arXiv:math/0702609v1 [math.ST] 21 Feb 2007

Local Functional Principal Component Analysis

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Abstract

Covariance operators of random functions are crucial tools to study the way random elements concentrate over their support. The principal component analysis of a random function X is well-known from a theoretical viewpoint and extensively used in practical situations. In this work we focus on local covariance operators. They provide some pieces of information about the distribution of X around a fixed point of the space x_0. A description of the asymptotic behaviour of the theoretical and empirical counterparts is carried out. Asymptotic developments are given under assumptions on the location of x_0 and on the distributions of projections of the data on the eigenspaces of the (non-local) covariance operator.

1 Introduction

1.1 The general framework

A recent and considerable interest has been given along the last years to the statistical analysis for functional data. Usually mathematical statistics or probability theory deal with data modelled as random variables i.e. measurable mappings from an abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to a finite dimensional space. If a sample is denoted $X_1, .., X_n$, the X's take values classically in $\mathbb{R}$ or $\mathbb{R}^p$. In our framework we turn to random elements with values in an infinite-dimensional function space denoted $\mathcal{F}$. In the sequel $\mathcal{F}$

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will be endowed at least with a separable Banach space structure with norm $\|\cdot\|$. In everyday's life, situation where such data appear are quite common : monitoring the value of a share yields a random curve $X(t)$ where t runs along the quotation time. Even more basically, observing the temperature at a given place along the day and during n consecutive days provides a "theoretical" sample $X_1(t), ..., X_n(t)$ where $t \in [0, 24]$. Here "theoretical" means that temperatures will be recorded each day at fixed moments. For instance X_1 will be observed at $t_1, ...t_{m_1}$, hence the true curve $X_1(t)$ will have to be reconstructed from the finite real valued sample $X_1(t_1), ..., X_1(t_{m_1})$ by interpolation techniques such as splines, wavelets, cosine bases, etc. For further references about this topic see Chui (1992) and Antoniadis, Oppenheim (1995) about wavelets and de Boor (1978) ou Dierckx (1993), about splines.

It turns out that probabilists have studied such random elements for a much longer time than statisticians (first works on the Brownian motion date back to the XIXth century), the first monograph dedicated to functional data was published in 1991 (Ramsay and Silverman (1991)). Modern computers make it now possible to carry out calculations for very high dimensional vectors and practical statisticians have shifted their interests to functional data or to so-called "high-dimensional problems" that fall within the scope of this paper. However a theoretical gap remains because not all asymptotic results have been given yet for such data. Besides probability theory unfortunately sometimes just give clues and not solutions to typically statistical issues (see later the paragraph devoted to small ball problems).

The interested reader could get familiar with the applied aspects by reading the monographs by Ramsay and Silverman or by Ferraty and Vieu (2006). Many probabilistic results will be found in Vakhania, Tarieladze and Chobanian (1987) or in Ledoux and Talagrand (1991).

In this setting we can define the expectation of X as :

$$\mathbb{E}X = \int_\Omega X d\mathbb{P} = \int_\mathcal{F} x d\mu_X(x) \in \mathcal{F}$$

where $\mu_X = \mathbb{P} \circ X^{-1}$ is the image of measure of $\mathbb{P}$ through the mapping X. The integral is of Bochner type and is defined whenever the real valued random variable $\|X\|$ is integrable.

In this article we focus on a very useful and common statistical technique : principal component analysis (PCA for short). The functional version of the PCA was initially studied by Dauxois, Pousse and Romain (1982). We refer to this seminal article for a complete mathematical definition. Briefly speaking functional PCA of a process $X(\cdot)$ comes down to the spectral analysis of the covariance operator associated to the sample (see below for definitions). We refer for instance to Silverman (1996), Ocaña, Aguilera and Valderrama

(1999), Kneip and Utikal (2001), Yao, Muller and Wang (2005), or Cardot, Mas and Sarda (2007) to overview some applications and extensions of the functional PCA. Also note that He, Muller and Wang (2003) introduced a version of canonical analysis for random functions.

1.2 The Hilbert setting

The abstract framework defined above is too general for statistical purposes. We will restrict ourselves to special spaces $\mathcal{F}$ but we will gain in terms of interpretation of our assumptions and results. Indeed we will assume once and for all that $\mathcal{F} = H$ is a separable Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. Several reasons can explain this tightening. First of all we know that each function X may be decomposed in denumerable bases. In practical situations these bases enable to get observation-curves from the discretized ones (see first paragraph above) by interpolation methods mentioned above. The following step is then achieved :

$$[X_1 (t_1), ..., X_1 (t_{m_1})] \Rightarrow X_1 (t) = \sum_{k=1}^{N} c_k e_k (t) .$$

The Sobolev spaces $W^{m,2}$ are classical examples of such Hilbert spaces :

$$W^{m,2} ([0, T]) = \left\{ f \in L^2 ([0, T]) : \sum_{k=0}^{m} \int_0^T \left(f^{(k)} (s) \right)^2 ds < +\infty \right\}$$

where $T > 0$ and $f^{(k)}$ denotes here the derivative of order k of f.

Besides as we will focus on covariance operators the Hilbert setting yields considerable simplifications. Bounded linear operators were extensively studied as well as their spectral properties (see references below). We introduce the two following operator spaces and associated norms.

The Banach space $\mathcal{L}(H, H) = \mathcal{L}$ is the classical space of bounded operator endowed with the norm defined for each T in $\mathcal{L}$ by :

$$\|T\|_\infty = \sup_{x \in \mathcal{B}_1} \|Tx\| ,$$

where $\mathcal{B}_1$ is the unit sphere of H. The Hilbert space $\mathcal{L}_2$ is the space of Hilbert-Schmidt operators, ($\mathcal{L}_2 \subset \mathcal{L}$) i.e. the space of those operators T such that, given a basis of H, say $(e_k)_{k \in \mathbb{N}}$,

$$\|T\|_2 = \sum_{k=1}^{+\infty} \|Te_k\|^2 < +\infty.$$

It is a well-known fact that $\mathcal{L}_2$ is a separable Hilbert space whenever H is. The inner product in $\mathcal{L}_2$ is :

$$\langle T, S \rangle_2 = \sum_{k=1}^{+\infty} \langle Te_k, Se_k \rangle$$

and does not depend on the choice of the basis $(e_k)_{k \in \mathbb{N}}$. The space of trace-class (or nuclear) operators $\mathcal{L}_1$ endowed with norm $\|\cdot\|_1$ will be mentioned sometimes in the paper. These norms are not equivalent and

$$\|\cdot\|_\infty \leq \|\cdot\|_2 \leq \|\cdot\|_1 .$$

The canonical injections from $\mathcal{L}_1$ onto $\mathcal{L}_2$ and from $\mathcal{L}_2$ onto $\mathcal{L}$ are consequently continuous. For further information on linear operators we refer to Schmeidler (1965), Weidman (1980), Dunford-Schwartz (1988), Gohberg, Goldberg and Kaashoek (1991) amongst many others.

2 Covariance and local covariance operators of random Hilbert elements

Let the tensor product between u and v in H stand for the one-rank operator from H to H by :

$$(u \otimes v)(t) = \langle u, t \rangle v$$

for all t in H.

Since covariance operators are undern concern within functional PCA, we should first of all define them and give some of their main features. The theoretical covariance operator Γ and its empirical counterpart, Γ_n, based on the independent and identically distributed sample $X_1, ..., X_n$ are symmetric positive trace class operators from H to H defined by :

$$\Gamma = \mathbb{E}\left((X_1 - \mathbb{E}X_1) \otimes (X_1 - \mathbb{E}X_1)\right), \tag{1}$$

$$\Gamma_n = \frac{1}{n} \sum_{k=1}^{n} \left(X_k - \overline{X}_n\right) \otimes \left(X_k - \overline{X}_n\right) \tag{2}$$

where

$$\overline{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k.$$

If X is centered $\mathbb{E}X = 0$ and $\Gamma = \mathbb{E}\left(X \otimes X\right)$. By (λ_k, e_k) we denote the k^{th} eigenelements (eigenvalues/eigenvectors) of Γ. The λ_k's are positive and we set $\lambda_1 \leq \lambda_2 \leq ...$, and $(\lambda_k)_{k \in \mathbb{N}} \in l_1$.

In the Hilbert setting, the distribution of a centered random element say X may be characterized a very simple way. Indeed if $=_d$ denotes equality in distribution :

$$X =_d \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \xi_k e_k \tag{3}$$

where the ξ_k's are centered non-correlated real random variables with unit variance. The above decomposition is often referred to as the Karhunen-Loève development or development of X with respect to its reproducing kernel Hilbert space (RKHS). For definition and studies of RKHS we refer to Berlinet, Thomas-Agnan (2004).

Now we introduce the definition of local covariance operator. Let K be a kernel, that is a nonnegative function defined on $\mathbb{R}^+$, and such that $\int K(s)\,ds = 1$ and let $h = h(n)$ be a bandwidth. Pick x_0 a fixed vector in H (x_0 will be a function if H is a space of functions). In the sequel we will often need to decompose x_0 in the basis e_k :

$$x_0 = \sum_{k=1}^{+\infty} \langle x_0, e_k \rangle e_k$$

where

$$\sum_{k=1}^{+\infty} \langle x_0, e_k \rangle^2 < +\infty.$$

Definition 1 *The theoretical local covariance operator of X at $x_0 \in H$ based on the kernel K and its empirical counterpart are respectively defined by :*

$$\Gamma_K = \mathbb{E}\left(K\left(\frac{\|X_1 - x_0\|}{h} \right) ((X_1 - x_0) \otimes (X_1 - x_0)) \right), \tag{4}$$

$$\Gamma_{K,n} = \frac{1}{n} \sum_{k=1}^{n} K\left(\frac{\|X_k - x_0\|}{h} \right) ((X_k - x_0) \otimes (X_k - x_0)). \tag{5}$$

Even if Γ_K and $\Gamma_{K,n}$ may have already been introduced elsewehere in articles dealing with functional data (see for instance Ferraty, Mas, Vieu (2007)), it is the first attempt, up to the author's knowledge, to provide these operators with a name. Note that Γ_K implicitely depends on n through h, even if this index does not explicitely appears. These operators are crucial in the nonparametric estimation of the regression function by local linear methods amongst others (see the conclusion for further details).

Proposition 1 *The local covariance operators Γ_K and $\Gamma_{K,n}$ are positive, selfadjoint. Besides they are trace class whenever K is bounded and*

$$\mathbb{E}\left(\|X_1\|^2\right) < +\infty$$

or when K is bounded and compactly supported.

The proposition is plain since the trace-class norm of Γ_K for instance is bounded by :

$$\begin{aligned}
&\mathbb{E}\left\|K\left(\frac{\|X_1 - x_0\|}{h}\right)((X_1 - x_0)\otimes(X_1 - x_0))\right\|_1 \\
&= \mathbb{E}\left[K\left(\frac{\|X_1 - x_0\|}{h}\right)\|X_1 - x_0\|^2\right].
\end{aligned}$$

In the sequel by $(\lambda_k)_{k\in\mathbb{N}}$ (resp. $(\lambda_{k,n})_{k\in\mathbb{N}}$) and $(\pi_k)_{k\in\mathbb{N}}$ (resp. $(\pi_{k,n})_{k\in\mathbb{N}}$) we denote the eigenvalues and the associated eigenvectors of Γ_K (resp. $\Gamma_{K,n}$)

The main goal of this paper is to describe the asymptotic behaviour of Γ_K and $\Gamma_{K,n}$ and to derive results for their eigenelements.

3 Intermezzo about Gamma varying functions and the small ball problem

It may be proved that whenever the random variable X takes values in a finite dimensional space, say $\mathbb{R}^p$, the covariance operators defined at displays (4) and (5) depend on the value of the density of X, if we assume that X admits a density at point x_0. Besides one may use a larger class of kernels (here since $\|X_1 - x_0\|$ is positive we are restricted to kernels with positive support which damages the rates of convergence). It is simple to prove that when X is a real-valued random variable that exhibits a non-null density f at x_0 with some regularity around x_0 :

$$\mathbb{E}\left[K\left(\frac{X_1 - x_0}{h}\right)(X_1 - x_0)^2\right] \sim h^3 f_X(x_0)\int u^2 K(u)\,du.$$

We refer to Fan (1993) for illustrating the issues of asymptotics for truncated moments in nonparametric regression estimation.

In our infinite dimensional framework the situation is quite different : since Lebesgue's measure is not defined on Hilbert spaces, the notion of 'density' cannot be defined either. It turns out that the density will be replaced by the "small ball probability of X", which is nothing but the cumulative density function for the norm of X (or $X - x_0$) in a neighborhood of 0 i.e. $\mathbb{P}(\|X - x_0\| < \varepsilon)$ for $\varepsilon \downarrow 0$. The random variable X may be replaced by any process $Z_t, t \in T$. The study of small ball probabilites is not new and is connected with the theory of large deviations (since $\|X - x_0\| / \varepsilon$ will be large when ε decays to zero). We refer to Ledoux-Talagrand (1991), Li, Linde (1999), Li, Shao (2001) for some more information about this topic.

Since these small ball probabilities appear in the main results of this article and are of much importance within the proofs, we collect a few results about them in order to be more illustrative. We provide two examples directly based on display (3). The small probability then heavily depends on the rate of decay of the eigenvalues of the covariance operator, namely the λ_k's.

If the rate of decay is arithmetic $\lambda_k \asymp k^{-(1+\alpha)}$. The problem was solved in Mayer-Wolf, Zeitouni (1993). They get :

$$\mathbb{P}(\|X\| < \varepsilon) \sim \exp\left(-\frac{C(\alpha)}{\varepsilon^{1/\alpha}}\right). \tag{6}$$

where $C(\alpha)$ is some positive constant.

When the rate of decay is exponential : $\lambda_k \asymp \exp(-ak)$, the calculations may not have been carried out. I did not find them elsewhere. They may be derived from formula (10) in Dembo, Mayer-Wolf, Zeitouni (1995).

Proposition 2 *When $\lambda_k \asymp \exp(-ak)$ in (3), then for $\varepsilon \to 0$:*

$$\mathbb{P}(\|X\| < \varepsilon) \sim \sqrt{\frac{\alpha}{-\pi \log(\varepsilon)}} \exp\left(-\frac{1}{4\alpha}[\log(\varepsilon)]^2\right) \tag{7}$$

The proof of this apparently new result is postponed to the end of the last section.

Let us leave the small ball probability for a moment. At this point we need to give some properties of a class of real functions. The statistician may be familiar with the definition of functions with regular variations since they appear in the theory of extremes. For instance $f : \mathbb{R} \to \mathbb{R}$ is regularly varying at 0 with index d if , for all fixed x in $\mathbb{R}$:

$$\lim_{h \to 0} \frac{f(hx)}{f(h)} = x^d.$$

A less known class of functions studied in the theory of regular variations is the so-called "class Γ". This class Γ will be of much use in the sequel. It was introduced by de Haan (1971), see also de Haan (1974) in connection with the theory of extemes. But Gaïffas (2005) used it to model the distribution of "rare" inputs in a non-parametric regression model : a density which is null and Γ-varying at x_0 will generate a distribution which rarely visit x_0.

Definition 2 *A function f belongs to the class Γ at 0 (or is Γ-varying at 0) if there exists a measurable positive function ρ such that for all $x \in \mathbb{R}$:*

$$\lim_{h \to 0^+} \frac{f\left(h + \rho\left(h\right) x\right)}{f\left(h\right)} = \exp\left(x\right). \tag{8}$$

The function ρ is called the auxiliary function of f.

We refer to Chapter 3.10 in Bingham, Goldie, Teugels (1987) for a deeper presentation and the essential properties of regularly varying functions and of the class Γ (see p.174-180). The reader should be aware that, in this book, the authors consider only functions that are Γ-varying at infinity. Their definitions and properties must be adapted to our setting : here functions are Γ-varying at 0. We collect now only those properties which will be used throughout the proofs :

Fact 1 : If $f \in \Gamma$, for all $x \in [0, 1[$

$$\lim_{h \to 0^+} \frac{f\left(hx\right)}{f\left(h\right)} = 0. \tag{9}$$

Fact 2 : If ρ is the auxiliary function of $f \in \Gamma$, then

$$\frac{\rho\left(s\right)}{s} \underset{s \to 0}{\to} 0, \tag{10}$$

$$\frac{\rho\left(s + x\rho\left(s\right)\right)}{\rho\left(s\right)} \underset{s \to 0}{\to} 1 \tag{11}$$

when s goes to 0 and for all $x \in \mathbb{R}$.

Fact 3 : If $f \in \Gamma$ then $F\left(x\right) = \int_0^x f\left(s\right) ds$ belongs to the class Γ too and

$$\int_0^h f\left(s\right) ds \underset{h \to 0}{\sim} f\left(h\right) \rho\left(h\right). \tag{12}$$

Now we turn again to the small ball probabilities. The following Proposition explains why the class Γ was introduced.

Proposition 3 *Functions defined on displays (6) and (7) are both Γ-varying at 0 with auxiliary functions :*

$$\rho\left(s\right) = \frac{\alpha}{C\left(\alpha\right)} s^{1+1/\alpha}$$

and

$$\rho\left(s\right) = -s / \left(2\alpha \log s\right)$$

respectively.

The proof is omitted since it is straightforward. The previous Proposition is quite important for the sequel.

It is seen from (4) for instance that the random element X is shifted from the origin by $-x_0$. Obviously small ball probabilites defined at displays (6) or (7) do not exactly match our goals. The shift, $-x_0$ is nonrandom but however the small balls probabilities may tremendously differ from those given above. It turns out that when x_0 lies in the reproducing kernel Hilbert space of X :

$$\mathbb{P}\left(\|X - x_0\| < \varepsilon\right) \sim C\left(x_0\right)\mathbb{P}\left(\|X\| < \varepsilon\right) \tag{13}$$

where $C\left(x_0\right)$ is a constant which depends only on x_0 and $\asymp$ should replace $\sim$ in (6) and (7). The articles by Li, Linde (1993) or Kuelbs, Li, Linde (1994) deal with the small ball problem for shifted balls.Since the small ball probability of X appears explicitely in the main results of this paper it is denoted for simplicity :

$$F_{x_0}\left(\varepsilon\right) = F\left(\varepsilon\right) = \mathbb{P}\left(\|X - x_0\| < \varepsilon\right).$$

Within the proofs several calculations must be carried out that involve some analytic properties of F and we need to announce the following claim, inspired by Proposition 3 and (13)

Claim 1 *The small ball probability functions F of shifted random elements in H (here $X - x_0$) are assumed to belong to the class Γ.*

The assumptions needed to define correctly $F_{x_0}\left(\varepsilon\right)$ that may appear in Li, Linde (1993) or Kuelbs, Li, Linde (1994) are supposed to hold in addition to those that will be given below.

Remark 1 *This claim directly leads us to arising the following question : is the small probability (at 0) of any process defined by (3) Γ-varying at zero ? Answering yes would provide a universal "representation" of these small ball probability functions (see Theorem 3.10.8 p.178 in Bingham, Goldie, Teugels (1987)). At this point we can answer only in the two important special cases mentioned above but this issue is under investigation (see Mas (2007)).*

4 Main results

We study convergence for random (or not) operators. This section is tiled into three subsections. In the first one we provide theorems dealing with asymptotics for the cells of some infinite matrix. Exact constants are computed. In the second subsection we get bounds in supremum or Hilbert-Schmidt norm for the operator(s) under concern. The third deals with the empirical local covariance operator $\Gamma_{K,n}$.

4.1 Cell-by-cell results

First let us introduce the assumptions needed in the sequel.

Assumption $\mathbf{A}_1$: *There exists a basis, say e_p in which the finite dimensional distributions of X, the $\langle X, e_p\rangle$'s are independent and all have a density f_p. This density is such that $(f_p)^{(i)}(\langle x_0, e_p\rangle) \neq 0$ for $i \in \{0,1,2\}$. Besides the density of the nonnegative real variable*

$$\sqrt{\sum_{k=1}^{+\infty} \langle X - x_0, e_k\rangle^2} = \|X - x_0\|$$

exists in a neighborhood of 0 and *belongs to the class Γ with auxiliary function ρ.*

Remark 2 *The notation e_p should not be misleading. The basis involved in Assumption $\mathbf{A}_1$ needs not to be the basis of eigenvalues of the operator* Γ. *But since in the important case of a gaussian random element X -with eigenvalues decaying at an arithmetic or geometric rate- $\mathbf{A}_1$ always holds for this special basis we will abusively keep the same notation.*

Assuming that the finite dimensional distributions are independent is needed to alleviate the proofs and to get exact constant in asymptotic expansions. Milder hypotheses on the joint distribution of the couple $(\langle X, e_k\rangle, \|X\|)$ could certainly prevail at the expense of more tedious caclulations as will be seen from the proofs.

Remark 3 *Within $\mathbf{A}_1$ the assumption $(f_p)^{(i)}(\langle x_0, e_p\rangle) \neq 0$ for $i \in \{0,1,2\}$ could be replaced by the more general one : "Let us denote*

$$\begin{aligned}
N_p^0 &= \inf\left\{k : f_p^{(2k)}(\langle x_0, e_p\rangle) \neq 0\right\}, \\
N_p^1 &= \inf\left\{k : f_p^{(2k+1)}(\langle x_0, e_p\rangle) \neq 0\right\}, \\
N_p^2 &= \inf\left\{k > N_p^0 : f_p^{(2k)}(\langle x_0, e_p\rangle) \neq 0\right\}
\end{aligned}$$

and assume that N_p^1 and N_p^2 are finite for all p". But once more we prefer to lose generality and gain readability. Also note that switching Assumption 1 to the one involving the N_p^k's leads to modified results in Theorem 1 : indeed the speed of convergence would then depend on N_p^0, N_p^1 and N_p^2.

Assumption $\mathbf{A_2}$: *The kernel K is bounded, $[0,1]$-supported, $K(1) > 0$* and

$$\sup_{s\in[0,1]} |K'(s)| < +\infty$$

This assumption is not too restrictive and could certainly be replaced by a milder one. But it is out of the scope of this article to provide minimal conditions on the kernel K.

We start with a development of Γ_K. Let $\delta_{i,j}$ be the Kronecker symbol ($\delta_{i,j} = 1$ if and only if $i = j$, 0 otherwise).

Theorem 1 *Assume $\mathbf{A}_1$ and $\mathbf{A_2}$. When h goes to 0 the operator Γ_K also tends to zero. And the following holds : for fixed i and j in $\mathbb{N}$,*

$$\langle \Gamma_K(h)(e_i), e_j\rangle \sim v(h)\,\delta_{i,j} + w(h)\,\mathcal{R}_{ij}, \tag{14}$$

where $v(h)$ and $w(h)$ are two real nonnegative sequences defined by :

$$v(h) = \mathbb{E}\left(K\left(\frac{\|X - x_0\|}{h}\right)\|X - x_0\|\,\rho(\|X - x_0\|)\right), \tag{15}$$

$$w(h) = \mathbb{E}\left(K\left(\frac{\|X - x_0\|}{h}\right)\|X - x_0\|^2\,\rho^2(\|X - x_0\|)\right) \tag{16}$$

and where the doubly indexed field $\mathcal{R}$ is defined by :

$$\mathcal{R}_{ii} = \frac{f_i''(\langle x_0, e_i\rangle)}{f_i(\langle x_0, e_i\rangle)},$$
$$\mathcal{R}_{ij} = \frac{f_i'(\langle x_0, e_i\rangle)}{f_i(\langle x_0, e_i\rangle)}\frac{f_j'(\langle x_0, e_j\rangle)}{f_j(\langle x_0, e_j\rangle)}, \quad i \neq j.$$

Theorem 1 provides asymptotics for each cell of the infinite dimensional matrix Γ_K when expressed in the basis $(e_i)_{1\le i\le n}$. Introducing the operator $\mathcal{R}$ defined in the basis $(e_i)_{1\le i\le n}$ by $\langle \mathcal{R}e_i, e_j\rangle = \mathcal{R}_{ij}$ we could rephrase this theorem by saying that "Γ_K is asymptotically equivalent "cell by cell" to the operator $v(h)\,I + w(h)\,\mathcal{R}$".

At this point the reader is not given much information on both sequences $v(h)$ and $w(h)$. It is actually basic to see that both sequences tend to zero. The next subsection will provide the reader with a more explicit description of the rate of decrease.

The next Proposition and the two next remarks give seminal properties of operator $\mathcal{R}$.

Proposition 4 *If both following conditions hold*

$$\mathbf{C}_1 : \sum_{i=1}^{+\infty} \left(\frac{f_i''(\langle x_0, e_i \rangle)}{f_i(\langle x_0, e_i \rangle)} \right)^2 < +\infty, \tag{17}$$

$$\mathbf{C}_2 : \sum_{i=1}^{+\infty} \left(\frac{f_i'(\langle x_0, e_i \rangle)}{f_i(\langle x_0, e_i \rangle)} \right)^2 < +\infty, \tag{18}$$

$\mathcal{R}$ is Hilbert-Schmidt. If (17) is replaced with

$$\mathbf{C}_1' : \left(\frac{f_i''(\langle x_0, e_i \rangle)}{f_i(\langle x_0, e_i \rangle)} \right)_{i \in \mathbb{N}} \in l_\infty \quad (\text{resp } c_0) \tag{19}$$

the operator $\mathcal{R}$ is bounded (resp. compact).
When either (18) or (19) does not hold, $\mathcal{R}$ is a symmetric unbounded operator.

The proof of this Proposition is omitted since it is a consequence of the following remark.

Remark 4 *The operator $\mathcal{R}$ may be rewritten :*

$$\mathcal{R} = \tau \otimes \tau + \operatorname{diag}(s_i)$$

where

$$\begin{aligned} \tau &= \left(\frac{f_1'(\langle x_0, e_1 \rangle)}{f_1(\langle x_0, e_1 \rangle)}, \frac{f_2'(\langle x_0, e_2 \rangle)}{f_2(\langle x_0, e_2 \rangle)}, \ldots \right) \\ &= \left((\ln f_1)'(\langle x_0, e_1 \rangle), (\ln f_2)'(\langle x_0, e_2 \rangle), \ldots \right), \\ s_i &= (\ln f_i)''(\langle x_0, e_i \rangle), \end{aligned}$$

and $\operatorname{diag}(s_i)$ denotes a diagonal operator expressed in the basis e_i with i^{th} term s_i. When (18) holds $\tau \in H$.

Before going into deeper details we should examine a typical situation, namely the case when X is a gaussian random element. The following Proposition shows that even in this basic situation, serious problems occur.

Proposition 5 *If X is gaussian and centered :*

$$\begin{aligned} \frac{f_i'(\langle x_0, e_i \rangle)}{f_i(\langle x_0, e_i \rangle)} &= -\frac{\langle x_0, e_i \rangle}{\lambda_i}, \\ \frac{f_i''(\langle x_0, e_i \rangle)}{f_i(\langle x_0, e_i \rangle)} &= \left(\frac{\langle x_0, e_i \rangle}{\lambda_i} \right)^2 - \frac{1}{\lambda_i} \end{aligned}$$

and conditions (18) and (19) of Proposition 4 cannot hold together which also means that the operator $\mathcal{R}$ is always unbounded in this setting.

Let us focus on these conditions since they may be easily interpreted. Indeed assuming (18) :

$$\sum_{i=1}^{+\infty}\left(\frac{\langle x_0, e_i\rangle}{\lambda_i}\right)^2 < +\infty$$

means that the coordinates of x_0 should decrease much quicker to zero than the eigenvalues which also means that x_0 should be smoother (more regular) than X itself. On the other hand (19) will hold whenever $\langle x_0, e_i\rangle^2 = \lambda_i + \lambda_i^2\tau_i$ (where $\tau \in l_\infty$ (resp c_0)), and x_0 should then be as regular as X but -surprisingly- not more...

Considering again the unusual conditions $\mathbf{C}_1, \mathbf{C}_2$ the reader may become suspicious and the situation in the gaussian framework makes it legitimate to wonder whether there exists a family of densities and an x_0 such that these conditions hold. The answer is positive and gives birth to the following Proposition.

Proposition 6 *Let f_i be the symmetric density defined on $\mathbb{R}$ by :*

$$f_i(x) = \frac{6}{3^6}\frac{1}{\lambda_i^2}\left(27\lambda_i^{3/2} - |x|^3\right)1\mathrm{I}_{\left\{x\leq 3\lambda_i^{1/2}\right\}}$$

and take $\langle x_0, e_i\rangle = x_i$ such that

$$\sum_{i=1}^{+\infty}\frac{x_i^2}{\lambda_i^3} < +\infty.$$

Then (17) and (18) both hold.

The proof of the Proposition is omitted since it stems from straightforward computations.

4.2 Norm results for the local covariance operator

The next issue is obviously to strengthen Theorem 1 : Is it possible to replace the "cell by cell" or "componentwise" convergence by convergence in norm ? First of all note that we may expect the rate of convergence to be $v(h)$ but we have to be cautious for several reasons :

- First of all $\mathcal{R}$ may be unbounded. In that situation we cannot expect a result such as :

$$\Gamma_K - \{v(h)I + w(h)\mathcal{R}\} \to 0$$

  in norm since $w(h)\mathcal{R}$ may not even be bounded whereas Γ_K is.

- We may have $\Gamma_K - v(h) I \to 0$ but we cannot get

$$\left\| \frac{\Gamma_K}{v(h)} - I \right\|_\infty \to 0$$

for topological reasons : $\frac{\Gamma_K}{v(h)}$ is for all h a compact operator and cannot converge to the identity operator (which is not compact) since $\mathcal{L}_c$ is a closed subspace of $\mathcal{L}$.

- Even worse : $\frac{\Gamma_K}{v(h)}$ may be asymptotically bounded or unbounded i.e.

$$\limsup_{h\to 0} \left\| \frac{\Gamma_K}{v(h)} \right\|_\infty < M \quad \text{or} \quad \limsup_{h\to 0} \left\| \frac{\Gamma_K}{v(h)} \right\|_\infty = +\infty.$$

The following example will illustrate the points above. Take T a diagonal operator expressed in a basis of H and defined this way : $T(h) = diag(a_i(h))$ where

$$a_i(h) = h\frac{\lambda_i}{\lambda_i + h} + \frac{h^{3/2}}{\lambda_i + h},$$

$h \downarrow 0$ and $\lambda \in l_1$. The reader will be easily convinced that we are in a situation similar to the one of theorem 1 : the i^{th} cell of the bounded operator T is asymptotically equivalent with $h + h^{3/2}/\lambda_i$. Here $v(h) = h$, $w(h) = h^{3/2}$ and $\mathcal{R} = diag(\lambda_i^{-1})$ is unbounded. Then it is elementary algebra to prove that :

$$\begin{aligned} \|T(h) - hI\|_\infty &\leq h^{1/2} \to 0, \\ \left\| \frac{T(h)}{h} \right\|_\infty &= 1 + h^{-1/2} \to +\infty. \end{aligned} \tag{20}$$

However if

$$\begin{aligned} a_i(h) &= h\frac{\lambda_i}{\lambda_i + h} + \frac{h^2}{\lambda_i + h}, \\ \left\| \frac{T(h)}{h} \right\|_\infty &= 2. \end{aligned}$$

Let us focus again on the local covariance operator Γ_K. If one tries to bound its norm a first attempt gives :

$$\begin{aligned} \|\Gamma_K\|_\infty &\leq \mathbb{E} \left\| K\left(\frac{\|X - x_0\|}{h}\right) ((X - x_0) \otimes (X - x_0)) \right\|_\infty \\ &= \mathbb{E}\left[K\left(\frac{\|X - x_0\|}{h}\right) \|X - x_0\|^2 \right]. \end{aligned} \tag{21}$$

The next theorem assesses that under mild conditions that hold in the gaussian framework this bound is not sharp. At this point we should turn back to Theorem 1, especially to the sequences $v(h)$ and $w(h)$ mentioned within this Theorem. The next Proposition provides first a bound then under an additional assumption an equivalent sequence for $v(h)$. The case of $w(h)$ will not be treated since the inspection of the method of proof would easily lead to similar results.

Proposition 7 *Let as above*

$$v(h) = \mathbb{E}\left(K\left(\frac{\|X - x_0\|}{h}\right)\|X - x_0\|\,\rho(\|X - x_0\|)\right)$$

then

$$\frac{v(h)}{\mathbb{E}\left[K\left(\frac{\|X-x_0\|}{h}\right)\|X - x_0\|^2\right]} \to 0.$$

Besides if ρ is regularly varying at 0 with positive index

$$v(h) \sim K(1)\,h\rho(h)\,F(h).$$

Remark 5 *The auxiliary functions appearing within Proposition 3 are both regularly varying with indices :*

$$d = (3 + 4\alpha)/(1 + 2\alpha)$$

for the first and

$$d = 1$$

for the second.

It is time for us to state the second main result of this paper. This Theorem is in a way complementary to the previous one. It provides an asymptotic first order development of Γ_K.

Theorem 2 *Suppose $\mathbf{A}_1$ and $\mathbf{A_2}$ hold. Let $\mathcal{V}_0$ be a fixed neighborhood of 0 and*

$$a_i = \sup_{t\in\mathcal{V}_0}\left|\frac{f_i(t + \langle x_0, e_i\rangle) - f_i(\langle x_0, e_i\rangle)}{f_i(\langle x_0, e_i\rangle)}\right|.$$

If

$$\sum_{i=1}^{+\infty} a_i^2 < +\infty \tag{22}$$

we have :

$$\|\Gamma_K - v(h)\,I\|_\infty = O(v(h)).$$

Remark 6 *By Proposition 7, the Theorem just provides a bound sharper than the rather "naïve" one at display (21) since obviously due to (10). A better result would be to obtain a second order term which would mean here to provide the explicit operator hidden behind the "$O(v(h))$". This has still to be done and holds perhaps under reinforced assumptions on both x_0 and the f_i's. But Theorem 1, as well as the examples treated above (see display (20)) let us claim that : "if ever $[\Gamma_K - v(h) I]/s(h)$ converges in norm, then necessarily $s(h) = w(h)$ and the limiting operator is then $\mathcal{R}$".*

Remark 7 *It has been seen above that we could not expect to obtain a $O(w(h))$ on the right, instead of $O(v(h))$ because in many situations the operator $\mathcal{R}$ will be unbounded. We see that the price to pay to enhance a "weak" result such as Theorem 1 to a "uniform" one such as Theorem 2 is a slower rate of decrease since obviously*

$$\frac{w(h)}{v(h)} \to 0.$$

Assumption (22) must be commented and illustrated by investigating some examples.

Example 1 *(Gauss) If X is gaussian, straightforward computations lead to :*

$$\frac{f_i(t + \langle x_0, e_i\rangle) - f_i(\langle x_0, e_i\rangle)}{f_i(\langle x_0, e_i\rangle)} = \exp\left(\frac{-t^2 - 2\langle x_0, e_i\rangle t}{2\lambda_i}\right) - 1$$

and if i is large enough (hence $\langle x_0, e_i\rangle$ small enough),

$$a_i = \left|\exp\left(\frac{\langle x_0, e_i\rangle^2}{2\lambda_i}\right) - 1\right|.$$

Then if $\frac{\langle x_0, e_i\rangle^2}{\lambda_i}$ tends to zero,

$$a_i \leq \frac{\langle x_0, e_i\rangle^2}{4\lambda_i}$$

and (22) holds when

$$\sum_{i=1}^{+\infty} \frac{\langle x_0, e_i\rangle^4}{\lambda_i^2} < +\infty. \tag{23}$$

Example 2 *(Laplace) If*

$$f_i(t) = \frac{1}{2\lambda_i} \exp\left(-\frac{|t|}{\lambda_i}\right)$$

we have

$$\frac{f_i(t + \langle x_0, e_i\rangle) - f_i(\langle x_0, e_i\rangle)}{f_i(\langle x_0, e_i\rangle)} = \exp\left(\frac{|\langle x_0, e_i\rangle| - |\langle x_0, e_i\rangle - t|}{2\lambda_i}\right) - 1$$

and assumption (22) holds when

$$\sum_{i=1}^{+\infty} \left[\exp\left(\frac{|\langle x_0, e_i\rangle|}{2\lambda_i}\right) - 1\right]^2 < +\infty,$$

hence when

$$\sum_{i=1}^{+\infty} \frac{\langle x_0, e_i\rangle^2}{\lambda_i^2} < +\infty.$$

Example 3 *(Unimodal densities) Since X is assumed to be centered and the f_i's are the densities of the random variables $\langle X - x_0, e_i\rangle$, we may extend both previous examples to a slightly more general situation. Indeed both variance and expectation of $\langle X - x_0, e_i\rangle$ tend to zero (they are respectively λ_i and $-\langle x_0, e_i\rangle$) and we can consider the case when f_i features a single peak (a mode) at $-\langle x_0, e_i\rangle$) and concentrates around* 0. *Then*

$$a_i = \left|\frac{f_i(0) - f_i(\langle x_0, e_i\rangle)}{f_i(\langle x_0, e_i\rangle)}\right|.$$

If we try to go beyond this relationship, a simple development of f_i around zero provides :

$$a_i = -\langle x_0, e_i\rangle \frac{f_i'(\langle x_0, e_i\rangle)}{f_i(\langle x_0, e_i\rangle)} + \frac{\langle x_0, e_i\rangle^2}{2} \frac{f_i''(c_i)}{f_i(\langle x_0, e_i\rangle)}$$

where c_i lies somewhere between 0 *and $\langle x_0, e_i\rangle$ Then it is plain to see that assumption (22) turns out to hold when : on a first hand (applying Cauchy-Schwartz inequality) condition* $\mathbf{C}_2$ *holds and on the other hand when*

$$\sup_i \frac{f_i''(c_i)}{f_i(\langle x_0, e_i\rangle)} < +\infty$$

which is not exactly assumption $\mathbf{C}_1'$ but which is not that far. Developping f_i up tof_i''' would let $\mathbf{C}_1'$ appear but would also creates an additional term. This does not prove that (22) is a necessary and sufficient condition for convergence of Γ_K in the sense of Theorem 2. But the closeness of (22) with the "weak" conditions $\mathbf{C}_1'$ and $\mathbf{C}_2$ shows that Theorem 2 is obtained under rather mild assumptions.

Remark 8 *It has been assumed throughout the paper that X has null expectation. When $\mathbb{E}X \neq 0$, the theorems continue to hold but assumptions such as (22) may implicitely involve this expectation itself. For instance (23) is replaced by :*

$$\sum_{i=1}^{+\infty} \frac{\langle x_0 + \mu, e_i \rangle^4}{\lambda_i^2} < +\infty$$

where $\mu = \mathbb{E}X$. More generally speaking in the situation when the X's are not centered, a condition involving x_0 should be replaced by the same condition involving $x_0 + \mu$.

4.3 Convergence of the empirical covariance operator

We go on with asymptotics for the empirical covariance operators, namely mean square error. This part is short since the most delicate issue was sorted out in the previous section.

Theorem 3 *When assumptions $\mathbf{A}_1$and $\mathbf{A}_2$ hold the following asymptotic results are true :*

$$\mathbb{E} \left\| \Gamma_{K,n} - \Gamma_K \right\|_\infty^2 = \frac{h^4 F(h)}{n} K^2(1)(1 + o(1))$$

The same sort of results hold for the eigenelements. Recall that $(\lambda_{n,p})_{p\in\mathbb{N}}$ (resp. $(\lambda_p)_{p\in\mathbb{N}}$) stands for the eigenvalues of $\Gamma_{K,n}$ (resp. Γ_K) and $(\pi_{n,p})_{p\in\mathbb{N}}$ (resp. $(\pi_p)_{p\in\mathbb{N}}$) stands for the associated eigenprojectors. The following Theorem estimates the rate of decrease to zero for the eigenelements.

Corollary 1 *Under the same conditions as in Theorem 3, for fixed $p \in \mathbb{N}$,*

$$\mathbb{E}(\lambda_{n,p} - \lambda_p)^2 = O\left(\frac{h^4 F(h)}{n}\right)$$
$$\mathbb{E} \left\| \pi_{n,p} - \pi_p \right\|_\infty^2 = O\left(\frac{h^4 F(h)}{n}\right)$$

Only a sketch of the proof of this Corollary is given since it may be seen as a by-product of an article by Mas and Menneteau (2003). Under simple additional assumptions, exact constants could be computed in both above displays by applying the formulas that appear in Theorem 1.2 p.129 in their article. But these computations are beyond the scope of this article : they make it necessary to introduce and explain Kato's perturbation theory as well as the associated functional calculus for linear operators. The interested reader is referred to Kato (1976), Dunford-Schwartz (1988), Gohberg, Goldberg and Kaashoek (1991).

4.4 Conclusion and perspectives

We proposed a first approach to what we named "local covariance operators". This article does not aim at giving an exhaustive list of their features but provides some clues for further and deeper study. For instance we can take for granted that small probabilities naturally appear when estimating the rates of convergence and that the class of Γ-varying functions (at 0) provides an accurate setting. It also turns out from Theorem 1 that the asymptotic behaviour of Γ_K is quite unusual and let appear, through $\mathcal{R}$, an unbounded operator operator whereas Γ_K tend s to zero.

Several issues will have to be addressed in the future. We can list some of them. The case of first order truncated moments, that is :

$$\mathbb{E}\left[(X-x_0)\,K\left(\left\|\frac{X-x_0}{h}\right\|\right)\right],$$

could certainly be studied in the framework of this article and with the same computational techniques. Almost sure convergence as well as weak convergence (convergence in distribution) could be adressed by following the same lines.

Another crucial issue is the existence and the properties of the inverse of Γ_K when it exists. Indeed let us introduce the nonparametric regression model for functional random variables :

$$y = r\,(X) + \varepsilon,$$

where $(y, X) \in \mathbb{R} \times H$. Investigating a pointwise estimate $\widehat{r}\,(x_0)$ of $r\,(x_0)$ by local linear methods leads to finding the inverse (or a pseudo-inverse) of Γ_K.

5 Mathematical derivations

For any $x = \sum x_k e_k$ in H and for $(i, j) \in \mathbb{N}^2$, $i \neq j$, set

$$\|x\|_{\neq i}^2 = \sum_{k\neq i} x_k^2$$
$$\|x\|_{\neq ij}^2 = \sum_{k\neq i,j} x_k^2$$

and denote $f_{\neq i}$ the density of $\|X\|_{\neq i}$ as well as $f_{\neq ij}$ the density of $\|X\|_{\neq ij}$. It is clear that when assumption $\mathbf{A}_1$ holds $\langle X, e_i\rangle$ and $\|X\|_{\neq i}$ are independent random variables.

In order to alleviate the notations, within the proofs -unless explicitely mentioned- X will stand for $X-x_0$ (x_0 is dropped since it is fixed but we keep aware that all our results and notations, especially small ball probabilities depend on x_0) and f_i for the density of $\langle X-x_0, e_i\rangle$.

5.1 Preliminary material

We begin with preliminary Lemmas which are assessed in a general setting and will be applied later. We recall the definition of the Gamma function Γ :

$$\Gamma(u)=\int_0^{+\infty} s^{u-1}\exp(-s)\,ds.$$

Lemma 1 *If f belongs to the class Γ with auxiliary function ρ, then for all $p\in\mathbb{N}$,*

$$\int_0^1 \frac{t^p}{\sqrt{1-t^2}} f\left(s\sqrt{1-t^2}\right) dt \underset{s\to 0}{\sim} 2^{\frac{p-1}{2}}\Gamma\left(\frac{p+1}{2}\right) f(s)\left(\frac{\rho(s)}{s}\right)^{\frac{p+1}{2}}.$$

Proof. We start with the following change of variable : $s\sqrt{1-t^2}=s-\rho(s)\,x$

$$\int_0^1 \frac{t^p}{\sqrt{1-t^2}} f\left(s\sqrt{1-t^2}\right) dt$$

$$=\int_0^{s/\rho(s)} \frac{\rho(s)}{s}\left(\frac{\rho(s)}{s}x\left(2+\frac{\rho(s)}{s}\right)\right)^{\frac{p-1}{2}} f(s-\rho(s)\,x)\,dx$$

$$=2^{\frac{p-1}{2}}\left(\frac{\rho(s)}{s}\right)^{\frac{p+1}{2}} f(s)\int_0^{s/\rho(s)}\left(x\left(1+\frac{\rho(s)}{2s}\right)\right)^{\frac{p-1}{2}}\frac{f(s-\rho(s)\,x)}{f(s)\exp(-x)}\exp(-x)\,dx.$$

To conclude it suffices to prove that

$$\frac{\int_0^{s/\rho(s)}\left(x\left(1+\frac{\rho(s)}{2s}\right)\right)^{\frac{p-1}{2}}\frac{f(s-\rho(s)x)}{f(s)\exp(-x)}\exp(-x)\,dx-\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}\underset{s\to 0}{\to} 0.$$

But the numerator may be rewritten

$$\int_0^{s/\rho(s)}\left[\left(\left(1+\frac{\rho(s)}{2s}\right)\right)^{\frac{p-1}{2}}\frac{f(s-\rho(s)\,x)}{f(s)\exp(-x)}-1\right]x^{\frac{p-1}{2}}\exp(-x)\,dx. \qquad (24)$$

Let us study briefly the sequence of functions $\frac{f(s-\rho(s)x)}{f(s)\exp(-x)}$. The sign of the first order derivative is the sign of :

$$\frac{f(s-\rho(s)\,x)}{f'(s-\rho(s)\,x)}-\rho(s)$$

By Theorem 3.10.11 in Bingham, Goldie, Teugels (1987), f' is Γ-varying with same auxiliary function as f and by Corollary 3.10.5 (b) p.177 ibidem we know that

$$\rho\left(s-\rho\left(s\right)x\right)=\frac{f\left(s-\rho\left(s\right)x\right)}{f'\left(s-\rho\left(s\right)x\right)}$$

Then since $\rho\left(0\right)=0$ and $\rho\geq 0$, ρ is strictly increasing in a neighborhood of zero and $\rho\left(s-\rho\left(s\right)x\right)\leq\rho\left(s\right)$, $\frac{f(s-\rho(s)x)}{f(s)\exp(-x)}$ is nonincreasing on $\left[0,s/\rho\left(s\right)\right]$ (as a function of x) hence

$$\sup_{x\in[0,s/\rho(s)]}\frac{f\left(s-\rho\left(s\right)x\right)}{f\left(s\right)\exp\left(-x\right)}=1$$

Now together with display (8) and (10) we can apply Lebesgue's dominated convergence Theorem to which completes the proof of (24) the Lemma. ■

The next lemma

If U and $V$are two real valued random variables, $f_{U,V}$ denotes the joint density of the couple (U,V). We need to compute four densities :

Lemma 2 *We have :*

$$f_{\langle X,e_i\rangle,\|X\|}\left(u,v\right)=\frac{v}{\sqrt{v^2-u^2}}f_i\left(u\right)f_{\neq i}\left(\sqrt{v^2-u^2}\right)1\!\!1_{\{v\geq|u|\}},\tag{25}$$

$$f_{\|X\|}\left(v\right)=v\int_{-1}^{1}\frac{f_i\left(vt\right)}{\sqrt{1-t^2}}f_{\neq i}\left(v\sqrt{1-t^2}\right)dt\tag{26}$$

and

$$f_{\langle X,e_i\rangle,\langle X,e_j\rangle,\|X\|}\left(t,u,v\right)$$
$$=\frac{v}{\sqrt{v^2-u^2-t^2}}f_i\left(t\right)f_j\left(u\right)f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right)1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}},\tag{27}$$
$$f_{\|X\|}\left(v\right)$$
$$=v^2\int_0^{2\pi}\int_0^1\frac{x}{\sqrt{1-x^2}}f_i\left(vx\cos\theta\right)f_j\left(vx\sin\theta\right)f_{\neq ij}\left(v\sqrt{1-x^2}\right)dxd\theta.$$

Proof. We only compute the first density since we could get the second by integration. The third and fourth could be obtained the same way. Let h be any bounded measurable function.

$$\begin{aligned}\mathbb{E}h\left(\langle X,e_i\rangle,\|X\|\right)&=\int h\left(x_i,\sqrt{x_i^2+y_i^2}\right)f_i\left(x_i\right)f_{\neq i}\left(y_i\right)dx_idy_i\\&=\int v\frac{h\left(u,v\right)}{\sqrt{v^2-u^2}}f_i\left(u\right)f_{\neq i}\left(\sqrt{v^2-u^2}\right)1\!\!1_{\{v\geq|u|\}}dudv\\&=\int h\left(u,v\right)f_{\langle X,e_i\rangle,\|X\|}\left(u,v\right)dudv.\end{aligned}$$

Identifying both last terms we get

$$f_{\langle X,e_i\rangle,\|X\|}(u,v)=\frac{v}{\sqrt{v^2-u^2}}f_i(u)\,f_{\neq i}\left(\sqrt{v^2-u^2}\right)1\!\!1_{\{v\geq|u|\}}.$$

Integrating this density with respect to the variable u yields $f_{\|X\|}(v)$ as in (26). ■

Lemma 3 *The following hold :*

$$f_{\|X\|}(v)\underset{0}{\sim}\Gamma\left(\frac{1}{2}\right)\sqrt{2v\rho(v)}f_i(0)\,f_{\neq i}(v),$$
$$f_{\|X\|}(v)\underset{0}{\sim}2\pi f_i(0)\,f_j(0)\,v\rho(v)\,f_{\neq ij}(v).$$

Besides if $f_{\|X\|}$, $f_{\neq i}$ and $f_{\neq ij}$ are Γ-varying for all i and j then they have all ρ as auxiliary function.

Proof. We restrict to proving the Lemma for $f_{\neq i}$. From Lemma 1 and (26) we get

$$\begin{aligned}f_{\|X\|}(v)&=v\int_{-1}^{1}\frac{f_i(vt)}{\sqrt{1-t^2}}f_{\neq i}\left(v\sqrt{1-t^2}\right)dt\\&\underset{0}{\sim}2vf_i(0)\int_{0}^{1}\frac{1}{\sqrt{1-t^2}}f_{\neq i}\left(v\sqrt{1-t^2}\right)dt\\&\underset{0}{\sim}\Gamma\left(\frac{1}{2}\right)\sqrt{2v\rho_i(v)}f_i(0)\,f_{\neq i}(v)\end{aligned}$$

where ρ_i denotes the auxiliary function of $f_{\neq i}$. Now we also have for all $x\geq 0$:

$$\begin{aligned}&\frac{f_{\|X\|}(v+\rho_i(v)\,x)}{f_{\|X\|}(v)}\underset{0}{\sim}\frac{\sqrt{2\,(v+\rho_i(v)\,x)\,\rho_i(v+\rho_i(v)\,x)}f_i(0)\,f_{\neq i}(v+\rho_i(v)\,x)}{\sqrt{2v\rho_i(v)}f_i(0)\,f_{\neq i}(v)}\\&\underset{0}{\sim}\sqrt{\frac{(v+\rho_i(v)\,x)\,\rho_i(v+\rho_i(v)\,x)}{v\rho_i(v)}}\exp(x).\end{aligned}$$

The term

$$\frac{(v+\rho_i(v)\,x)\,\rho_i(v+\rho_i(v)\,x)}{v\rho_i(v)}=\left(1+\frac{\rho_i(v)\,x}{v}\right)\left(\frac{\rho_i(v+\rho_i(v)\,x)}{\rho_i(v)}\right)$$

tends to 1 by **Fact 2**. Finally

$$\frac{f_{\|X\|}(v+\rho_i(v)\,x)}{f_{\|X\|}(v)}\to\exp(x)$$

and ρ_i is also the auxiliary function for $f_{\|X\|}$. Since $f_{\|X\|}$ is also Γ-varying with auxiliary function ρ, we can set $\rho_i = \rho$ (the auxiliary function is unique up to an asymptotic equivalence, see Corollary 3.10.5 (b) p.177 in Bingham, Goldie, Teugels (1987)). The same steps would lead us to the second part of the Lemma. ■

5.2 Proof of cell-by-cell results

By "cell" we just mean that, identifiying Γ_K with an infinite matrix, we consider in this subsection asymptotics for $\langle \Gamma_K e_i, e_j \rangle$. We study first the diagonal of Γ_K :

Proposition 8 *Fix the index $i \in \mathbb{N}$:*

$$\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X, e_i\rangle^2\right] \underset{h\to 0}{\sim} \int_0^h v\rho(v) K\left(\frac{v}{h}\right) f_{\|X\|}(v)\,dv + \frac{f_i''(0)}{f_i(0)}\int_0^h v^2\rho^2(v) K\left(\frac{v}{h}\right) f_{\|X\|}(v)\,dv. \tag{28}$$

Corollary 2 *From the above we deduce that :*

$$\mathbb{E}\left[\langle X, e_i\rangle^2 \mid \|X\| = v\right] \sim v\rho(v).$$

Proof. We start from the joint density at display (25) :

$$\begin{aligned}
&\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X, e_i\rangle^2\right] \\
&= \int\int K\left(\frac{v}{h}\right) u^2 \frac{v}{\sqrt{v^2-u^2}} f_i(u) f_{\neq i}\left(\sqrt{v^2-u^2}\right) 1\!\!1_{\{h\geq v\geq |u|\}}\,du\,dv \\
&= \int_0^h vK\left(\frac{v}{h}\right)\left(\int_{-v}^{v} \frac{u^2}{\sqrt{v^2-u^2}} f_i(u) f_{\neq i}\left(\sqrt{v^2-u^2}\right) du\right) dv \qquad (29) \\
&= \int_0^h v^3 K\left(\frac{v}{h}\right)\left(\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} f_i(xv) f_{\neq i}\left(v\sqrt{1-x^2}\right) dx\right) dv.
\end{aligned}$$

Setting :

$$I_i(v) = \frac{\int_{-1}^{1} \frac{vx^2}{\sqrt{1-x^2}} f_i(xv) f_{\neq i}\left(v\sqrt{1-x^2}\right) dx}{\int_{-1}^{1} \frac{v}{\sqrt{1-x^2}} f_i(vx) f_{\neq i}\left(v\sqrt{1-x^2}\right) dx},$$

we have

$$\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X, e_i\rangle^2\right] = \int_0^h v^2 K\left(\frac{v}{h}\right) I_i(v) f_{\|X\|}(v)\, dv. \tag{30}$$

We focus on $I(v)$ and prove that :

$$I_i(v) \sim \frac{\rho(v)}{v}\left(1 + v\rho(v)\frac{f_i''(0)}{f_i(0)}\right). \tag{31}$$

It is clear that from (30) and (31) we can derive (28).
Now from $f_i(xv) = f_i(0) + vxf_i'(0) + \frac{v^2x^2}{2} f_i''(0) + o(v^2)$, we can develop $I_i(v)$. Setting :

$$\mathcal{J}_p(v) = \left(\int_{-1}^{1} \frac{x^p}{\sqrt{1-x^2}} f_{\neq i}\left(v\sqrt{1-x^2}\right) dx\right) \quad p \in \mathbb{N},$$

we get :

$$\begin{aligned}
I_i(v) &= \frac{f_i(0)\mathcal{J}_2(v) + \frac{v^2}{2} f_i''(0)\mathcal{J}_4(v)}{f_i(0)\mathcal{J}_0(v) + \frac{v^2}{2} f_i''(0)\mathcal{J}_2(v)}(1+o(1)) \\
&= \frac{\mathcal{J}_2(v)}{\mathcal{J}_0(v)} \frac{1 + \frac{v^2}{2}\frac{f_i''(0)}{f_i(0)}\frac{\mathcal{J}_4(v)}{\mathcal{J}_2(v)}}{1 + \frac{v^2}{2}\frac{f_i''(0)}{f_i(0)}\frac{\mathcal{J}_2(v)}{\mathcal{J}_0(v)}}(1+o(1)) \\
&= \frac{\mathcal{J}_2(v)}{\mathcal{J}_0(v)}\left(1 + \frac{v^2}{2}\frac{f_i''(0)}{f_i(0)}\left(\frac{\mathcal{J}_4(v)}{\mathcal{J}_2(v)} - \frac{\mathcal{J}_2(v)}{\mathcal{J}_0(v)}\right)\right)(1+o(1)).
\end{aligned}$$

Then we invoke Lemma 1 and Lemma 3 to get :

$$\begin{aligned}
\frac{\mathcal{J}_2(v)}{\mathcal{J}_0(v)} &= \frac{\rho(v)}{v}(1+o(1)), \\
\frac{\mathcal{J}_4(v)}{\mathcal{J}_2(v)} &= 3\frac{\rho(v)}{v}(1+o(1)),
\end{aligned}$$

hence

$$I_i(v) = \frac{\rho(v)}{v}\left(1 + \frac{f_i''(0)}{f_i(0)} v\rho(v)\right)(1+o(1))$$

which finishes the proof of the Proposition 8. ■

Proposition 9 *Let us take $i \neq j$ in $\mathbb{N}$, we have :*

$$\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X,e_i\rangle\langle X,e_j\rangle\right]\sim\frac{f_i'(0)\,f_j'(0)}{f_i(0)\,f_j(0)}\int_0^h K\left(\frac{v}{h}\right)v^2\rho^2(v)\,f_{\|X\|}(v)\,dv.$$

Proof. The proof basically follows the same lines as the previous Proposition with a few changes :

$$\begin{aligned}
&\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X,e_i\rangle\langle X,e_j\rangle\right]\\
&=\int\int\int K\left(\frac{v}{h}\right)\frac{utv}{\sqrt{v^2-u^2-t^2}}f_i(t)\,f_j(u)\,f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right)1\!\!1_{\left\{h\geq v\geq\sqrt{u^2+t^2}\right\}}dudvdt\\
&=\int_0^h vK\left(\frac{v}{h}\right)\left(\int\int\frac{ut}{\sqrt{v^2-u^2-t^2}}f_i(t)\,f_j(u)\,f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right)1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}}dudt\right)dv\\
&=\int_0^h vK\left(\frac{v}{h}\right)\\
&\times\left(\int\int\frac{r^3\sin\theta\cos\theta}{\sqrt{v^2-r^2}}f_i(r\cos\theta)\,f_j(r\sin\theta)\,f_{\neq ij}\left(\sqrt{v^2-r^2}\right)1\!\!1_{\{v\geq r\}}drd\theta\right)dv
\end{aligned}\tag{32a}$$

$$\begin{aligned}
&=\int_0^h v^4K\left(\frac{v}{h}\right)\\
&\times\left(\int\int\frac{x^3\sin\theta\cos\theta}{\sqrt{1-x^2}}f_i(xv\cos\theta)\,f_j(xv\sin\theta)\,f_{\neq ij}\left(v\sqrt{1-x^2}\right)1\!\!1_{\{1\geq x\geq 0\}}dxd\theta\right)dv\\
&=\int_0^h v^2K\left(\frac{v}{h}\right)J(v)\,f_{\|X\|}(v)\,dv,
\end{aligned}\tag{32b}$$

where

$$J(v)=\frac{\int_0^{2\pi}\int_0^1\frac{x^3\sin\theta\cos\theta}{\sqrt{1-x^2}}f_i(xv\cos\theta)\,f_j(xv\sin\theta)\,f_{\neq ij}\left(v\sqrt{1-x^2}\right)dxd\theta}{\int_0^{2\pi}\int_0^1\frac{x}{\sqrt{1-x^2}}f_i(vx\cos\theta)\,f_j(vx\sin\theta)\,f_{\neq ij}\left(v\sqrt{1-x^2}\right)dxd\theta}.\tag{33}$$

At last we prove that :

$$J(v)\sim\frac{f_i'(0)\,f_j'(0)}{f_i(0)\,f_j(0)}\rho^2(v)$$

and go quickly through it :

$$
\begin{aligned}
J(v) &= \frac{\int_0^{2\pi}\int_0^1 \frac{x^3\sin\theta\cos\theta}{\sqrt{1-x^2}} f_i(xv\cos\theta)\, f_j(xv\sin\theta)\, f_{\neq ij}\left(v\sqrt{1-x^2}\right) 1\!\!1_{\{1\geq x\geq 0\}} dxd\theta}{\int_0^{2\pi}\int_0^1 \frac{x}{\sqrt{1-x^2}} f_i(vx\cos\theta)\, f_j(vx\sin\theta)\, f_{\neq ij}\left(v\sqrt{1-x^2}\right) dxd\theta} \\
&= \frac{\int_0^{2\pi}\int_0^1 \frac{x^3\sin\theta\cos\theta}{\sqrt{1-x^2}} f_i(xv\cos\theta)\, f_j(xv\sin\theta)\, f_{\neq ij}\left(v\sqrt{1-x^2}\right) 1\!\!1_{\{1\geq x\geq 0\}} dxd\theta}{f_i(0)\, f_j(0)\int_0^{2\pi}\int_0^1 \frac{x}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dxd\theta}(1+o(1)) \\
&= \frac{v^2}{2\pi}\frac{f_i'(0)\, f_j'(0)}{f_i(0)\, f_j(0)}\frac{\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta\int_0^1 \frac{x^5}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx}{\int_0^1 \frac{x}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx}(1+o(1)) \\
&= \frac{v^2}{2\pi}\frac{f_i'(0)\, f_j'(0)}{f_i(0)\, f_j(0)}\frac{\pi}{4}\frac{\mathcal{J}_5(v)}{\mathcal{J}_1(v)}(1+o(1)) \\
&= \frac{v^2}{8}\frac{f_i'(0)\, f_j'(0)}{f_i(0)\, f_j(0)}\frac{2^2\Gamma(3)\left(\frac{\rho(v)}{v}\right)^3}{\Gamma(1)\left(\frac{\rho(v)}{v}\right)}(1+o(1)) \\
&= \frac{f_i'(0)\, f_j'(0)}{f_i(0)\, f_j(0)}\rho^2(v)(1+o(1))
\end{aligned}
$$

This last step ends the proof of Proposition 9. ■

Proof of Theorem 1 : The proof of the Theorem stems from Propositions 8 and 9. For instance,

$$\int_0^h v\rho(v)\, K\left(\frac{v}{h}\right) f_{\|X\|}(v)\, dv = \mathbb{E}\left[\|X\|\,\rho(\|X\|)\, K\left(\frac{\|X\|}{h}\right)\right].$$

5.3 Proof of norm results

We decompose Γ_K into two terms : a purely diagonal one and non-diagonal one. In fact :

$$
\begin{aligned}
\left\langle \Gamma_K^d e_i, e_i\right\rangle &= \left\langle \Gamma_K e_i, e_i\right\rangle \\
\left\langle \Gamma_K^d e_i, e_j\right\rangle &= 0 \quad i\neq j
\end{aligned}
$$

and

$$\Gamma_K^{\#d} = \Gamma_K - \Gamma_K^d. \tag{34}$$

We first prove that :

Lemma 4

$$\left\|\Gamma_K^d - v(h)\, I\right\|_\infty = O(v(h))$$

It suffices to prove that :

$$\sup_{i\in\mathbb{N}}\left|\left\langle \Gamma_K e_i,e_i\right\rangle - v\left(h\right)\right| = O\left(v\left(h\right)\right)$$

Let us denote :

$$\varphi_i\left(t\right) = \frac{f_i\left(t\right) - f_i\left(0\right)}{f_i\left(0\right)}$$

We have

$$\sup_{t\in\mathcal{V}_0}\left|\varphi_i\left(t\right)\right| = a_i$$

where a_i was introduced in Theorem 2.

We start from (30) and (31) :

$$\begin{aligned}
&\left\langle \Gamma_K e_i,e_i\right\rangle - v\left(h\right)\\
&= \mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\left\langle X,e_i\right\rangle^2\right] - v\left(h\right)\\
&= \int_0^h t^2 K\left(\frac{t}{h}\right) I_i\left(t\right) f_{\|X\|}\left(t\right)dt - \int_0^h t\rho\left(t\right) K\left(\frac{t}{h}\right) f_{\|X\|}\left(t\right)dt\\
&= \int_0^h t^2 K\left(\frac{t}{h}\right)\left(I_i\left(t\right) - \frac{\rho\left(t\right)}{t}\right) f_{\|X\|}\left(t\right)dt \qquad (35)
\end{aligned}$$

We will first focus on :

$$I_i\left(t\right) - \frac{\rho\left(t\right)}{t} = I_i\left(t\right) - \frac{\mathcal{J}_2\left(t\right)}{\mathcal{J}_0\left(t\right)} + \frac{\mathcal{J}_2\left(t\right)}{\mathcal{J}_0\left(t\right)} - \frac{\rho\left(t\right)}{t}$$

Let us develop

$$\begin{aligned}
&I_i\left(t\right) - \frac{\mathcal{J}_2\left(t\right)}{\mathcal{J}_0\left(t\right)}\\
&= \frac{\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} f_i\left(xt\right) f_{\neq i}\left(t\sqrt{1-x^2}\right)dx}{\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} f_i\left(tx\right) f_{\neq i}\left(t\sqrt{1-x^2}\right)dx} - \frac{\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} f_{\neq i}\left(t\sqrt{1-x^2}\right)dx}{\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} f_{\neq i}\left(t\sqrt{1-x^2}\right)dx}
\end{aligned}$$

It is plain that

$$I_i\left(t\right) = \frac{\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}}\left(1+\varphi_i\left(tx\right)\right) f_{\neq i}\left(t\sqrt{1-x^2}\right)dx}{\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\left(1+\varphi_i\left(tx\right)\right) f_{\neq i}\left(t\sqrt{1-x^2}\right)dx}$$

Now denote

$$\mathcal{J}_0^*\left(t\right) = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\varphi_i\left(tx\right) f_{\neq i}\left(t\sqrt{1-x^2}\right)dx$$

$$\mathcal{J}_2^*\left(t\right) = \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}}\varphi_i\left(tx\right) f_{\neq i}\left(t\sqrt{1-x^2}\right)dx$$

then

$$I_i(t) = \frac{\mathcal{J}_0(t) + \mathcal{J}_0^*(t)}{\mathcal{J}_2(t) + \mathcal{J}_2^*(t)}$$

hence

$$I_i(t) - \frac{\mathcal{J}_2(t)}{\mathcal{J}_0(t)} = \frac{\mathcal{J}_2^*(t)\,\mathcal{J}_0(t) - \mathcal{J}_2(t)\,\mathcal{J}_0^*(t)}{\mathcal{J}_0(t)\left(\mathcal{J}_0(t) + \mathcal{J}_0^*(t)\right)}$$

We are going to use the following inequalities :

$$\begin{gathered}
\left|\mathcal{J}_0^*(t)\right| \leq a_i \mathcal{J}_0(t) \\
\left|\mathcal{J}_2^*(t)\right| \leq a_i \mathcal{J}_2(t) \\
\mathcal{J}_0(t) + \mathcal{J}_0^*(t) \geq \mathcal{J}_0(t)(1 - a_i) \geq 0.
\end{gathered}$$

They yield :

$$\left|I_i(t) - \frac{\mathcal{J}_2(t)}{\mathcal{J}_0(t)}\right| \leq \frac{2a_i \mathcal{J}_2(t)}{(1 - a_i)\,\mathcal{J}_0(t)}$$

Turning back to (35) we get :

$$\begin{aligned}
&\left|\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X, e_i\rangle^2\right] - v(h)\right| \\
&\leq \frac{2a_i}{1 - a_i}\int_0^h t^2 K\left(\frac{t}{h}\right)\frac{\mathcal{J}_2(t)}{\mathcal{J}_0(t)} f_{\|X\|}(t)\,dt + \int_0^h t^2 K\left(\frac{t}{h}\right)\left(\frac{\mathcal{J}_2(t)}{\mathcal{J}_0(t)} - \frac{\rho(t)}{t}\right) f_{\|X\|}(t)\,dt
\end{aligned}$$

Remind that, in order to alleviate notations we remove the index i $\mathcal{J}_0(t)$ and $\mathcal{J}_2(t)$. Assume that

$$\limsup_{t \to 0} \sup_i \frac{\mathcal{J}_2(t)}{\mathcal{J}_0(t)} \frac{t}{\rho(t)} \leq M \tag{36}$$

then we get

$$\begin{aligned}
\limsup_{h \to 0} \sup_i \left|\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X, e_i\rangle^2\right] - v(h)\right| &\leq \left(\frac{2a_i M}{1 - a_i} + M + 1\right)\int_0^h t\rho(t)\,K\left(\frac{t}{h}\right) f_{\|X\|}(t)\,dt \\
&\leq M' v(h)
\end{aligned}$$

where M' is some constant which does not depend on i or h. This finally entails Lemma 4.In order to finish the proof we prove (36) now as a Lemma.

Lemma 5 *We have*

$$\limsup_{t \to 0} \sup_i \frac{\mathcal{J}_2(t)}{\mathcal{J}_0(t)} \frac{t}{\rho(t)} \leq M$$

Proof. We need to turn back to the proof of Lemma (1) from which we pick :

$$\frac{\mathcal{J}_2(t)}{\mathcal{J}_0(t)} = 2\frac{\rho(t)}{t}\frac{\int_0^{t/\rho(t)}\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx}{\int_0^{t/\rho(t)}\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx}$$

And it suffices to get :

$$\limsup_{t\to 0}\sup_i \frac{\int_0^{t/\rho(t)}\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx}{\int_0^{t/\rho(t)}\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx} \leq M \tag{37}$$

We start with noting that

$$\begin{aligned}
&\frac{\int_0^{t/\rho(t)}\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx}{\int_0^{t/\rho(t)}\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx}\\
&= \frac{\int_0^1\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx+\int_1^{t/\rho(t)}\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx}{\int_0^1\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx+\int_1^{t/\rho(t)}\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx}\\
&\leq \frac{\int_0^1\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx+\int_1^{t/\rho(t)}\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx}{\int_0^1\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx+\int_1^{t/\rho(t)}\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx}\\
&\leq 1+\frac{\int_1^{t/\rho(t)}\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx}{\int_0^1\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx}.
\end{aligned}$$

We deal with the denumerator

$$\int_0^1\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx \geq f_{\neq i}(t-\rho(t))\int_0^1\frac{1}{\sqrt{x}}dx = 2f_{\neq i}(t-\rho(t))$$

and

$$\frac{\int_1^{t/\rho(t)}\sqrt{x}f_{\neq i}(t-\rho(t)x)\,dx}{\int_0^1\frac{1}{\sqrt{x}}f_{\neq i}(t-\rho(t)x)\,dx} \leq \frac{1}{2}\int_1^{t/\rho(t)}\sqrt{x}\frac{f_{\neq i}(t-\rho(t)x)}{f_{\neq i}(t-\rho(t))}dx$$

Then we we rewrite

$$\begin{aligned}
\frac{f_{\neq i}(t-\rho(t)x)}{f_{\neq i}(t-\rho(t))} &= \frac{f_{\neq i}\left((t-\rho(t))+\rho(t-\rho(t))\frac{\rho(t)}{\rho(t-\rho(t))}(1-x)\right)}{f_{\neq i}(t-\rho(t))}\\
&= \frac{f_{\neq i}((t-\rho(t))+\rho(t-\rho(t))\,y_t)}{f_{\neq i}(t-\rho(t))}
\end{aligned}$$

with $y_t = \frac{\rho(t)}{\rho(t-\rho(t))}(1-x)$. It is plain that, if $y_t = y$ does not depend on t that

$$\frac{f_{\neq i}\left(\left(t-\rho\left(t\right)\right)+\rho\left(t-\rho\left(t\right)\right)y\right)}{f_{\neq i}\left(t-\rho\left(t\right)\right)} \underset{t\to 0}{\to} \exp(y)$$

Since $\frac{\rho(t)}{\rho(t-\rho(t))} \to 1$ (see display 2.11.2 in Bingham, Goldie, Teugels (1987)) and by Proposition 3.10.2 ibidem,

$$\frac{f_{\neq i}\left(\left(t-\rho\left(t\right)\right)+\rho\left(t-\rho\left(t\right)\right)y_t\right)}{f_{\neq i}\left(t-\rho\left(t\right)\right)} \underset{t\to 0}{\to} \exp(1-x).$$

From this remark, proving the Lemma finally comes down to proving

$$\sup_{i,t}\int_1^{t/\rho(t)} \sqrt{x}\frac{f_{\neq i}\left(t-\rho\left(t\right)x\right)}{f_{\neq i}\left(t\right)\exp\left(-x\right)}\exp\left(-x\right)\exp dx \leq M. \tag{38}$$

We focus on

$$\frac{f_{\neq i}\left(t-\rho\left(t\right)x\right)}{f_{\neq i}\left(t\right)\exp\left(-x\right)}.$$

By the representation Theorem 3.10.8 in Bingham, Goldie, Teugels (1987) and since all functions $f_{\neq i}$ have he same auxiliary function ρ :

$$\frac{f_{\neq i}\left(t-\rho\left(t\right)x\right)}{f_{\neq i}\left(t\right)\exp\left(-x\right)} = \exp\left[x - \int_{t-\rho(t)x}^{t}\frac{du}{\rho\left(u\right)}\right].$$

Now it is easily seen that ρ is continuous and nondecreasing in a neighborhood of 0 (see the remark about this fact within the proof of Lemma 1). Hence for t small enough

$$-x\frac{\rho\left(t\right)}{\rho\left(t-\rho\left(t\right)x\right)} \leq -\int_{t-\rho(t)x}^{t}\frac{du}{\rho\left(u\right)} \leq -x$$

and

$$\frac{f_{\neq i}\left(t-\rho\left(t\right)x\right)}{f_{\neq i}\left(t\right)\exp\left(-x\right)} \leq 1$$

for all $0 \leq x \leq t/\rho\left(t\right)$ and for t close to zero. At last (38) holds, hence (37) hence Lemma 5. ■

Proof of Theorem 2 :

In order to finish the proof of the Theorem we have to cope now with $\Gamma_K^{\#d}$ (see (34)) since Lemma 4 provides a fair estimate with Γ_K^d. We have to prove that :

$$\left\|\Gamma_K^{\#d}\right\|_\infty = O\left(v\left(h\right)\right)$$

Actually we will prove that this bound is true in Hilbert-Schmidt norm since the Hilbert-Schmidt norm of $\Gamma_K^{\#d}$ is easier to handle than its sup-norm.

We have (32b) in mind and we start from :

$$J(v) = \frac{\int\int \frac{ut}{\sqrt{v^2-u^2-t^2}} f_i(t) f_j(u) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}{\int\int \frac{v}{\sqrt{v^2-u^2-t^2}} f_i(t) f_j(u) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}$$

This display was obtained from (33) by a change of variable. Introducing again the function φ_i (see above) :

$$f_i(t) f_j(u) = f_i(0) f_j(0) (1+\varphi_i(t)) (1+\varphi_j(u))$$

hence

$$\begin{aligned} J(v) &= \frac{\int\int \frac{ut}{\sqrt{v^2-u^2-t^2}} (1+\varphi_i(t)) (1+\varphi_j(u)) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}{\int\int \frac{v}{\sqrt{v^2-u^2-t^2}} (1+\varphi_i(t)) (1+\varphi_j(u)) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt} \\ &= \frac{\int\int \frac{ut}{\sqrt{v^2-u^2-t^2}} \varphi_i(t) \varphi_j(u) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}{v\int\int \frac{(1+\varphi_i(t)+\varphi_j(u)+\varphi_i(t)\varphi_j(u))}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt} \end{aligned}$$

We see that

$$J(v) = \frac{\int\int \frac{ut}{\sqrt{v^2-u^2-t^2}} \varphi_i(t) \varphi_j(u) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}{\int\int \frac{v}{\sqrt{v^2-u^2-t^2}} (1+\varphi_i(t)) (1+\varphi_j(u)) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}$$

since obviously

$$\int\int \frac{ut}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt = 0$$

$$\int\int \frac{ut}{\sqrt{v^2-u^2-t^2}} \varphi_i(t) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt = 0.$$

We treat the numerator and the denumerator separatedly. Let

$$\mathcal{N} = \int\int \frac{ut}{\sqrt{v^2-u^2-t^2}} \varphi_i(t) \varphi_j(u) f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt$$

$$\mathcal{D} = \int\int v \frac{(1+\varphi_i(t)+\varphi_j(u)+\varphi_i(t)\varphi_j(u))}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt.$$

Denoting again

$$a_i = \sup_{t\in\mathcal{V}_0} |\varphi_i(t)|,$$

we have

$$|\mathcal{N}| \leq a_i a_j \int\int \frac{|ut|}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt$$

and

$$\begin{aligned}\mathcal{D} &\geq \int\int v\frac{(1-|\varphi_i(t)|-|\varphi_j(u)|+|\varphi_i(t)\varphi_j(u)|)}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt \\ &\geq \int\int v\frac{(1-a_i-a_j-a_ia_j)}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt\end{aligned}$$

It is plain that for sufficiently large i and j,

$$1-a_i-a_j-a_ia_j>0$$

hence

$$J(v) \leq \frac{a_ia_j}{(1-a_i-a_j-a_ia_j)} \frac{\int\int \frac{|ut|}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}{\int\int \frac{v}{\sqrt{v^2-u^2-t^2}} f_{\neq ij}\left(\sqrt{v^2-u^2-t^2}\right) 1\!\!1_{\left\{v\geq\sqrt{u^2+t^2}\right\}} dudt}$$

and $\lim_{i,j\to+\infty} a_i+a_j+a_ia_j=0$ hence is smaller than 0.5 for large i and j. Then we apply the same change of variable as in (32a). We get for i and j large enough :

$$\begin{aligned}J(v) &\leq 2a_ia_j \frac{\int_0^v\int_0^{2\pi} \frac{r^3|\cos\theta\sin\theta|}{\sqrt{v^2-r^2}} f_{\neq ij}\left(\sqrt{v^2-r^2}\right) drd\theta}{\int_0^v\int_0^{2\pi} \frac{vr}{\sqrt{v^2-r^2}} f_{\neq ij}\left(\sqrt{v^2-r^2}\right) drd\theta} \\ &\leq 4\pi a_ia_j v \frac{\int_0^1 \frac{x^3}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx}{\int_0^1 \frac{x}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx}\end{aligned}$$

Now we invoke Lemma 1 to get :

$$\frac{\int_0^1 \frac{x^3}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx}{\int_0^1 \frac{x}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx} \sim 2\frac{\rho(v)}{v}$$

We should check now that :

$$\limsup_{v\to 0}\sup_{i,j} \frac{v}{\rho(v)} \frac{\int_0^1 \frac{x^3}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx}{\int_0^1 \frac{x}{\sqrt{1-x^2}} f_{\neq ij}\left(v\sqrt{1-x^2}\right) dx} < M$$

The detailks of this step are omiied since they copy almost verbatim those of Lemma 5.

At last :

$$J(v) \leq 10\pi a_i a_j \rho(v)$$

for large i and j and small v.

With this inequality in hand, we go back to (32b)

$$\mathbb{E}\left[K\left(\frac{\|X\|}{h}\right)\langle X, e_i\rangle\langle X, e_j\rangle\right] \leq 10\pi a_i a_j \int_0^h v^2 \rho(v) K\left(\frac{v}{h}\right) f_{\|X\|}(v)\, dv$$

from which we deduce that

$$\left\|\Gamma_K^{\#d}\right\|_\infty^2 \leq \left\|\Gamma_K^{\#d}\right\|_2^2 \leq \left(10\pi \int_0^h v^2 \rho(v) K\left(\frac{v}{h}\right) f_{\|X\|}(v)\, dv\right)^2 \left(\sum_{i=1}^{+\infty} a_i^2\right)^2$$

and

$$\left\|\Gamma_K^{\#d}\right\|_\infty \leq 10\pi \left(\sum_{i=1}^{+\infty} a_i^2\right) \mathbb{E}\left(\|X\|^2 \rho(\|X\|) K\left(\frac{\|X\|}{h}\right)\right).$$

Since K has compact support we may say that $\|X\| \leq h$ hence :

$$\mathbb{E}\left(\|X\|^2 \rho(\|X\|) K\left(\frac{\|X\|}{h}\right)\right) \leq hv(h),$$

$$\left\|\Gamma_K^{\#d}\right\|_\infty \leq 10\pi \left(\sum_{i=1}^{+\infty} a_i^2\right) hv(h).$$

Together with Lemma 4 and assumption (22) this last inequality yields Theorem 2.

We are going to prove Proposition 7 but first we give a Lemma :

Lemma 6 *Let $m, p \in \mathbb{N}$:*

$$\mathbb{E}\left[K^m\left(\frac{\|X_1 - x_0\|}{h}\right)\|X_1 - x_0\|^p\right] \sim K^m(1) h^p F(h)$$

Proof.

$$\mathbb{E}\left[K^m\left(\frac{\|X_1 - x_0\|}{h}\right)\|X_1 - x_0\|^p\right] = \int_0^h u^p K^m\left(\frac{u}{h}\right) d\mathbb{P}^{\|X_1 - x_0\|}(u)$$

$$= h^p \int_0^1 u^p K^m(u)\, d\mathbb{P}^{\|X_1 - x_0\|/h}(u)$$

We apply Fubini's theorem. It is plain that :

$$K^m(u) u^p = K^m(1) - \int_u^1 \left[s^p K^m(s)\right]' ds,$$

hence

$$\int_0^1 K^m(u)\, u^p d\mathbb{P}^{\|X_1-x_0\|/h}(u)$$
$$= K^m(1)\int_0^1 d\mathbb{P}^{\|X_1-x_0\|/h}(u) - \int\int [K^m(s)\, s^p]' \, 1\!\!1_{\{0\le u\le s\le 1\}} d\mathbb{P}^{\|X_1-x_0\|/h}(u)\, ds$$
$$= K^m(1)\, F(h) - \int_0^1 [s^p K^m(s)]' \, F(hs)\, ds$$
$$= K^m(1)\, F(h)\left(1 - \int_0^1 [s^p K^m(s)]' \frac{F(hs)}{F(h)} ds\right)$$

which finally gives

$$\mathbb{E}\left[K^m\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^p\right] = h^p K^m(1)\, F(h)\left(1 - \int_0^1 [s^p K^m(s)]' \frac{F(hs)}{F(h)} ds\right).$$

We deal with $\int_0^1 [K^m(s)\, s^p]' \frac{F(hs)}{F(h)} ds$ and just have to show that this integral goes to zero when h does to prove the Lemma. Remind that assumption $\mathbf{A}_2$ ensures that $K(1) > 0$ and that $\sup_s |K(s)'| < +\infty$. Hence :

$$\sup_s |s^p K^m(s)'| < +\infty.$$

At last Fact 1 (see display (9)) ensures that $F(hs)/F(h) \to 0$ when s is fixed. Noticing that $F(hs)/F(h) \le 1$ and Lebesgue's dominated convergence Theorem are enough to get the desired result and to complete the proof of the Lemma. ■

Proof of Proposition 7 : We begin with the first part of the Proposition. We must prove that :

$$v(h) = \mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|\,\rho(\|X_1-x_0\|)\right]$$
$$= o\left(\mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^2\right]\right)$$

By (10) and as ρ is a positive function, we can assume that ρ is non-decreasing in a neighborhood of 0 hence that :

$$\mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|\,\rho(\|X_1-x_0\|)\right]$$
$$\le \rho(h)\,\mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|\right]$$

since the support of K is $[0,1]$. Then applying Lemma 6 with $m=1$ and $p=1$ we get

$$\mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|\right]\sim K(1)\,hF(h).$$

Then for h small enough :

$$\frac{\mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|\,\rho\left(\|X_1-x_0\|\right)\right]}{\mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^2\right]}\leq\frac{2K(1)\,\rho(h)\,hF(h)}{\mathbb{E}\left[K\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^2\right]} \tag{39}$$

By Lemma 6 again with $m=1$ and $p=2$, the denumerator behaves like $K(1)\,h^2F(h)$. At last since $\rho(h)/h\to 0$, the display above tends to zero. We go on with the second part of the Proposition. It is not difficult by copying the proof of Lemma 6 to show that :

$$v(h)=K(1)\,\rho(h)\,hF(h)\left(1-\int_0^1\frac{[sK(s)\,\rho(hs)]'}{\rho(h)}\frac{F(hs)}{F(h)}ds\right) \tag{40}$$

and

$$\frac{[sK(s)\,\rho(hs)]'}{\rho(h)}=\frac{K(s)\,\rho(hs)}{\rho(h)}+\frac{sK'(s)\,\rho(hs)}{\rho(h)}+\frac{hsK(s)\,\rho'(hs)}{\rho(h)}.$$

The first and second term on the right are uniformly bounded with respect to h and s since ρ is non decreasing in a neighborhood of zero and K and K' are bounded. We turn to the last. Remind that here ρ is assumed to be regularly varying at zero. From (10) we now that the index of regular variation of ρ is $d\geq 0$. Now open the book by Bingham, Goldie and Teugels (1987). The definition of regular variation is given p.18. from Theorem 1.7.2 p.39 we deduce that ρ' is regularly varying with index $d-1$ and also that :

$$\limsup_{t\to 0}\left|\frac{t\rho'(t)}{\rho(t)}\right|=d$$

which means that $hsK(s)\,\rho'(hs)/\rho(h)$ is unformly bounded with respect to h and s for small h and $0\leq s\leq 1$. Applying the dominated convergence theorem to (40) yields the desired result.

Proof of Theorem 3 :

Simple calculations give :

$$\begin{aligned}
&\mathbb{E}\left\|\Gamma_{K,n}-\Gamma_K\right\|^2\\
&=\frac{1}{n}\left(\mathbb{E}\left[K^2\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^4\right]-\|\Gamma_K\|^2\right)
\end{aligned}$$

We begin with computing $\mathbb{E}\left[K^2\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^4\right]$. Lemma 6 gives :

$$\mathbb{E}\left[K^2\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^4\right]\sim K^2(1)\,h^4F(h)$$

Under the assumptions of Theorem 2 we also have :

$$\|\Gamma_K\|=O\left(v\left(h\right)\right)\tag{41}$$

where (see (39)) :

$$\frac{v(h)}{K(1)\,\rho(h)\,hF(h)}\to 1$$

and

$$\|\Gamma_K\|^2=O\left(\left[\rho(h)\,hF(h)\right]^2\right)=o\left(h^4K^2(1)\,F(h)\right)$$

which also means that

$$\begin{aligned}
\mathbb{E}\left\|\Gamma_{K,n}-\Gamma_K\right\|^2 &\sim \frac{1}{n}\mathbb{E}\left[K^2\left(\frac{\|X_1-x_0\|}{h}\right)\|X_1-x_0\|^4\right]\\
&\sim K^2(1)\,\frac{h^4F(h)}{n}.
\end{aligned}$$

Theorem 3 is now proved.

Proof of Corollary 1 :
The proof of the first display, related with the eigenvalues, stems from the famous bound :

$$\sup_{p\in\mathbb{N}}\left|s_p(T)-s_p(U)\right|\le\|U-T\|_\infty$$

where U and Y are compact linear operators from and onto H and $s_p(T)$ stands for the p^{th} characteristic number of the operator T. The proof of the second display, namely on eigenprojections, will be deduced from an article by Mas and Menneteau (2003) as announced sooner in the paper.

Proof of Proposition 2 : We want to estimate $P(\|X\|\le\varepsilon)$ when ε goes to 0. In order to understand these few lines, the reader is referred to

Dembo et alii (1995). We have to compute or to give asymptotic equivalents for formulas (3), (4) and (5) in their article. Let $\theta = \theta(\varepsilon)$ be the unique solution of :

$$\mu(\theta) = \varepsilon$$

where here

$$\mu(\theta) = \sum_{i=1}^{+\infty} \frac{1}{\exp(\alpha i) + 2\theta}.$$

We begin with :

$$\begin{aligned} \mu(\theta) &= \sum_{i=1}^{+\infty} \frac{1}{\exp(\alpha i) + 2\theta} \sim \int_1^{+\infty} \frac{dx}{\exp(\alpha x) + 2\theta} \\ &= \frac{1}{\alpha} \int_0^{\exp(-\alpha)} \frac{du}{1 + 2\theta u} = \frac{\log(1 + 2\theta \exp(-\alpha))}{2\alpha\theta}. \end{aligned} \tag{42}$$

Then

$$\begin{aligned} \psi(\theta) &= \sqrt{2 \sum_{i=1}^{+\infty} \left(\frac{\theta}{\exp(\alpha i) + 2\theta} \right)^2} \sim \theta \sqrt{2 \int_1^{+\infty} \frac{dx}{(\exp(\alpha x) + 2\theta)^2}} \\ &= \theta \sqrt{2 \frac{1}{\alpha} \int_0^{\exp(-\alpha)} \frac{u du}{(1 + 2\theta u)^2}} \\ &= \sqrt{\frac{1}{2\alpha}} \sqrt{\log(1 + 2\theta \exp(-\alpha)) - \frac{2\theta \exp(-\alpha)}{1 + 2\theta \exp(-\alpha)}} \end{aligned}$$

At last

$$I(\theta) = \frac{1}{2} \sum_{i=1}^{+\infty} \log(1 + 2\theta \exp(-\alpha i)) - \theta \mu(\theta). \tag{43}$$

We focus on

$$\frac{1}{2} \sum_{i=1}^{+\infty} \log(1 + 2\theta \exp(-\alpha i)) \sim \frac{1}{2} \int_1^{+\infty} \log(1 + 2\theta \exp(-\alpha x)) \, dx.$$

Setting $u = 2\theta \exp(-\alpha x)$ this last integral becomes

$$\begin{aligned}
&\frac{1}{2}\int_1^{+\infty} \log(1+2\theta\exp(-\alpha x))\,dx \\
&= \frac{1}{2\alpha}\int_0^{2\theta\exp(-\alpha)} \frac{\log(1+u)}{u}du \\
&\sim \frac{1}{2\alpha}\int_1^{2\theta\exp(-\alpha)} \frac{\log(1+u)}{u}du \\
&\sim \frac{1}{2\alpha}\int_1^{2\theta\exp(-\alpha)} \frac{\log(u)}{u}du = \frac{1}{4\alpha}\left[\log(2\theta\exp(-\alpha))\right]^2 \\
&\sim \frac{1}{4\alpha}\left[\log\theta\right]^2.
\end{aligned}$$

From (43) we see that :

$$\frac{I(\theta)}{\frac{1}{4\alpha}\left[\log\theta\right]^2} = p(\theta) - \frac{\theta\mu(\theta)}{\frac{1}{4\alpha}\left[\log\theta\right]^2},$$

where $p(\theta)$ tends to 1 when θ goes to infinity. Then from (42) we get :

$$\frac{\theta\mu(\theta)}{\frac{1}{4\alpha}\left[\log\theta\right]^2} \sim \frac{4\alpha}{\log\theta} \to 0$$

and

$$\frac{I(\theta)}{\frac{1}{4\alpha}\left[\log\theta\right]^2} \to 1.$$

Collecting our results we have the following final asymptotic equivalence

$$\begin{aligned}
\theta &\sim -\frac{\log(\varepsilon)}{2\alpha\varepsilon} \\
\mu(\theta) = \varepsilon &\sim \frac{\log(\theta)}{2\alpha\theta} \\
\psi(\theta) &\sim \sqrt{\frac{1}{2\alpha}\log(\theta)} \sim \sqrt{\frac{-\log(\varepsilon)}{2\alpha}} \\
I(\theta) &\sim \frac{1}{4\alpha}\left[\log(\theta)\right]^2 \sim \frac{1}{4\alpha}\left[\log(\varepsilon)\right]^2
\end{aligned}$$

Collecting these formulas together with display (10) in the above-mentioned article we get :

$$\mathbb{P}(\|X\| < \varepsilon) \sim \sqrt{\frac{\alpha}{-\pi\log(\varepsilon)}}\exp\left(-\frac{1}{4\alpha}\left[\log(\varepsilon)\right]^2\right).$$

Acknowledgements : I am sincerely grateful to S. Gaïffas who pointed out to me the existence of de Haan's class Γ.

References

[1] Akhiezer N.I, Glazman I.M. (1981) : *Theory of Linear Operators in Hilbert Spaces*, Vol 1, Monographs and Studies in Mathematics, **10**, Pitman.

[2] Antoniadis A., Oppenheim G. (1995) : *Wavelets and Statistics*, Lecture Notes in Statistics, **103**, Springer.

[3] Berlinet A., Thomas-Agnan C. (2004) : *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, Kluwer Academic Publishers, Boston.

[4] Bingham N.H., Goldie C.M., Teugels J.L. (1987) : *Regular Variations.* Encyclopedia of Mathematics and Its Applications, Cambridge University Press.

[5] de Boor C. (1978) : *A Practical Guide to Splines*, Springer.

[6] Cardot H., Mas A., Sarda P. (2007) : CLT in functional linear models, to appear in *Probab. Theory and Related Fields.*

[7] Dauxois J., Pousse A., Romain Y. (1982) : Asymptotic theory for the principal component analysis of a random vector function : some applications to statistical inference, *Journal of Multivariate Analysis*, **12**, 136-154.

[8] Dembo A., Meyer-Wolf E., Zeitouni O. (1995) : Exact behavior of gaussian semi-norms, *Satistics and Probability Letters*, **23**, 275-280.

[9] Dierckx P. (1993) : *Curves and Surface Fitting with Splines*, Clarendon Press.

[10] Dunford N., Schwartz, J.T. (1988) : *Linear Operators, Vol. I & II.* Wiley Classics Library.

[11] Fan J. (1993) : Local linear regression smoothers and their minimax efficiencies, *Ann. Stat*, **21**, 196-216.

[12] Ferraty F., Mas A., Vieu, P. (2007) : Advances in nonparametric regression for functional variables, to appear in *Australian and New-Zealand Journal of Statistics*.

[13] Gaiffas S. (2005) : Convergence rates for pointwise curve estimation with a degenerate design, *Mathematical Methods of Statistics*, **14**.

[14] Gohberg I., Goldberg S., Kaashoek, M.A. (1991) : *Classes of Linear Operators Vol I & II*, Operator Theory : Advances and Applications, Birkhaüser.

[15] de Haan L. (1971) : A form of regular variation and its application to the domain of attraction of the double exponential distribution, *Z. Wahrscheinlichkeitstheorie.verw. Geb.* **17**, 241-258.

[16] de Haan L. (1974) : Equivalence classes of regularly varying functions, *Stochastic Processes and their Applications*, **2**, 243-259.

[17] He, G., Muller, H.G. and Wang, J.L. (2003) : Functional canonical analysis for square integrable stochastic processes. *J. Mult. Analysis*, **85**, 54-77

[18] Kato T. (1976) : *Perturbation theory for linear operators*. Grundlehren der mathematischen Wissenschaften. 132. Berlin -Heidelberg - New York: Springer-Verlag. 2nd Ed.

[19] Kneip A., Utikal, K.J. (2001) : Inference for density families using functional principal component analysis. *J. Amer. Statist. Assoc.*, **96**, 519–542.

[20] Kuelbs J., Li, W.V., Linde W. (1994) : The Gaussian measure of shifted balls. *Probab. Theory Related Fields*, **98**, 143–162.

[21] Ledoux M., Talagrand M. (1991) : *Probability in Banach Spaces. Isoperimetry and Processes,* Ergebnisse der Mathematik und ihrer Grenzgebiete, 23, Springer-Verlag.

[22] Li W.V., Linde W. (1993) : Small ball problems for non-centered gaussian measures, *Probab. Math. Stat*, **14**, 231-251.

[23] Li W.V., Linde W. (1999) : Approximation, metric entropy and small ball estimates for Gaussian measures. *Ann. Probab.*, **27**, 1556–1578

[24] Li, W.V., Shao Q.-M. (2001) : Gaussian processes : Inequalities, small ball probabilities and applications, Handbook of Statistics 19, 533-597.

[25] Mas A. (2007) : A representation for gaussian small ball probabilities, preprint.

[26] Mas A., Menneteau L. (2003) : Perturbation approach applied to the asymptotic study of random operators, *Progress in Probability*, **55**, 127-134, Birkhäuser.

[27] Meyer-Wolf E., Zeitouni O (1993) : The probability of small gaussian ellipsoïds, Annals of Probability, **21**, n°1, 14-24.

[28] Ocaña F.A., Aguilera A.M., Valderrama M.J. (1999) : Functional principal components analysis by choice of norm. *J. Multivariate Anal.*, **71**, 262–276.

[29] Schmeidler W. (1965) : *Linear Operators in Hilbert Spaces*, Academic Press.

[30] Silverman, B.W.(1996) : Smoothed principal component analysis by choice of norm, *Ann. Statist.* **24** , 1–24.

[31] Weidman J. (1980) : *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics , Springer.

[32] Yao, F., Muller, H.-G. and Wang, J.-L. (2005). Functional Data Analysis for Sparse Longitudinal Data, *J. Amer. Statist. Assoc.*, **100**, 577-590