common way is to use the filtering theory, so as to reduce the stochastic control problem with partial information to one with full information as in Pham and Quenez (2001) or Roland (2007). Then we can apply the results of the full information problem.

The outline of this paper is organized as follows. In Section 2, we describe the model and formulate the optimization problem. In Section 3, we solve the logarithmic utility function with a direct approach. In Section 4, we consider the power utility function by giving a characterization of the value function by a BSDE thanks to the dynamic programming principle, then we approximate the value function by a sequence of solutions of Lipschitz BSDEs. In Section 5, we use results from filtering theory to reduce the stochastic control problem with partial information to one with full information, then we apply the results of the full information problem to the partial information problem. Finally we study the indifference price for a contingent claim and the information price linked to the insider information.

In all this paper, elements of $\mathbb{R}^n$, $n \geq 1$, are identified to column vectors, the superscript $'$ stands for the transposition, $||.||$ the square norm, $\mathbb{1}$ the vector of $\mathbb{R}^n$ such that each component of this vector is equal to 1. Let U and V two vectors of $\mathbb{R}^n$, $U * V$ denotes the vector such that $(U * V)_i = U_i V_i$ for each $i \in \{1, \dots, n\}$. Given a vector $X \in \mathbb{R}^n$, $|X|^2$ denotes the vector of $\mathbb{R}^n$ such that $|X|^2_i = |X_i|^2$ for each $i \in \{1, \dots, n\}$. For a function $f : \mathbb{R} \to \mathbb{R}$ and a vector $X \in \mathbb{R}^n$, we denote by $f(X)$ the vector of $\mathbb{R}^n$ such that $f(X)_i = f(X_i)$ for each $i \in \{1, \dots, n\}$. Let $X \in \mathbb{R}^n$, $diag(X)$ is the matrix such that $diag(X)_{ij} = X_i$ if $i = j$ else $diag(X)_{ij} = 0$.

2 The model

We start with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a time horizon $T \in (0, \infty)$. We assume throughout that all processes are defined on the finite time interval $[0, T]$. Suppose that this space is equipped with two stochastic processes: a n-dimensional Brownian motion (W_t) and a p-dimensional jump process $(N_t) = ((N^i_t), 1 \leq i \leq p)$ with $N^i_t = \mathbb{1}_{\tau_i \leq t}$, where $(\tau_i)_{1 \leq i \leq p}$ are p default times. We make the following assumptions on the default times:

Assumption 2.1. (i) The defaults do not appear simultaneously: $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$.

(ii) Each default can appear at any time: $\mathbb{P}(\tau_i > t) > 0$.

We denote by $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ the filtration generated by these processes, which is assumed to satisfy the usual conditions of right-continuity and completeness. We denote for each $i \in \{1, \dots, p\}$ by (M_t^i) the compensated martingale of the process (N_t^i) and by (Λ_t^i) its compensator in the filtration $\mathbb{F}$. We assume that the compensator (Λ_t^i) is absolutely continuous with respect to the Lebesgue measure, so that there exists a process (λ_t^i) such that $\Lambda_t^i = \int_0^t \lambda_s^i ds$. We can see that for each $i \in \{1, \dots, p\}$

$$M_t^i = N_t^i - \int_0^t \lambda_s^i ds \tag{2.1}$$

is an $\mathbb{F}$-martingale. We assume that the process (λ_t^i) is uniformly bounded. It should be noted that the construction of such process (N_t^i) is fairly standard; see, for example, Bielecki and Rutkowski (2004).

We introduce some sets used throughout the paper:

- $L^{1,+}$ is the set of positive $\mathbb{F}$-adapted càd-làg processes on $[0, T]$ such that $\mathbb{E}[Y_t] < \infty$ for any $t \in [0, T]$.
- $\mathcal{S}^2$ is the set of $\mathbb{F}$-adapted càd-làg processes on $[0, T]$ such that $\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^2] < \infty$.
- $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{F}$-predictable processes on $[0, T]$ such that

$$\mathbb{E}\Big[\int_0^T ||Z_t||^2 dt\Big] < \infty \quad (\text{resp. } \int_0^T ||Z_t||^2 dt < \infty, \ \mathbb{P} - a.s. \ ).$$

- $L^2(M)$ (resp. $L^1_{loc}(M)$) is the set of $\mathbb{F}$-predictable processes on $[0, T]$ such that

$$\mathbb{E}\Big[\int_0^T \lambda'_t |U_t|^2 dt\Big] < \infty \quad (\text{resp. } \int_0^T \lambda'_t |U_t| dt < \infty, \ \mathbb{P} - a.s. \ ).$$

We consider a financial market consisting of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at each date, and n risky assets with n-dimensional price process $S = (S^1, \dots, S^n)'$ evolving according to the following model

$$dS_t = \text{diag}(S_t)(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \quad 0 \le t \le T, \tag{2.2}$$

We shall make the following standing assumptions:

Assumption 2.2. – μ (resp. σ, β) is an $\mathbb{R}^n$ (resp. $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times p}$)-valued uniformly bounded predictable stochastic process.

- For all t, the $n \times n$ matrix σ_t is nonsingular, and we assume that $\sigma\sigma'$ is uniformly elliptic, i.e. $\epsilon I_n \le \sigma\sigma' \le K I_n$, $\mathbb{P} - a.s.$ for constants $0 < \epsilon < K$.
- We suppose that the process (S_t) is positive $\forall\, t \in [0, T]$, $\mathbb{P} - a.s.$

Remark 2.1. The assumption $\sigma\sigma'$ is uniformly elliptic implies that the predictable $\mathbb{R}^n$-valued process $\theta_t = \sigma'_t(\sigma_t\sigma'_t)^{-1}\mu_t$ is uniformly bounded.

An n-dimensional $\mathbb{F}$-predictable process $\pi = (\pi_t)_{0\le t\le T}$ is called trading strategy if $\int \frac{\pi^i_t X_t}{S^i_t} dS^i_t$ is well defined for each $i = 1, \ldots, n$. For $i = 1, \ldots, n$, the process π^i_t describes the part of the wealth invested in asset i. The number of shares of asset i is given by $\frac{\pi^i_t X_t}{S^i_t}$. The wealth process $X^{x,\pi}$ of a self-financing trading strategy π with initial capital x satisfies the equation

$$X^{x,\pi}_t = x \exp\Big(\int_0^t (\pi'_s \mu_s - \frac{||\pi'_s\sigma_s||^2}{2}) ds + \int_0^t \pi'_s \sigma_s dW_s \Big) \prod_{j=1}^{p} (1 + \pi'_{\tau^j} \beta^{.,j}_{\tau^j} N^j_t). \tag{2.3}$$

For a given initial time t and an initial capital x, the associated wealth process is denoted by $X^{t,x,\pi}_s$.

Now let $U : \mathbb{R} \to \mathbb{R}$ be a utility function. The optimization problem consists in maximizing the expected utility from terminal wealth over the class $\mathcal{A}(x)$ of admissible portfolios (which will be defined in the sequel). More precisely, we want to characterize the value function of this problem, which is defined by

$$V(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\Big[U(X^{x,\pi}_T)\Big], \tag{2.4}$$

and we also want to give the optimal strategy when it exists. We begin by the simple case when U is the logarithmic utility function, then we study the case of power utility function.

3 Logarithmic utility function

In this section, we specify the meaning of optimality for trading strategies by stipulating that the agent wants to maximize his expected utility from his terminal wealth $X^{x,\pi}_T$ with respect to the logarithmic utility function

$$U(x) = \log(x),\ x > 0.$$

Our goal is to solve the following optimization problem (we take $n = p = 1$ for the sake of simplicity)

$$V(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\big[\log(X^{x,\pi}_T)\big], \tag{3.1}$$

with $\mathcal{A}(x)$ the set of admissible portfolios defined by:

Definition 3.1. The set of admissible trading strategies $\mathcal{A}(x)$ consists of all $\mathbb{F}$-predictable processes (π_t) satisfying $\mathbb{E}\big[\int_0^T |\pi_t\sigma_t|^2 dt\big] + \mathbb{E}\big[\int_0^T \lambda_t |\log(1+\pi_t\beta_t)| dt\big] < \infty$, and such that $\pi_t\beta_t > -1,\ \mathbb{P}-a.s.$ for any $0 \le t \le \tau$.

We can see from (3.1) that $V(x) = \log(x) + V(1)$. Hence, we only study the case $x = 1$. And for the sake of brevity, we shall denote X^π_t instead of $X^{1,\pi}_t$ and $\mathcal{A}$ instead of $\mathcal{A}(1)$.

Remark 3.1. The condition $\pi_t\beta_t > -1,\ \mathbb{P}-a.s.$ for any $0 \le t \le \tau$ is stronger than $X^{x,\pi}_t > 0,\ \mathbb{P}-a.s.$ for any $0 \le t \le T$, but it is necessary to be able to write

$$\log(X^\pi_t) = \int_0^t (\pi_s\mu_s - \frac{|\pi_s\sigma_s|^2}{2}) ds + \int_0^t \pi_s\sigma_s dW_s + \int_0^t \log(1+\pi_s\beta_s)(dM_s + \lambda_s ds). \tag{3.2}$$

As in [21], we assume that $\sup_{\pi\in\mathcal{A}} \mathbb{E}[\log(X_T^\pi)] < \infty$.

We add the following assumption on the coefficients to be able to solve the optimization problem (3.1) directly:

Assumption 3.1. The process (β_t^{-1}) is uniformly bounded.

With this assumption, we get easily the value function $V(x)$ and the optimal strategy:

Theorem 3.1. *The solution of the optimization problem (3.1) is given by*

$$V(x) = \log(x) + \mathbb{E}\Big[\int_0^T \big(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t \log(1+\hat{\pi}_t\beta_t)\big)dt\Big],$$

with $\hat{\pi}$ the optimal trading strategy defined by

$$\hat{\pi}_t = \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\mu_t\beta_t+\sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{if } t<\tau \text{ and } \beta_t \neq 0, \\ \dfrac{\mu_t}{\sigma_t^2} \quad \text{if } t<\tau \text{ and } \beta_t = 0 \text{ or } t \geq \tau. \end{cases} \tag{3.3}$$

Proof. With (3.2) and Definition 3.1, we get the following expression for $V(1)$

$$V(1) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[\int_0^T \big(\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t \log(1+\pi_t\beta_t)\big)dt\Big],$$

which implies that

$$V(1) \leq \mathbb{E}\Big[\int_0^T \operatorname*{ess\,sup}_{\pi_t\beta_t>-1} \Big\{\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t \log(1+\pi_t\beta_t)\Big\}dt\Big]. \tag{3.4}$$

For any $t \in [0,T]$ and any $\omega \in \Omega$, we have

$$\operatorname*{ess\,sup}_{\pi_t\beta_t>-1} \Big\{\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t \log(1+\pi_t\beta_t)dt\Big\} = \hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t \log(1+\hat{\pi}_t\beta_t),$$

with $\hat{\pi}_t$ defined by (3.3). Then from inequality (3.4), we can see that

$$V(1) \leq \mathbb{E}\Big[\int_0^T \big(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t \log(1+\hat{\pi}_t\beta_t)\big)dt\Big].$$

It now is sufficient to show that the strategy $(\hat{\pi}_t)$ is admissible. It is clearly the case with Assumption 3.1. Thus the previous inequality is an equality

$$V(1) = \mathbb{E}\Big[\int_0^T \big(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t \log(1+\hat{\pi}_t\beta_t)\big)dt\Big],$$

and the strategy $(\hat{\pi}_t)$ is optimal. ☐

Remark 3.2. Assumption 3.1 can be reduced to

$$\mathbb{E}\Big[\int_0^T |\hat{\pi}_t\sigma_t|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \lambda_t |\log(1+\hat{\pi}_t\beta_t)|dt\Big] < \infty.$$

Remark 3.3. Recall that in the case without default, the optimal strategy is given by $\pi_t^0 = \mu_t/\sigma_t$. Thus, in the case of default, the optimal strategy can be written under the form

$$\hat{\pi}_t = \pi_t^0 - \epsilon_t,$$

where ϵ_t is an additional term given by

$$\epsilon_t = \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} + \dfrac{1}{2\beta_t} - \dfrac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{si } t < \tau \text{ et } \beta_t \neq 0, \\ 0 \quad \text{si } t < \tau \text{ et } \beta_t = 0 \text{ ou } t \geq \tau. \end{cases}$$

Note that if we assume that β_t is negative (resp. β_t is positive), i.e. the asset price (S_t) has a negative jump (resp. a positive jump) at default time τ, ϵ_t is positive (resp. ϵ_t is negative), i.e. the agent has to invest less (resp. more) in the risky asset than in the case of a market without default.

4 Power utility

In this section, we keep the notation of Section 3 and we shall study the case of the power utility function defined by

$$U(x) = x^\gamma, \quad x \geq 0, \quad \gamma \in (0,1).$$

In order to formulate the optimization problem we first define the set of admissible trading strategies.

Definition 4.1. The set of admissible strategies $\mathcal{A}(x)$ consists of all $\mathbb{F}$-predictable processes $\pi = (\pi_t)_{0\leq t\leq T}$ such that $\int_0^T ||\pi_t'\sigma_t||^2 dt + \int_0^T |\pi_t'\beta_t|\lambda_t dt < \infty$, $\mathbb{P}-a.s.$ and such that $\pi'_{\tau_j}\beta_{\tau_j}^{:,j} \geq -1$, $\mathbb{P}-a.s.$ for each $j \in \{1,\ldots,p\}$.

Remark 4.1. From expression (2.3), it is obvious that the condition $\pi'_{\tau_j}\beta_{\tau_j}^{:,j} \geq -1$, $\mathbb{P}-a.s.$ for each $j \in \{1,\ldots,p\}$ is equivalent to $X_t^{x,\pi} \geq 0$, $\mathbb{P}-a.s.$ for any $t \in [0,T]$.

The portfolio optimization problem consists in determining a predictable portfolio $\pi_t = (\pi_t^1,\ldots,\pi_t^n)'$ which attains the optimal value

$$V(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{E}[(X_T^{x,\pi})^\gamma]. \tag{4.1}$$

Problem (4.1) can be clearly written as $V(x) = x^\gamma V(1)$. Therefore, it is sufficient to study the case $x = 1$. As in [21], we assume that $\sup_{\pi\in\mathcal{A}(1)} \mathbb{E}[(X_T^{1,\pi})^\gamma] < \infty$. To solve the optimization problem, we give a dynamic extension of the initial problem. For any initial time $t \in [0,T]$, we define the value function $J(t)$ by the following random variable

$$J(t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t(1)} \mathbb{E}\Big[\big(X_T^{t,1,\pi}\big)^\gamma \Big| \mathcal{F}_t\Big],$$

with $\mathcal{A}_t(1)$ the set of $\mathbb{F}$-predictable processes $\pi = (\pi_s)_{t\leq s\leq T}$ such that $\int_t^T ||\pi_s'\sigma_s||^2 ds + \int_t^T |\pi_s'\beta_s|\lambda_s ds < \infty$, $\mathbb{P}-a.s.$ and such that $\pi'_{\tau_j}\beta_{\tau_j}^{:,j} \geq -1$, $\mathbb{P}-a.s.$ for each $j \in \{1,\ldots,p\}$.

For the sake of brevity, we shall denote X_s^π (resp. $X_s^{t,\pi}$) instead of $X_s^{1,\pi}$ (resp. $X_s^{t,1,\pi}$) and $\mathcal{A}$ (resp. $\mathcal{A}_t$) instead of $\mathcal{A}(1)$ (resp. $\mathcal{A}_t(1)$). And to simplify the notation, we suppose in the remainder of this section that $n = p = 1$, we give the generalization of the results in Part 4.3.

In the sequel, we will use the martingale representation theorem (see Jeanblanc *et al.* (2009)) to characterize the value function $J(t)$:

Lemma 4.1. *Any $(\mathbb{P}, \mathbb{F})$-local martingale has the representation*

$$m_t = m_0 + \int_0^t a_s dW_s + \int_0^t b_s dM_s, \ \forall\, t \in [0, T], \ \mathbb{P} - a.s., \tag{4.2}$$

where $a \in L^2_{loc}(W)$ and $b \in L^1_{loc}(M)$. If (m_t) is a square integrable martingale, each term on the right-hand side of the representation (5.11) is square integrable.

4.1 Optimization over bounded strategies

Let us fix $k \in \mathbb{N}$. Before studying the value function $J(t)$, we study the value functions $(J^k(t))_{k\in\mathbb{N}}$ defined by

$$J^k(t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t^k} \mathbb{E}\big[(X_T^{t,\pi})^\gamma \big| \mathcal{F}_t\big], \ \forall\, t \in [0, T], \ \mathbb{P} - a.s., \tag{4.3}$$

where $\mathcal{A}_t^k$ is the set of strategies of $\mathcal{A}_t$ uniformly bounded by k. This means that the parts of the wealth invested in the assets have to be bounded by a constant k (which makes sense in finance, because the ratio of the amount of money invested or borrowed to the wealth must be bounded according to the financial legislation). We want to characterize the value function $J^k(t)$ by a BSDE. For that we introduce for any $\pi \in \mathcal{A}^k$ the càd-làg process J_t^π defined for all $t \in [0, T]$ by

$$J_t^\pi = \mathbb{E}\big[(X_T^{t,\pi})^\gamma \big| \mathcal{F}_t\big].$$

The family $((J_t^\pi))_{\pi\in\mathcal{A}^k}$ is uniformly bounded:

Lemma 4.2. *The process (J_t^π) is uniformly bounded by a constant independent of π.*

Proof. Fix $t \in [0, T]$. We have

$$J_t^\pi = \mathbb{E}\Big[\exp\Big(\gamma \int_t^T (\mu_s\pi_s - \frac{|\sigma_s\pi_s|^2}{2})ds + \int_t^T \gamma\sigma_s\pi_s dW_s\Big)(1 + \pi_\tau\beta_\tau \mathbb{1}_{t<\tau\leq T})^\gamma \Big| \mathcal{F}_t\Big],$$

since the coefficients μ_t, σ_t and β_t are supposed to be bounded, we have

$$J_t^\pi \leq (1 + k\,|\beta|_\infty)^\gamma \exp\Big((\gamma\, k\,|\mu|_\infty + \gamma^2 \frac{(k\,|\sigma|_\infty)^2}{2})\,T\Big).$$

□

Classically, for any $\pi \in \mathcal{A}^k$ the process (J_t^π) can be shown to be the solution of a linear BSDE. More precisely, there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$, such that $(J_t^\pi, Z_t^\pi, U_t^\pi)$ is the unique solution in $\mathcal{S}^2 \times L^2(W) \times L^2(M)$ of the linear BSDE with bounded coefficients

$$\begin{cases} -dJ_t^\pi &= -Z_t^\pi dW_t - U_t^\pi dM_t + \Big\{\gamma\pi_t(\mu_t J_t^\pi + \sigma_t Z_t^\pi) + \dfrac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 J_t^\pi \\ &\qquad + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t^\pi + U_t^\pi)\Big\}dt, \\ J_T^\pi &= 1. \end{cases} \tag{4.4}$$

Using the fact that for any $t \in [0,T]$, $J^k(t) = \operatorname{ess\,sup}_{\pi \in \mathcal{A}_t^k} J_t^\pi$, we derive that $(J^k(t))$ corresponds to the solution of a BSDE, whose generator is the essential supremum over π of the generators of $(J_t^\pi)_{\pi\in\mathcal{A}^k}$. More precisely,

Proposition 4.1. *The following properties hold:*

- *Let (Y_t, Z_t, U_t) be the solution in $\mathcal{S}^2 \times L^2(W) \times L^2(M)$ of the following Lipschitz BSDE*

$$\begin{cases} -dY_t = -Z_t dW_t - U_t dM_t + \operatorname*{ess\,sup}_{\pi\in\mathcal{A}^k}\Big\{\gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 Y_t \\ \qquad + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(Y_t + U_t)\Big\}dt, \\ Y_T = 1. \end{cases} \tag{4.5}$$

  Then, for any $t \in [0,T]$, $J^k(t) = Y_t$, $\mathbb{P} - a.s.$

- *There exists a unique optimal strategy for $J^k(0) = \sup_{\pi\in\mathcal{A}^k}\mathbb{E}[(X_T^\pi)^\gamma]$.*

- *A strategy $\hat\pi \in \mathcal{A}^k$ is optimal for $J^k(0)$ if and only if it attains the essential supremum of the generator in (4.5) $dt \otimes d\mathbb{P} - a.e.$*

Proof. Since for any $\pi \in \mathcal{A}^k$ there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$ such that $(J_t^\pi, Z_t^\pi, U_t^\pi)$ is the solution of the BSDE

$$-dJ_t^\pi = f^\pi(t, J_t^\pi, Z_t^\pi, U_t^\pi)dt - Z_t^\pi dW_t - U_t^\pi dM_t \;;\; J_T^\pi = 1,$$

with $f^\pi(s,y,z,u) = \frac{\gamma(\gamma-1)}{2}\pi_s^2\sigma_s^2 y + \gamma\pi_s(\mu_s y + \sigma_s z) + \lambda_s\big((1+\pi_s\beta_s)^\gamma - 1\big)(y+u)$. Let us introduce the generator f which satisfies $ds \otimes d\mathbb{P} - a.e.$

$$f(s,y,z,u) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}^k} f^\pi(s,y,z,u).$$

Note that f is Lipschitz, since the supremum of affine functions, whose coefficients are bounded by a constant $c > 0$, is Lipschitz with Lipschitz constant c. Hence, by results of [36], the BSDE with Lipschitz generator f

$$-dY_t = f(y, Y_t, Z_t, U_t)dt - Z_t dW_t - U_t dM_t \;;\; Y_T = 1$$

admits a unique solution denoted by (Y_t, Z_t, U_t).
By the comparison theorem in case of jumps (see for example Royer (2006)) $Y_t \geq J_t^\pi$, $\forall\, t \in [0,T]$, $\mathbb{P} - a.s.$ As this inequality is satisfied for any $\pi \in \mathcal{A}^k$, it is obvious that

$Y_t \geq \operatorname{ess\,sup}_{\pi\in\mathcal{A}^k} J_t^\pi$, $\mathbb{P}-a.s.$ Also, by applying a predictable selection theorem, one can easily show that there exists $\hat{\pi} \in \mathcal{A}^k$ such that for any $t \in [0,T]$, we have

$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}^k}\left\{\gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 Y_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(Y_t+U_t)\right\}$$
$$= \gamma\hat{\pi}_t(\mu_t Y_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\hat{\pi}_t^2\sigma_t^2 Y_t + \lambda_t((1+\hat{\pi}_t\beta_t)^\gamma - 1)(Y_t+U_t).$$

Thus (Y_t, Z_t, U_t) is a solution of BSDE (4.4) associated with $\hat{\pi}$. Therefore by uniqueness of the solution of BSDE (4.4), we have $Y_t = J_t^{\hat{\pi}}$ and thus $Y_t = \operatorname{ess\,sup}_{\pi\in\mathcal{A}_t^k} J_t^\pi = J_t^{\hat{\pi}}$, $\forall\, t \in [0,T]$, $\mathbb{P}-a.s.$
The uniqueness of the optimal strategy is due to the strict concavity of the function $x \mapsto x^\gamma$. □

4.2 General case

In this part, we characterize the value function $J(t)$ by a BSDE, but the general case is more complicated than the case with bounded strategies and it needs more technical tools. Note that the random variable $J(t)$ is defined uniquely only up to $\mathbb{P}$-almost sure equivalent and that the process $(J(t))$ is adapted but not necessarily progressive. Using dynamic control technics, we derive the following characterization of the value function:

Proposition 4.2. *$(J(t))$ is the smallest $\mathbb{F}$-adapted process such that $((X_t^\pi)^\gamma J(t))$ is a supermartingale for any $\pi \in \mathcal{A}$ with $J(T) = 1$. More precisely, if $(\bar{J}_t)$ is an $\mathbb{F}$-adapted process such that $((X_t^\pi)^\gamma(\bar{J}_t))$ is a supermartingale for any $\pi \in \mathcal{A}$ with $\bar{J}_T = 1$, then for any $t \in [0,T]$, we have $J(t) \leq \bar{J}_t$, $\mathbb{P}-a.s.$*

From [21], there exists an optimal strategy $\hat{\pi} \in \mathcal{A}$ such that $J(0) = \mathbb{E}[(X_T^{\hat{\pi}})^\gamma]$. And with the dynamic programming principle, we have the following optimality criterion:

Proposition 4.3. *The following assertions are equivalent:*

i) $\hat{\pi}$ is an optimal strategy, that is $\mathbb{E}[(X_T^{\hat{\pi}})^\gamma] = \sup_{\pi\in\mathcal{A}} \mathbb{E}[(X_T^\pi)^\gamma]$.

ii) The process $((X_t^{\hat{\pi}})^\gamma J(t))$ is a martingale.

The proof of these propositions is given in Appendix A.

By Proposition 4.2, $(J(t))$ is a supermartingale. Hence $\mathbb{E}[J(t)] \leq J(0) < \infty$ that for any $t \geq 0$.

Proposition 4.4. *There exists a càd-làg modification of $J(t)$ which is denoted by (J_t).*

Proof. By Proposition 4.3, we know that $J(t) = \mathbb{E}[(X_T^{\hat{\pi}})^\gamma|\mathcal{F}_t]/(X_t^{\hat{\pi}})^\gamma$, $\mathbb{P}-a.s.$ Which implies the desired result. □

This càd-làg process is characterized by a BSDE. More precisely,

Theorem 4.1. *There exist $Z \in L^2_{loc}(W)$ and $U \in L^1_{loc}(M)$ such that the process (J_t, Z_t, U_t) is the minimal solution*[1] *in $L^{1,+} \times L^2_{loc}(W) \times L^1_{loc}(M)$ of the following BSDE*

$$\begin{cases} -\,dJ_t &= -\,Z_t dW_t - U_t dM_t + \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \Big\{ \gamma \pi_t (\mu_t J_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 J_t \\ & \qquad + \lambda_t ((1 + \pi_t \beta_t)^\gamma - 1)(J_t + U_t) \Big\} dt, \\ J_T &= 1. \end{cases} \tag{4.6}$$

If a strategy $\hat{\pi} \in \mathcal{A}$ is optimal for $J_0 = \sup_{\pi \in \mathcal{A}} \mathbb{E}[(X_T^\pi)^\gamma]$ then $\hat{\pi}$ attains the essential supremum in the generator of BSDE (4.6) $dt \otimes d\mathbb{P}$ a.s.

The proof of this theorem is postponed in Appendix B.

There exists another characterization of the value function J_t as the limit of processes $(J_t^k)_{k \in \mathbb{N}}$ as k tends to $+\infty$, with (J_t^k) is the value function in the case where the strategies are bounded by k:

Theorem 4.2. *For any $t \in [0, T]$, we have*

$$J_t = \lim_{k \to \infty} \uparrow J^k(t), \ \mathbb{P} - a.s.$$

The proof of this theorem is given in Appendix C.

This allows to approximate the value function J_t by numerical computation, since the value functions (J_t^k) are the solution of Lipschitz BSDEs and the results of Bouchard and Elie (2008) can be applied.

4.3 Several default times and several assets

In this part, we only give the BSDEs in the case of several default times and several assets. The proofs are not given, but they are identical to the proofs for $n = p = 1$.

– BSDE (4.5) is written

$$\begin{cases} -\,dY_t &= -\,Z_t' dW_t - U_t' dM_t + \operatorname*{ess\,sup}_{\pi \in \mathcal{A}^k} \Big\{ \gamma \pi_t' (Y_t \mu_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2} ||\pi_t' \sigma_t||^2 Y_t \\ & \qquad + [(\mathbb{1} + \pi_t' \beta_t)^\gamma - \mathbb{1}](Y_t \lambda_t + \lambda_t * U_t)) \Big\} dt, \\ Y_T &= 1, \end{cases}$$

– and BSDE (4.6) is written

$$\begin{cases} -\,dY_t &= -\,Z_t' dW_t - U_t' dM_t + \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \Big\{ \gamma \pi_t' (Y_t \mu_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2} ||\pi_t' \sigma_t||^2 Y_t \\ & \qquad + [(\mathbb{1} + \pi_t' \beta_t)^\gamma - \mathbb{1}](Y_t \lambda_t + \lambda_t * U_t)) \Big\} dt, \\ Y_T &= 1. \end{cases}$$

[1] That is for any solution $(\bar{J}_t, \bar{Z}_t, \bar{U}_t)$ of BSDE (4.6) in $L^{1,+} \times L^2_{loc}(W) \times L^1_{loc}(M)$, we have $J_t \le \bar{J}_t$, $\forall \, t \in [0, T]$, $\mathbb{P} - a.s.$

5 The partial information case

The difference between this section and the previous sections is that here we require the investment process to be adapted to the natural filtration generated by the price process and the default time process. This requirement means that the only available information for agents in this economy at a certain time are the prices of the financial assets up to that time and the default times. The underlying Brownian motion, the drift process and the compensator process in the system of equations for the asset prices are not directly observable.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability triplet and $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ a filtration in $\mathcal{F}$ satisfying the usual conditions (augmented and right continuous). Suppose that this space is equipped with (W_t) and (N_t) as in Section 2. We also assume there are one risk-free asset and n risky assets on the market. As in Section 2, we assume that the price process (S_t) evolves according to the following model

$$dS_t = \text{diag}(S_t)(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \quad 0 \leq t \leq T, \tag{5.1}$$

moreover we assume that $\sigma_t = \sigma(t, S_{t-}, t \wedge \tau)$ and $\beta_t = \beta(t, S_{t-}, t \wedge \tau)$, with $t \wedge \tau = (t \wedge \tau_1, \ldots, t \wedge \tau_p)'$. The known functions $\sigma(t, s, h)$ and $\beta(t, s, h)$ are measurable mappings from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^p$ into $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{n \times p}$. We make the hypotheses of Assumption 2.2 and we add the following assumption:

Assumption 5.1. The functions $s\sigma(t, s, h)$ and $s\beta(t, s, h)$ are Lipschitz in $s \in \mathbb{R}^n$, uniformly in $t \in [0, T]$ and $h \in \mathbb{R}^p$.

We now consider an agent in this market who can observe neither the Brownian motion (W_t) nor the drift (μ_t) and the process (λ_t), but only the asset price process (S_t) and the default times $(\tau_i)_{1 \leq i \leq p}$. We shall denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the $\mathbb{P}$-filtration augmented by the price process (S_t) and the default process (N_t). The trading strategies are defined as in Section 2, but we add the condition that they are $\mathbb{G}$-predictable. We now want to solve the problem of maximization of expected utility from terminal wealth for logarithmic, power and exponential utility functions. It is not possible to use directly the results of the full information case because we do not know the Brownian motion, the drift and the compensator. Moreover there exists no martingale representation theorem for the $\mathbb{G}$-martingale. Thus before to study the problem of maximization, we begin by an operation of filtering as in Pham and Quenez (2001).

5.1 Filtering

Let us define the process $\rho_t = \sigma_t^{-1} \mu_t$, assumed to satisfy the integrability condition

$$\int_0^T ||\rho_t||^2 dt < \infty, \ \mathbb{P} - a.s. \tag{5.2}$$

Consider the positive local martingale defined by $L_0 = 1$ and $dL_t = -L_t \rho_t' dW_t$. It is explicitly given by

$$L_t = \exp\Big(-\int_0^t \rho_s' dW_s - \frac{1}{2}\int_0^t ||\rho_s||^2 ds\Big). \tag{5.3}$$

We shall make the usual standing assumption on filtering theory:

Assumption 5.2. The process (L_t) is a martingale, i.e. $\mathbb{E}[L_T] = 1$.

Under this last assumption, one can define a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ characterized by

$$\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\Big|_{\mathcal{F}_t} = L_t,\ \ 0 \le t \le T. \tag{5.4}$$

By Girsanov's theorem, the n-dimensional process defined by

$$\tilde{W}_t = W_t + \int_0^t \rho_s ds \tag{5.5}$$

is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion and the compensated martingale (M_t) is still a $(\mathbb{Q}, \mathbb{F})$-martingale. The dynamic of (S_t) under $\mathbb{Q}$ is given by

$$dS_t = \mathrm{diag}(S_t)(\sigma(t, S_{t^-}, t \wedge \tau)d\tilde{W}_t + \beta(t, S_{t^-}, t \wedge \tau)dN_t). \tag{5.6}$$

We begin by proving a lemma which will be of paramount importance in the sequel:

Proposition 5.1. *Under Assumptions 2.2 and 5.2, the filtration $\mathbb{G}$ is the augmented filtration of $(\tilde{W}, N)$.*

Proof. Let $\mathbb{F}^{\tilde{W},N}$ be the augmented filtration of $(\tilde{W}, N)$. From (5.6), we have

$$\tilde{W}_t = \int_0^t \sigma(s, S_{s^-}, s \wedge \tau)^{-1}\mathrm{diag}(S_t^{-1})dS_s - \int_0^t \sigma(s, S_{s^-}, s \wedge \tau)^{-1}\beta(s, S_{s^-}, s \wedge \tau)dN_s,$$

for all $t \in [0, T]$, which implies that $(\tilde{W}_t)$ is $\mathbb{G}$-adapted and $\mathbb{F}^{\tilde{W},N} \subset \mathbb{G}$. Conversely, under the assumptions on the coefficients, by a classical result of stochastic differential equation (see [31], Theorem V 3.7), the unique solution of (5.6) is $\mathbb{F}^{\tilde{W},N}$-adapted, hence $\mathbb{G} \subset \mathbb{F}^{\tilde{W},N}$ and finally $\mathbb{G} = \mathbb{F}^{\tilde{W},N}$. □

We now make the standing assumption on the risk premia process (ρ_t):

Assumption 5.3. For all $t \in [0, T]$, $\mathbb{E}|\rho_t| < \infty$.

Since the processes (ρ_t) and (λ_t) are not $\mathbb{G}$-predictable, it is natural to introduce the $\mathbb{G}$-conditional law of the random variables ρ_t and λ_t, say

$$\tilde{\lambda}_t = \mathbb{E}\big[\lambda_t \big| \mathcal{G}_t\big] \text{ and } \tilde{\rho}_t = \mathbb{E}\big[\rho_t \big| \mathcal{G}_t\big].$$

Consider the couple of processes $(\bar{W}_t, \bar{M}_t)$ defined by

$$\begin{cases} \bar{W}_t = \tilde{W}_t - \displaystyle\int_0^t \tilde{\rho}_s ds, \\ \bar{M}_t = N_t - \displaystyle\int_0^t \tilde{\lambda}_s ds. \end{cases} \tag{5.7}$$

These are the so-called innovation processes of filtering theory. By classical results in filtering theory (see for example [28], Proposition 2.27), we have:

Proposition 5.2. *The process* $(\bar{M}_t)$ *is a* $(\mathbb{Q}, \mathbb{G})$*-martingale.*

Proof. Since the process (N_t) and the intensity $(\tilde{\lambda}_t)$ are $\mathbb{G}$-adapted, the process $(\bar{M}_t)$ is $\mathbb{G}$-adapted. We can write from (2.1)

$$\bar{M}_t = M_t + \int_0^t (\lambda_s - \tilde{\lambda}_s) ds.$$

By the law of iterated conditional expectation, it is easy to check that $(\bar{M}_t)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale. □

Remark 5.1. From Proposition 5.1 and (5.7), the filtration $\mathbb{G}$ is equal to the augmented filtration of $(\tilde{W}, \bar{M})$, since $[\bar{M}]_t = N_t$.

We have also the following property about the process $(\bar{W}_t)$:

Proposition 5.3. *Under Assumptions 5.2 and 5.3, the process* $(\bar{W}_t)$ *is a* $(\mathbb{P}, \mathbb{G})$*-Brownian motion.*

Proof. We can write with (5.5)

$$\bar{W}_t = W_t + \int_0^t \sigma(s, S_s, s \wedge \tau)^{-1} (\mu_s - \tilde{\mu}_s) ds, \tag{5.8}$$

where $\tilde{\mu}_t = \mathbb{E}\big[\mu_t \big| \mathcal{G}_t\big]$. By Proposition 5.1, $\bar{W}$ is $\mathbb{G}$-adapted. Moreover, we have $[\bar{W}^i, \bar{W}^j]_t = \delta_{ij} t$ for all $t \in [0, T]$, where δ_{ij} is the Kronecker notation. By the law of iterated conditional expectation, it is easy to check that $(\bar{W}_t)$ is a $\mathbb{G}$-martingale. We then conclude by Levy's characterization theorem on Brownian motion (see, e.g., Theorem 3.3.16 in [18]. □

Denote by (Λ_t), the $(\mathbb{Q}, \mathbb{F})$-martingale given by $\Lambda_t = 1/L_t$. We then have

$$\frac{d\,\mathbb{P}}{d\,\mathbb{Q}}\Big|_{\mathcal{F}_t} = \Lambda_t,\ \ 0 \le t \le T.$$

Let $(\tilde{\Lambda}_t)$ be the $(\mathbb{Q}, \mathbb{G})$-martingale given by $\tilde{\Lambda}_t = \mathbb{E}_{\mathbb{Q}}\big[\Lambda_t \big| \mathcal{G}_t\big]$. Recall the classical proposition (see for example [23] or [30]), which gives the expression of $(\tilde{\Lambda}_t)$:

Lemma 5.1. *Under Assumptions 5.2 and 5.3, we have*

$$\tilde{\Lambda}_t = \exp\Big(\int_0^t \tilde{\rho}_s' d\tilde{W}_s - \frac{1}{2} \int_0^t ||\tilde{\rho}_s||^2 ds \Big). \tag{5.9}$$

Proposition 5.4. *The process* $(\bar{M}_t)$ *is a* $(\mathbb{P}, \mathbb{G})$*-martingale.*

Proof. Since $\frac{d\mathbb{P}}{d\mathbb{Q}}\big|_{\mathcal{G}_t} = \tilde{\Lambda}_t$, we can apply Girsanov's theorem and we get that the process $(\bar{M}_t)$ is a $(\mathbb{P}, \mathbb{G})$-martingale. □

By means of innovation processes, we can describe from (5.1) and (5.8) the dynamics of the partially observed default model within a framework of full observation model

$$\begin{cases} dS_t = \tilde{\mu}_t dt + \sigma(t, S_{t^-}, t \wedge \tau) d\bar{W}_t + \beta(t, S_{t^-}, t \wedge \tau) dN_t, \\ d\bar{M}_t = dN_t - \tilde{\lambda}_t dt, \end{cases} \tag{5.10}$$

where $(\tilde{\mu}_t)$ and $(\tilde{\lambda}_t)$ are $\mathbb{G}$-predictable processes.
Hence, the operations of filtering and control can be put in sequence and thus separated.

5.2 Optimization problem for the logarithmic and power utility functions

To apply the results of Section 4 and of Lim and Quenez (2009), it is sufficient to have a martingale representation theorem for $(\mathbb{P},\mathbb{G})$-martingale with respect to $\bar{W}$ and $\bar{M}$. Notice it cannot be directly derived from the usual martingale representation theorem since $\mathbb{G}$ is not equal to the filtration generated by $\bar{W}$ and $\bar{M}$.

Lemma 5.2. *Any $(\mathbb{P},\mathbb{G})$-local martingale has the representation*

$$m_t = m_0 + \int_0^t a_s' d\bar{W}_s + \int_0^t b_s' d\bar{M}_s,\ \forall\, t\in[0,T],\ \mathbb{P}-a.s., \tag{5.11}$$

where $a\in L^2_{loc}(\bar{W})$ and $b\in L^1_{loc}(\bar{M})$. If (m_t) is a square integrable martingale, each term on the right-hand side of the representation (5.11) is square integrable.

The proof of this lemma is postponed in Appendix D.

It is now possible to apply the previous results because the price process evolves according to the equation

$$\begin{cases} dS_t = \operatorname{diag}(S_t)(\tilde{\mu}_t dt + \sigma_t d\bar{W}_t + \beta_t dN_t),\\ d\bar{M}_t = dN_t - \tilde{\lambda}_t dt, \end{cases}$$

where each coefficient is $\mathbb{G}$-predictable, and there exists a martingale representation theorem for $(\mathbb{P},\mathbb{G})$-martingale. We get the following characterization for the value functions and the optimal strategies when they exist.

For the logarithmic utility function, we have:

Theorem 5.1. *We assume that β_t^{-1} is uniformly bounded. Then, the solution of the optimization problem for the logarithmic utility function is given by*

$$V(x) = \log(x) + \mathbb{E}\Big[\int_0^T \big(\hat{\pi}_t\tilde{\mu}_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \tilde{\lambda}_t\log(1+\hat{\pi}_t\beta_t)\big)dt\Big],$$

with $\hat{\pi}$ the optimal trading strategy defined by

$$\hat{\pi}_t = \begin{cases} \dfrac{\tilde{\mu}_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\tilde{\mu}_t\beta_t+\sigma_t^2)^2 + 4\tilde{\lambda}_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{if } t<\tau \text{ and } \beta_t\neq 0,\\ \dfrac{\tilde{\mu}_t}{\sigma_t^2} & \text{if } t<\tau \text{ and } \beta_t = 0 \text{ or } t\geq\tau. \end{cases}$$

Therefore, we can see that the optimal portfolio in the case of partial information can be formally derived from the full information case by replacing the unobservable coefficients μ_t and λ_t by theirs estimates $\tilde{\mu}_t$ and $\tilde{\lambda}_t$.

For the power utility function, we have:

Theorem 5.2. – *Let $(\bar{Y}_t,\bar{Z}_t,\bar{U}_t)$ the minimal solution in $L^{1,+}\times L^2_{loc}(\bar{W})\times L^1_{loc}(\bar{M})$ of the BSDE (4.6) with $(W,\ M,\ \mu,\ \lambda)$ replaced by $(\bar{W},\ \bar{M},\ \tilde{\mu},\ \tilde{\lambda})$, then*

$$\bar{Y}_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t}\mathbb{E}[(X_T^{t,\pi})^\gamma|\mathcal{G}_t],\ \mathbb{P}-a.s.$$

– *If a strategy $\hat{\pi} \in \mathcal{A}$ is optimal for $J_0 = \sup_{\pi\in\mathcal{A}} \mathbb{E}[(X_T^\pi)^\gamma]$ then $\hat{\pi}$ attains the essential supremum in the generator of BSDE (4.6) $dt \otimes d\mathbb{P}$ a.s.*

– *Moreover the process $(\bar{Y}_t)$ is the nondecreasing limit of the process $(\bar{Y}_t^k)_{k\in\mathbb{N}}$, where $(\bar{Y}_t^k, \bar{Z}_t^k, \bar{U}_t^k)$ is the solution in $\mathcal{S}^2\times L^2(\bar{W})\times L^2(\bar{M})$ of the BSDE (4.5) with $(W,\, M,\, \mu,\, \lambda)$ replaced by $(\bar{W},\, \bar{M},\, \tilde{\mu},\, \tilde{\lambda})$.*

5.3 Optimization problem for the exponential utility function and indifference pricing

We can also apply the results of Lim and Quenez (2009) for the exponential utility function. In this case, we assume that the agent faces some liability, which is modeled by a random variable ζ (for example, ζ may be a contingent claim written on some default events affecting the prices of the underlying assets). We suppose that ζ is a non-negative $\mathcal{G}_T$-adapted process (note that all the results still hold under the assumption that ζ is only lower bounded). Without loss of generality we can use a somewhat different notion of trading strategy: ϕ_t corresponds to the amount of money invested in the assets. The number of shares i is ϕ_t^i/S_t^i. With this notation, under the assumption that the trading strategy is self-financing, the wealth process $(X_t^{x,\phi})$ associated with a trading strategy ϕ and an initial capital x is equal to

$$X_t^{x,\phi} = x + \int_0^t \phi_s' \tilde{\mu}_s ds + \int_0^t \phi_s' \sigma_s dW_s + \int_0^t \phi_s' \beta_s dN_s.$$

Our goal is to solve the optimization problem for an agent who buys a contingent claim ζ

$$V(x,\zeta) = \sup_{\phi\in\mathcal{A}(x)} \mathbb{E}\Big[-\exp\Big(-\gamma\big(X_T^{x,\phi} + \zeta\big)\Big)\Big] = \exp(-\gamma x)\mathcal{V}(0,\zeta), \tag{5.12}$$

where $\mathcal{A}(x)$ is defined by:

Definition 5.1. The set of admissible trading strategies $\mathcal{A}(x)$ consists of all $\mathbb{G}$-predictable processes $\phi = (\phi_t)_{0\le t\le T}$, which satisfy $\int_0^T ||\phi_t'\sigma_t||^2 ds + \int_0^T |\phi_t'\beta_t|^2\tilde{\lambda}_t dt < \infty$, $\mathbb{P}-a.s.$ and such that for any ϕ fixed and any $t \in [0,T]$, there exists a constant $K_{t,\pi}$ such that for any $s \in [t,T]$, we have $X_s^{t,\pi} \ge K_{t,\pi}$, $\mathbb{P}-a.s.$

To solve this problem, it is sufficient to study the case $x = 0$. For that we give a dynamic extension of the initial problem as in Section 4. For any initial time $t \in [0,T]$, we define the value function $J^\zeta(t)$ by the following random variable

$$J^\zeta(t) = \operatorname*{ess\,inf}_{\phi\in\mathcal{A}_t} \mathbb{E}\Big[\exp\Big(-\gamma\big(X_T^{t,0,\phi} + \zeta\big)\Big)\Big|\mathcal{G}_t\Big],$$

with $\mathcal{A}_t$ is the admissible portfolio strategies set defined by:

Definition 5.2. The set of admissible trading strategies $\mathcal{A}_t$ consists of all $\mathbb{G}$-predictable processes $\phi = (\phi_s)_{t\le s\le T}$, which satisfy $\int_t^T ||\phi_s'\sigma_s||^2 ds + \int_t^T |\phi_s'\beta_s|^2\tilde{\lambda}_s ds < \infty$, $\mathbb{P}-a.s.$ and such that for any ϕ fixed and any $s \in [t,T]$, there exists a constant $K_{s,\pi}$ such that for any $u \in [s,T]$, we have $X_u^{s,\pi} \ge K_{s,\pi}$, $\mathbb{P}-a.s.$

We introduce the two following sets:

- $\mathcal{S}^{+,\infty}$ is the set of positive $\mathbb{G}$-adapted $\mathbb{P}$-essentially bounded càd-làg processes on $[0,T]$.
- $\mathcal{A}^2$ is the set of the increasing adapted càd-làg processes K such that $K_0 = 0$ and $\mathbb{E}|K_T|^2 < \infty$.

By applying the results of the companion paper Lim and Quenez (2009), we get the following characterizations of the value function:

Theorem 5.3. – *Let* $(\bar{Y}_t, \bar{Z}_t, \bar{U}_t, \bar{K}_t)$ *the maximal solution*[2] *in* $\mathcal{S}^{+,\infty} \times L^2(\bar{W}) \times L^2(\bar{M}) \times \mathcal{A}^2$ *of*

$$\begin{cases} -d\bar{Y}_t = -\bar{Z}_t' d\bar{W}_t - \bar{U}_t' d\bar{M}_t - d\bar{K}_t + \operatorname*{ess\,inf}_{\phi\in\mathcal{A}} \Big\{ \dfrac{\gamma^2}{2} ||\phi_t'\sigma_t||^2 \bar{Y}_t - \gamma\phi_t'(\bar{Y}_t\tilde{\mu}_t + \sigma_t\bar{Z}_t) \\ \qquad\qquad - \big(1 - e^{-\gamma\phi_t'\beta_t}\big)(\bar{Y}_t\tilde{\lambda}_t + \tilde{\lambda}_t * \bar{U}_t)\Big\} dt, \\ \bar{Y}_T = \exp(-\gamma\zeta). \end{cases} \tag{5.13}$$

then $\bar{Y}_t = \bar{J}^\zeta(t),\ \mathbb{P}-a.s.$

- $\bar{J}^\zeta(t) = \lim_{n\to\infty} \downarrow \bar{J}^{\zeta,k}(t)$, *with* $\bar{J}^{\zeta,k}(t) = \operatorname{ess\,inf}_{\phi\in\mathcal{A}_t^k} \mathbb{E}[\exp(-\gamma(X_T^{t,0,\phi} + \zeta))|\mathcal{G}_t]$ *and* $\mathcal{A}_t^k$ *is the set of strategies of* $\mathcal{A}_t$ *uniformly bounded by* k.
- *Let* $(\bar{Y}_t^k, \bar{Z}_t^k, \bar{U}_t^k)$ *is the unique solution in* $\mathcal{S}^2 \times L^2(\bar{W}) \times L^2(\bar{M})$ *of the following BSDE*

$$\begin{cases} -d\bar{Y}_t^k = -\bar{Z}_t^{k'} d\bar{W}_t - \bar{U}_t^{k'} d\bar{M}_t + \operatorname*{ess\,inf}_{\phi\in\mathcal{A}^k} \Big\{ \dfrac{\gamma^2}{2} ||\phi_t'\sigma_t||^2 \bar{Y}_t^k - \gamma\phi_t'(\bar{Y}_t^k\tilde{\mu}_t + \sigma_t\bar{Z}_t^k) \\ \qquad\qquad - (1 - e^{-\gamma\phi_t'\beta_t})(\bar{Y}_t^k\tilde{\lambda}_t + \tilde{\lambda}_t * \bar{U}_t^k)\Big\} dt, \\ \bar{Y}_T^k = \exp(-\gamma\zeta), \end{cases} \tag{5.14}$$

then $\bar{Y}_t^k = \bar{J}^{\zeta,k}(t),\ \mathbb{P}-a.s.$

We can now define the indifference pricing of the contingent claim ζ. The Hodges approach to pricing of unhedgeable claims is a utility-based approach and can be summarized as follows: the issue at hand is to assess the value of some (defaultable) claim ζ as seen from the perspective of an investor who optimizes his behavior relative to some utility function, in our case we use the exponential utility function. The investor has two choices:

- he only invests in the risk-free asset and in the risky assets, in this case the associated optimization problem is

$$V(x,0) = \sup_{\phi\in\mathcal{A}(x)} \mathbb{E}[-\exp(-\gamma(X_T^{x,\phi}))],$$

[2] That is for any solution $(\bar{J}_t, \bar{Z}_t, \bar{U}_t, \bar{K}_t)$ of BSDE (5.13) in $\mathcal{S}^{+,\infty} \times L^2(\bar{W}) \times L^2(\bar{M}) \times \mathcal{A}^2$, we have $\bar{J}_t \le J_t,\ \forall\, t \in [0,T],\ \mathbb{P}-a.s.$

– he also invests in the contingent claim, whose price is $\bar{p}$ at 0, in this case the associated optimization problem is

$$V(x-\bar{p},\zeta) = \sup_{\phi\in\mathcal{A}(x)} \mathbb{E}[-\exp(-\gamma(X_T^{x-\bar{p},\phi}+\zeta))].$$

Definition 5.3. For a given initial capital x, the Hodges buying price of a defaultable claim ζ is the price $\bar{p}$ such that the investor's value functions are indifferent between holding and not holding the contingent claim, i.e.

$$V(x,0) = V(x-\bar{p},\zeta).$$

The Hodges price $\bar{p}$ can be derived explicitly by applying the results of Theorem 5.3. If the agent buys the contingent claim at the price $\bar{p}$ and invests the rest of his wealth in the risk-free asset and in the risky assets, the value function is equal to

$$V(x-\bar{p},\zeta) = -\exp(-\gamma(x-\bar{p}))\bar{J}^{\zeta}(0).$$

If he invests all his wealth in the risk-free asset and in the risky assets, the value function is equal to

$$V(x,0) = -\exp(-\gamma x)\bar{J}^{0}(0).$$

The Hodges price for the contingent claim ζ is clearly given by the formula

$$\bar{p} = \frac{1}{\gamma}\ln\Big(\frac{\bar{J}^0(0)}{\bar{J}^{\zeta}(0)}\Big).$$

Remark 5.2. If we restrict the admissible strategies to the bounded set $\mathcal{A}^k$, the indifference price $\bar{p}^k$ can also be defined by the same method. More precisely,

$$\bar{p}^k = \frac{1}{\gamma}\ln\Big(\frac{\bar{J}^{0,k}(0)}{\bar{J}^{\zeta,k}(0)}\Big),$$

where $\bar{J}^{\zeta,k}(0)$ is defined in Theorem 5.3.
Note that

$$\bar{p} = \lim_{k\to\infty}\bar{p}^k.$$

This allows to approximate the indifference price by numerical computation, since the value functions $(\bar{J}_t^{\zeta,k})_{k\in\mathbb{N}}$ are the solution of a Lipschitz BSDE and the results of Bouchard and Elie (2008) can be applied.

We assume that there are two kinds of agents in the market: the insider agents and the classical agents. We define the information price d for a contingent claim as the difference between the buying price for a classical agent and the buying price for an insider agent. The buying price, if the agent knows the full information, is defined by (see Lim and Quenez (2009))

$$p = \frac{1}{\gamma}\ln\Big(\frac{J^0(0)}{J^{\zeta}(0)}\Big),$$

where $(J^\zeta(t), Z_t, U_t, K_t)$ is the maximal solution of BSDE (5.13) with $(\bar{W},\ \bar{M},\ \tilde{\mu},\ \tilde{\lambda})$ replaced by $(W,\ M,\ \mu,\ \lambda)$.
Then the benefit of an insider agent who has a full information is the information price

$$d = \bar{p} - p.$$

This price can be computed as the limit of the information prices $(d^k)_{k\in\mathbb{N}}$, where d^k is the information price if we restrict the admissible strategies to the bounded set $\mathcal{A}^k$

$$d^k = \frac{1}{\gamma}\Big(\ln\Big(\frac{\bar{J}^{0,k}(0)}{J^{0,k}(0)}\Big) - \ln\Big(\frac{\bar{J}^{\zeta,k}(0)}{J^{\zeta,k}(0)}\Big)\Big),$$

where $(J^{\zeta,k}(t))$ is the solution of BSDE (5.14) with $(\bar{W},\ \bar{M},\ \tilde{\mu},\ \tilde{\lambda})$ replaced by $(W,\ M,\ \mu,\ \lambda)$.
Then we have

$$d = \lim_{k\to\infty} d^k.$$

Appendix

A Proof of Propositions 4.2 and 4.3

The proof of these propositions is based on the following lemma:

Lemma A.1. *The set $\{J_t^\pi,\ \pi\in\mathcal{A}_t\}$ is stable by supremum for any $t\in[0,T]$, i.e. for any $\pi^1,\ \pi^2\in\mathcal{A}_t$, there exists $\pi\in\mathcal{A}_t$ such that $J_t^\pi = J_t^{\pi^1}\vee J_t^{\pi^2}$.*
Furthermore, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}\in\mathcal{A}_t$ for any $t\in[0,T]$, such that

$$J(t) = \lim_{n\to\infty}\uparrow J_t^{\pi^n},\ \mathbb{P}-a.s.$$

Proof. Let us introduce the set $E = \{J_t^{\pi^1}\geq J_t^{\pi^2})\}$ which belongs to $\mathcal{F}_t$. Let us define the strategy π for any $s\in[t,T]$ by the formula $\pi_s = \pi_s^1 \mathbb{1}_E + \pi_s^2\mathbb{1}_{E^c}$. It is obvious that $\pi\in\mathcal{A}_t$. And by construction of π, it is clear that $J_t^\pi = J_t^{\pi^1}\vee J_t^{\pi^2}$.
The second part of the lemma follows by classical results on the essential supremum (see [27]). □

We first prove that the process $((X_t^\pi)^\gamma J(t))$ is a supermartingale for any $\pi\in\mathcal{A}$. For that it is sufficient to show for any $s\leq t$ that

$$\mathbb{E}\big[(X_t^{s,\pi})^\gamma J(t)\big|\mathcal{F}_s\big]\leq J(s),\ \mathbb{P}-a.s.$$

By Lemma A.1, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{A}_t$ such that $J(t) = \lim\uparrow J_t^{\pi^n},\ \mathbb{P}-a.s.$ We define the strategy $\tilde{\pi}^n$ by $\tilde{\pi}_u^n = \pi_u\mathbb{1}_{[s,t]}(u) + \pi_u^n\mathbb{1}_{]t,T]}(u)$, which is clearly admissible. By the monotone convergence theorem and using the definition of $J(s)$, one can easily show that

$$\mathbb{E}\big[(X_t^{s,\pi})^\gamma J(t)\big|\mathcal{F}_s\big] = \lim_{n\to\infty}\uparrow\mathbb{E}\big[(X_T^{s,\tilde{\pi}^n})^\gamma\big|\mathcal{F}_s\big]\leq J(s),\ \mathbb{P}-a.s.$$

Hence, the process $((X_t^\pi)^\gamma J(t))$ is a supermartingale for any $\pi \in \mathcal{A}$.

Second, we prove that $(J(t))$ is the smallest process satisfying $((X_t^\pi)^\gamma J(t))$ is a supermartingale for any $\pi \in \mathcal{A}$. For that we suppose that $(\bar{J}_t)$ is an $\mathbb{F}$-adapted process such that $((X_t^\pi)^\gamma(\bar{J}_t))$ is a supermartingale for any $\pi \in \mathcal{A}$ with $\bar{J}_T = 1$. Fix $t \in [0,T]$. For any $\pi \in \mathcal{A}$, we have $\mathbb{E}[(X_T^\pi)^\gamma|\mathcal{F}_t] \leq (X_t^\pi)^\gamma \bar{J}_t$, $\mathbb{P}-a.s.$ This inequality is equivalent to $\mathbb{E}[(X_T^{t,\pi})^\gamma|\mathcal{F}_t] \leq \bar{J}_t$. Which implies

$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t} \mathbb{E}\big[(X_T^{t,\pi})^\gamma\big|\mathcal{F}_t\big] \leq \bar{J}_t,\ \mathbb{P}-a.s.,$$

which clearly gives that $J_t \leq \bar{J}_t,\ \mathbb{P}-a.s.$

At last, we prove the optimality criterion, that is Proposition 4.3. Suppose that the strategy $\hat{\pi}$ is an optimal strategy, hence we have

$$J(0) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[(X_T^\pi)^\gamma\right] = \mathbb{E}\big[\big(X_T^{\hat{\pi}}\big)^\gamma\big].$$

As the process $((X_t^{\hat{\pi}})^\gamma J(t))$ is a supermartingale by Proposition 4.2 and that $J(0) = \mathbb{E}[(X_T^{\hat{\pi}})^\gamma]$, the process $((X_t^{\hat{\pi}})^\gamma J(t))$ is a martingale.
To show the converse, suppose that the process $((X_t^{\hat{\pi}})^\gamma J(t))$ is a martingale, then $\mathbb{E}[(X_T^{\hat{\pi}})^\gamma] = J(0)$. Moreover $\mathbb{E}[(X_t^\pi)^\gamma J(t)] \leq J(0)$ for any $\pi \in \mathcal{A}$ by Proposition 4.2. Which implies that

$$J(0) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[(X_T^\pi)^\gamma\right] = \mathbb{E}\big[\big(X_T^{\hat{\pi}}\big)^\gamma\big].$$

B Proof of Theorem 4.1

The proof of this theorem is based on Propositions 4.2 and 4.3, on Doob-Meyer's decomposition and on the martingale representation theorem.

Since the process (J_t) is a supermartingale, it can be written under the following form by using Doob-Meyer decomposition (see [3]) and the martingale representation theorem

$$dJ_t = Z_t dW_t + U_t dM_t - dA_t, \tag{B.1}$$

with $Z \in L^2_{loc}(W)$, $U \in L^1_{loc}(M)$, and (A_t) is a nondecreasing $\mathbb{F}$-adapted process and $A_0 = 0$. From product rule, the derivative of process $((X_t^\pi)^\gamma J_t)$ can be written under the form

$$d((X_t^\pi)^\gamma J_t) = (X_{t^-}^\pi)^\gamma\big(dA_t^\pi + dM_t^\pi\big),$$

with $A_0^\pi = 0$ and

$$\begin{cases} dA_t^\pi = \Big[\gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 J_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t + U_t)\Big]dt - dA_t, \\ dM_t^\pi = (\gamma\pi_t\sigma_t J_t + Z_t)dW_t + (U_t + ((1+\pi_t\beta_t)^\gamma - 1)(J_t+U_t))dM_t. \end{cases} \tag{B.2}$$

From Proposition 4.2, we have $dA_t^\pi \leq 0$ for any $\pi \in \mathcal{A}$, which implies

$$dA_t \geq \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 J_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t+U_t)\Big\}dt.$$

From [21], there exists an optimal strategy $\hat{\pi} \in \mathcal{A}$ to the optimization problem and from Proposition 4.3, we get

$$dA_t = \Big[\gamma\hat{\pi}_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\hat{\pi}_t^2\sigma_t^2 J_t + \lambda_t((1+\hat{\pi}_t\beta_t)^\gamma - 1)(J_t + U_t)\Big]dt.$$

Which imply that

$$dA_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 J_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t + U_t)\Big\}dt. \quad \text{(B.3)}$$

Therefore the process (J_t, Z_t, U_t) is a solution of BSDE (4.6).

We now prove that it is the minimal solution. Let $(\bar{J}_t, \bar{Z}_t, \bar{U}_t)$ be a solution of BSDE (4.6). Let us prove that $((X_t^\pi)^\gamma \bar{J}_t)$ is a supermartingale for any $\pi \in \mathcal{A}$. From the product rule, we can write the derivative of this process under the form

$$d\left((X_t^\pi)^\gamma\,\bar{J}_t\right) = (X_{t^-}^\pi)^\gamma\left[d\bar{M}_t^\pi + d\bar{A}_t^\pi - d\bar{A}_t\right], \quad \text{(B.4)}$$

where $\bar{A}_t$ (resp. $\bar{M}_t$) is given by (B.3) (resp. B.2) with (J, Z, U) replaced by $(\bar{J}, \bar{Z}, \bar{U})$, and $\bar{A}_0^\pi = 0$ and

$$d\bar{A}_t^\pi = \Big[\gamma\pi_t(\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 \bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t)\Big]dt.$$

By integrating (B.4), we get

$$(X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 = \int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi - \int_0^t (X_s^\pi)^\gamma (d\bar{A}_s - d\bar{A}_s^\pi).$$

As $d\bar{A}_s \geq d\bar{A}_s^\pi$, we have $\int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi \geq (X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 \geq -\bar{J}_0$. It implies that $(\bar{M}_t^\pi)$ is a supermartingale, since it is a lower bounded local martingale. Hence, the process $((X_t^\pi)^\gamma \bar{J}_t)$ is a supermartingale for any $\pi \in \mathcal{A}$, because it is the sum of a supermartingale and a nonincreasing process. Proposition 4.2 implies that $J_t \leq \bar{J}_t,\ \forall\, t \in [0,T],\ \mathbb{P} - a.s.$, which ends this proof.

C Proof of Theorem 4.2

We first remark that (J_t^k) satisfies the following property:

Lemma C.1. *The process (J_t^k) is the smallest $\mathbb{F}$-adapted process such that $((X_t^\pi)^\gamma J_t^k)$ is a supermartingale for any $\pi \in \mathcal{A}^k$ with $J_t^k = 1$.*

To prove this lemma, we use exactly the same arguments as in the proof of Proposition 4.2, since Lemma A.1 is still true with $\mathcal{A}_t^k$ instead of $\mathcal{A}_t$.

Fix $t \in [0,T]$. It is obvious with the definition of sets $\mathcal{A}_t$ and $\mathcal{A}_t^k$ that $\mathcal{A}_t^k \subset \mathcal{A}_t$ for each $k \in \mathbb{N}$ and hence

$$J_t^k \leq J_t,\ \mathbb{P} - a.s. \quad \text{(C.1)}$$

Moreover, since $\mathcal{A}_t^k \subset \mathcal{A}_t^{k+1}$ for each $k \in \mathbb{N}$, it follows that the positive sequence $\left(J_t^k\right)_{k\in\mathbb{N}}$ is nondecreasing. Let us define the random variable

$$\tilde{J}(t) = \lim_{k\to\infty} \uparrow J_t^k, \ \mathbb{P} - a.s.$$

It is obvious that the process $\tilde{J}(t) \leq J_t$, $\mathbb{P}-a.s.$ from (C.1) and this holds for any $t \in [0, T]$. It remains to prove that for any $t \in [0, T]$, $J_t \leq \tilde{J}(t)$, $\mathbb{P} - a.s.$ As in the proof of Theorem 4.2 of the companion paper Lim and Quenez (2009), we first prove that the process $\tilde{J}(t^+)$ is càd-làg and satisfies $\tilde{J}(t^+) \leq \tilde{J}(t)$, $\mathbb{P} - a.s.$ The process $((X_t^\pi)^\gamma \tilde{J}(t^+))$ is a supermartingale for any bounded strategy $\pi \in \mathcal{A}$. In the sequel, we shall denote $\bar{J}_t$ instead of $\tilde{J}(t^+)$. We now prove that $\bar{J}_t \geq J_t$, $\forall\, t \in [0, T]$, $\mathbb{P} - a.s.$ Since $(\bar{J}_t)$ is a càd-làg supermartingale, it admits the following Doob-Meyer decomposition

$$d\bar{J}_t = \bar{Z}_t dW_t + \bar{U}_t dM_t - d\bar{A}_t,$$

with $\bar{Z} \in L^2_{loc}(W)$, $\bar{U} \in L^1_{loc}(M)$ and $(\bar{A}_t)$ is a nondecreasing $\mathbb{G}$-adapted process with $\bar{A}_0 = 0$. As before, we use the fact that the process $((X_t^\pi)^\gamma \bar{J}_t)$ is a supermartingale for any bounded strategy $\pi \in \mathcal{A}$ to give some conditions satisfied by the process $(\bar{A}_t)$. Let $\pi \in \mathcal{A}$ be a uniformly bounded strategy, the product rule gives

$$d((X_t^\pi)^\gamma \bar{J}_t) = (X_{t^-}^\pi)^\gamma \big(d\bar{A}_t^\pi + d\bar{M}_t^\pi\big), \tag{C.2}$$

where $(\bar{A}_t^\pi)$ and $(\bar{M}_t^\pi)$ are given by (B.2) with $(J,\, Z,\, U,\, A)$ replaced by $(\bar{J},\, \bar{Z},\, \bar{U},\, \bar{A})$.

Let $\bar{\mathcal{A}}_t$ be the subset of uniformly bounded strategies of $\mathcal{A}_t$. Since the process $((X_t^\pi)^\gamma \bar{J}_t)$ is a supermartingale for any $\pi \in \bar{\mathcal{A}}$, we have

$$d\bar{A}_t \geq \operatorname*{ess\,sup}_{\pi\in\bar{\mathcal{A}}} \Big\{\gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t)\Big\}dt. \tag{C.3}$$

It is not possible to give an exact expression of $\bar{A}_t$ as in the previous proof, because we do not know if $\hat{\pi} \in \bar{\mathcal{A}}$. But this inequality is sufficient for the demonstration. Now, the following equality holds $dt \otimes d\mathbb{P}$ $a.s.$

$$\operatorname*{ess\,sup}_{\pi\in\bar{\mathcal{A}}} \Big\{\gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t)\Big\} =$$
$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t)\Big\}. \tag{C.4}$$

We now want to show that $((X_t^\pi)^\gamma \bar{J}_t)$ is a supermartingale for any $\pi \in \mathcal{A}$. Fix $\pi \in \mathcal{A}$ (not necessarily uniformly bounded), we get

$$(X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 = \int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi + \int_0^t (X_s^\pi)^\gamma d\bar{A}_s^\pi,$$

with $(\bar{A}_t^\pi)$ and $(\bar{M}_t^\pi)$ given by (B.2) with $(J,\, Z,\, U,\, A)$ replaced by $(\bar{J},\, \bar{Z},\, \bar{U},\, \bar{A})$.

Inequality (C.3) and equality (C.4) imply that $d\bar{A}_t^\pi \leq 0$, $\mathbb{P} - a.s.$ Therefore, we have

$$\int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi \geq (X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 \geq -\bar{J}_0.$$

Thus, $(\bar{M}_t^\pi)$ is a supermartingale, since it is a lower bounded local martingale. As $(\bar{M}_t^\pi)$ is a supermartingale and $(\bar{A}_t^\pi)$ is nonincreasing, the process $((X_t^\pi)^\gamma \bar{J}_t)$ is a supermartingale and this holds for any $\pi \in \mathcal{A}$. Since (J_t) is the smallest process (see Proposition 4.2) satisfying these properties, we have $J_t \leq \bar{J}_t,\ \mathbb{P}-a.s.$ Which ends the proof.

D Proof of Lemma 5.2

First, recall Bayes formula: for all $t \in [0,T]$ and $X \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$, one has

$$\mathbb{E}\big[X\big|\mathcal{G}_t\big] = \frac{\mathbb{E}_{\mathbb{Q}}\big[\Lambda_t X\big|\mathcal{G}_t\big]}{\tilde{\Lambda}_t}. \tag{D.1}$$

Let (ξ_t) be the optional projection of the $\mathbb{P}$-martingale (L_t) to $\mathbb{G}$, so

$$\xi_t = \mathbb{E}\big[L_t\big|\mathcal{G}_t\big].$$

By applying relation (D.1) to $X = L_t$, we immediately obtain $\xi_t = 1/\tilde{\Lambda}_t$ and thus

$$\xi_t = \exp\Big(-\int_0^t \tilde{\rho}_s' d\bar{W}_s - \frac{1}{2}\int_0^t ||\tilde{\rho}_s||^2 ds\Big).$$

Let (m_t) be a $(\mathbb{P},\mathbb{G})$-local martingale. From Bayes rule, the process $(\tilde{m}_t)$ given by

$$\tilde{m}_t = m_t \xi_t^{-1}, \quad 0 \leq t \leq T,$$

is a $(\mathbb{Q},\mathbb{G})$-local martingale. From Remark 5.1 and Lemma 5.2, there exists a couple of processes $(\tilde{a}_t, \tilde{b}_t)$ with $\tilde{a} \in L^2_{loc}(\tilde{W})$ and $\tilde{b} \in L^1_{loc}(\bar{M})$ such that

$$\tilde{m}_t = \int_0^t \tilde{a}_s' d\tilde{W}_s + \int_0^t \tilde{b}_s' d\bar{M}_s, \quad 0 \leq t \leq T.$$

By Ito's formula applied to $m_t = \tilde{m}_t \xi_t$, definition of $(\bar{W}_t)$ and $(\bar{M}_t)$ (see (5.7)), we obtain that

$$m_t = \int_0^t a_s' d\bar{W}_s + \int_0^t b_s' d\bar{M}_s,$$

with $a_t = \xi_t \tilde{a}_t - \tilde{m}_t \xi_t \tilde{\rho}_t$ and $b_t = \xi_{t^-} \tilde{b}_t$.

References

[1] Bellamy, N. (2001): "Wealth optimization in an incomplete market driven by a jump-diffusion process", *Journal of Mathematical Economics*, **35**, 259-287.

[2] Bielecki, T. and M. Rutkowski (2004): "Credit Risk: Modeling, Valuation and Hedging", *Springer Finance.*

[3] Dellacherie, C. and P.A. Meyer (1980): "Probabilités et potentiel", Théorie des martingales, *Hermann.*

[4] Detemple J. (1986): "Asset pricing in a production economy with incomplete information", *Journal of Finance*, **41**, 383-391.

[5] Dothan M.U and D. Feldman (1986): "Equilibrum interest rates and multiperiod bonds in a partially observable economy", *Journal of Finance*, **41**, 369-382.

[6] El Karoui, N. (1979): "Les Aspects Probabilistes du Contrôle Stochastique", Ecole d'Eté de Probabilités de Saint-Flour IX, *Springer-Verlag*.

[7] El Karoui N. and M.C. Quenez (1995): "Dynamic programming and pricing of contingent claims in an incomplete market", *SIAM Journal on Control and Optimization*, **33**, 29-66.

[8] El Karoui, N., Peng, S. and M.C. Quenez (1997): "Backward stochastic differential equations in finance", *Mathematical finance*, **7**, 1-71.

[9] Frey R. and W. Runggaldier (1999): "Risk-minimizing hedging strategies under restricted information: the case of stochastic volatility models observable onaly at discrete random times", *Preprint*.

[10] Gennotte G. (1986): "Optimal portfolio choice under incomplete information", *Journal of Finance*, **41**, 733-746.

[11] Hodges S.D. and A. Neuberger (1989): "Optimal replication of contingent claim under transaction costs", *Review of Futures Markets*, **8**, 222-239.

[12] Hu Y. , Imkeller P. and M. Muller (2005): "Utility maximization in incomplete markets", *The Annals of Applied Probability*, **15**, 1691-1712.

[13] Jeanblanc, M. and M. Pontier (1990): "Optimal portfolio for a small investor in a market with discontinuous prices", *Applied mathematics and optimization*, **22**, 287-310.

[14] Jeanblanc, M., Yor, M. and M. Chesney (2009): "Mathematical Methods for Financial Markets", *Springer Finance*.

[15] Jiao, Y. and H. Pham (2009): "Optimal investment with counterparty risk: a default-density modeling approach", *Preprint LPMA*.

[16] Karatzas, I., Lehoczky, J.P and S. Shreve (1987): "Optimal portfolio and consumption decisions for a small investor on a finite horizon", *SIAM Journal on Control and Optimization*, **25**, 1557-1586.

[17] Karatzas, I., Lehoczky, J.P, Shreve, S. and G. Xu (1991): "Martingale and duality methods for utility maximization in an incomplete market", *SIAM Journal on Control and Optimization*, **29**, 702-730.

[18] Karatzas I. and S. Shreve (1991): "Brownian motion and stochastic calculus", *Springer Verlag*.

[19] Karatzas I. and X. Xue (1991): "A note on utility maximization under partial observations", *Mathematical Finance*, **1**, 57-70.

[20] Karatzas I. and X. Zhao (1998): "Bayesian adaptative portfolio optimization", *Preprint*, Columbia University.

[21] Kramkov D. and W. Schachermayer (1999): "The asymptotic elasticity of utility functions and optimal investment in incomplete markets", *The Annals of Applied Probability*, **9**, 904-950.

[22] Lakner P. (1995): "Utility maximization with partial information", *Stochastic Processes and their Applications*, **56**: 247-273.

[23] Lakner P. (1998): "Optimal trading strategy for an investor: the case of partial information", *Stochastic Processes and their Applications*, **76**, 77-97.

[24] Lasry J.M. and P.L. Lions (1999): "Stochastic control under partial information and applications to finance", *Stochastic Processes and their Applications*, **56**, 247-273.

[25] Lim T. and M.C. Quenez (2009): "Utility maximization in incomplete market with default", *submitted to Finance and Stochastics*.

[26] Merton R.C. (1971): "Optimum consumption and portfolio rules in a continuous-time model", *Journal of Economic Theory*, **3**, 373-413.

[27] Neveu, J. (1975): "Discrete-Parameter Martingales", English translation, North-Holland, Amsterdam and American Elsevier, New York.

[28] Pardoux E. (1989): "Filtrage non linéaire et équations aux dérivées partielles stochastiques associées", Lect. Notes in Math., *Springer Verlag*, 1464, 67-163.

[29] Pardoux, E. (1997): "Generalized discontinuous backward stochastic differential equation", in: N, El Karoui, L. Mazliak (Eds.), Backward Stochastic Differential Equations, in: Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow, 207-219.

[30] Pham H. and M.C. Quenez (2001): "Optimal portfolio in partially observed stochastic volatility models", *Annals of Applied Probability*, **11**, 210-238.

[31] Protter P. (1990): "Stochastic integration and differential equations", *Springer Verlag*.

[32] Roland S. (2007): "Partially observed jump diffusions", *Thesis*.

[33] Royer, M. (2006): "Backward stochastic differential equations with jumps and related non-linear expectations", *Stochastic processes and their applications*, **116**, 1358-1376.

[34] Saas J. and U. Haussmann (2004): "Optimizing the terminal wealth under partial information: the drift process as a continuous markov chain", *Finance and Stochastics*, **8**, 553-577.

[35] Stricker C. (1983): "Quelques remarques sur les semi-martingales gaussiennes et le problème de l'innovation", *Lect. Notes in Cont. and Inf. Sci.*, *Springer Verlag*, **61**, 260-276.

[36] Tang S.H. and X. Li (1994): "Necessary conditions for optimal control of stochastic systems with random jumps", *SIAM Journal on Control and Optimization*, **32**, 1447-1475.