Integrable discretizations for the short wave model of the Camassa-Holm equation

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Abstract. The link between the short wave model of the Camassa-Holm equation (SCHE) and bilinear equations of the two-dimensional Toda lattice (2DTL) is clarified. The parametric form of N-cuspon solution of the SCHE in Casorati determinant is then given. Based on the above finding, integrable semi-discrete and full-discrete analogues of the SCHE are constructed. The determinant solutions of both semi-discrete and fully discrete analogues of the SCHE are also presented.

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1. Introduction

In the present paper, we consider integrable discretizations of the nonlinear partial differential equation

$$w_{TXX} - 2\kappa^2 w_X + 2w_X w_{XX} + w w_{XXX} = 0, \qquad (1)$$

which belongs to the Harry-Dym hierarchy [1, 2, 3]. Here κ is a real parameter and, as shown subsequently, can be normalized by the scaling transformation when $\kappa \neq 0$. A connection between Eq.(1) and the sinh-Gordon equation was established in [4]. When $\kappa = 0$, Eq.(1) is called the Hunter-Saxton equation and is derived as a model for weakly nonlinear orientation waves in massive nematic liquid crystals [5]. The Lax pair and bi-Hamiltonian structure were discussed by Hunter and Zheng [6]. The dissipative and dispersive weak solutions were discussed in details by the same authors [7, 8].

Equation (1) can be viewed as a short-wave model of the Camassa-Holm equation [9]

$$w_T + 2\kappa^2 w_X - w_{TXX} + 3ww_X = 2w_X w_{XX} + w w_{XXX}\,. \qquad (2)$$

Following the procedure in [10, 11, 12], we introduce the time and space variables $\tilde{T}$ and $\tilde{X}$

$$\tilde{T} = \varepsilon T\,, \quad \tilde{X} = \varepsilon^{-1} X\,,$$

where ε is a small parameter. Then w is expanded as $w = \varepsilon^2(w_0 + \varepsilon w_1 + \cdots)$ with w_i $(i = 0, 1, \cdots)$ being functions of $\tilde{T}$ and $\tilde{X}$. At the lowest order in ε, we obtain

$$w_{0,\tilde{T}\tilde{X}\tilde{X}} - 2\kappa^2 w_{0,\tilde{X}} + 2w_{0,\tilde{X}} w_{0,\tilde{X}\tilde{X}} + w_0 w_{0,\tilde{X}\tilde{X}\tilde{X}} = 0\,, \qquad (3)$$

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which is exactly Eq.(1) after writing back into the original variables. Based on this fact, Matsuno obtained the N-cuspon solution of Eq.(1) by taking the short-wave limit on the N-soliton solution of the Camassa-Holm equation [13, 14].

Note that the parameter κ of Eq.(1) can be normalized to 1 under the transformation

$$x = \kappa X\,, \quad t = \kappa T\,,$$

which leads to

$$w_{txx} - 2w_x + 2w_x w_{xx} + w w_{xxx} = 0\,. \tag{4}$$

We call Eq.(4) the short wave model of the Camassa-Holm equation (SCHE). Without loss of generality, we will focus on Eq. (4) and its integrable discretizations, since the solution of Eq.(1) with arbitrary nonzero κ, its integrable discretizations and the corresponding solutions can be recovered through the above transformation.

The reminder of the present paper is organized as follows. In section 2, we reveal a connection between the SCHE and the bilinear form two-dimensional Toda-lattice (2DTL) equations. The parametric form of N-cuspon solution expressed by the Casorti determinant is given, which is consistent with the solution given in [13]. Based on this fact, we propose an integrable semi-discrete analogue of the SCHE in section 3, and further its integrable full-discrete analogue in section 4. The concluding remark is given in section 5.

2. The connection with 2DTL equations, and N-cuspon solution in determinant form

2.1. The link of the SCHE with the two-reduction of 2DTL equations

In this section, we will show that the SCHE can be derived from the bilinear form of two-dimensional Toda lattice (2DTL) equations

$$-\left(\frac{1}{2}D_{-1}D_1 - 1\right)\tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1}\,, \tag{5}$$

where D_x is the Hirota D-derivative defined as

$$D_x^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^n f(x)g(y)|_{y=x}\,,$$

and D_{-1} and D_1 represent the Hirota D derivatives with respect to variables x_{-1} and x_1, respectively.

It is shown that the N-soliton solution of the 2DTL equations (5) can be expressed as the Casorati determinant [16, 17]

$$\tau_n = \left|\psi_i^{(n+j-1)}(x_1, x_{-1})\right|_{1\le i,j\le N} = \begin{vmatrix} \psi_1^{(n)} & \psi_1^{(n+1)} & \cdots & \psi_1^{(n+N-1)} \\ \psi_2^{(n)} & \psi_2^{(n+1)} & \cdots & \psi_2^{(n+N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N^{(n)} & \psi_N^{(n+1)} & \cdots & \psi_N^{(n+N-1)} \end{vmatrix}\,, \tag{6}$$

with $\psi_i^{(n)}$ satisfying the following dispersion relations:

$$\frac{\partial \psi_i^{(n)}}{\partial x_{-1}} = \psi_i^{(n-1)}, \qquad \frac{\partial \psi_i^{(n)}}{\partial x_1} = \psi_i^{(n+1)}\,.$$

A particular choice of $\psi_i^{(n)}$

$$\psi_i^{(n)} = a_{i,1} p_i^n e^{p_i^{-1} x_{-1} + p_i x_1 + \eta_{0i}} + a_{i,2} q_i^n e^{q_i^{-1} x_{-1} + q_i x_1 + \eta'_{0i}}\,, \tag{7}$$

automatically satisfies the above dispersion relations.

Applying the two-reduction $\tau_{n-1} = (\prod_{i=1}^{N} p_i^2)^{-1}\tau_{n+1}$, i.e., enforcing $p_i = -q_i$, $i = 1,\cdots,N$, we get

$$-\left(\frac{1}{2}D_{-1}D_1 - 1\right)\tau_n\cdot\tau_n = \tau_{n+1}^2\,, \tag{8}$$

where the gauge transformation $\tau_n \to (\prod_{i=1}^{N} p_i)^n\tau_n$ is used. Letting $\tau_0 = f$, $\tau_1 = g$ and $x_{-1} = s$, $x_1 = y$, the above bilinear equation (8) takes the following form:

$$-\left(\frac{1}{2}D_sD_y - 1\right) f\cdot f = g^2\,, \tag{9}$$

$$-\left(\frac{1}{2}D_sD_y - 1\right) g\cdot g = f^2\,. \tag{10}$$

Introducing $u = g/f$, Eqs.(9) and (10) can be converted into

$$-(\ln f)_{ys} + 1 = u^2\,, \tag{11}$$

$$-(\ln g)_{ys} + 1 = u^{-2}\,. \tag{12}$$

Subtracting Eq.(12) from Eq.(11), one obtains

$$\frac{\rho}{2}(\ln\rho)_{ys} + 1 = \rho^2\,, \tag{13}$$

by letting $\rho = u^2$.

Introducing the dependent variable transformation

$$w = -2(\ln g)_{ss}\,,$$

it then follows

$$\frac{1}{2}w_y = -\frac{\rho_s}{\rho^2}\,,$$

or

$$(\ln\rho)_s = -\frac{\rho}{2}w_y\,, \tag{14}$$

by differentiating Eq.(12) with respect to s.

In view of Eq.(14), Eq.(13) becomes

$$-\frac{\rho}{2}\left(\frac{\rho}{2}w_y\right)_y + 1 = \rho^2\,. \tag{15}$$

Introducing the hodograph transformation

$$\begin{cases} x = 2y - 2(\ln g)_s\,, \\ t = s\,, \end{cases}$$

and referring to Eq.(12), we have

$$\frac{\partial x}{\partial y} = 2 - 2(\ln g)_{ys} = \frac{2}{\rho}\,, \qquad \frac{\partial x}{\partial s} = -2(\ln g)_{ss} = w\,,$$

which implies

$$\begin{cases} \partial_y = \dfrac{2}{\rho}\partial_x\,, \\ \partial_s = \partial_t + w\partial_x\,. \end{cases}$$

Thus, Eqs.(14) and (15) can be cast into

$$\left\{ \begin{array}{l} (\partial_t + w\partial_x)\ln\rho = -w_x\,, \\ -w_{xx} + 1 = \rho^2\,. \end{array} \right. \tag{16}$$

By eliminating ρ, we arrive at

$$(\partial_t + w\partial_x)\ln(-w_{xx} + 1) = -2w_x\,,$$

or

$$(\partial_t + w\partial_x)w_{xx} - 2w_x(1 - w_{xx}) = 0\,,$$

which is actually the SCHE (4).

2.2. The N-cuspon solution of the SCHE

Based on the link of the SCHE with the two-reduction of 2DTL equations, the N-cuspon solution of the SCHE (4) is given as follows:

$$w = -2(\ln g)_{ss}\,,$$

$$\left\{ \begin{array}{l} x = 2y - 2(\ln g)_s\,, \\ t = s\,, \end{array} \right.$$

$$g = \left|\psi_i^{(j)}(y,s)\right|_{1\le i,j\le N}\,,$$

$$\psi_i^{(j)} = a_{i,1}p_i^j e^{p_i{}^{-1}s + p_iy + \eta_{0i}} + a_{i,2}(-p_i)^j e^{-p_i{}^{-1}s - p_iy + \eta'_{0i}}\,. \tag{17}$$

Moreover, the N-cuspon solution of the SCHE (1) with non-zero κ is given as follows:

$$w(y,T) = -2(\ln g)_{ss}, \tag{18}$$

$$\left\{ \begin{array}{l} X = \frac{2y}{\kappa} - \frac{2}{\kappa}(\ln g)_s, \\ T = \frac{s}{\kappa}, \end{array} \right. \tag{19}$$

where

$$g = \left|\psi_i^{(j)}(y,s)\right|_{1\le i,j\le N}\,,$$

with

$$\psi_i^{(n)} = a_{i,1}p_i^n e^{p_iy + s/p_i + \eta_{i0}} + a_{i,2}(-p_i)^n e^{-p_iy - s/p_i + \eta'_{i0}}\,.$$

We remark here that to assure the regularity of the solution, the τ-function is required to be positive definite. In what follows, we list the one-cuspon and two-cuspon solutions. For $N = 1$, the τ-function is

$$g = 1 + e^{2p_1(y + \kappa T/p_1^2 + y_0)}\,,$$

by choosing $a_{1,1}/a_{1,2} = -1$, which yields the one-cuspon solution

$$w(y,T) = -\frac{2}{p_1^2}\mathrm{sech}^2\left[p_1(y + \kappa T/p_1^2 + y_0)\right]\,,$$

$$X = \frac{2y}{\kappa} - \frac{2}{\kappa p_1}\left\{1 + \tanh\left[p_1(y + \kappa T/p_1^2 + y_0)\right]\right\}\,.$$

The profiles of one-cuspon with $\kappa = 1.0$ and $\kappa = 0.1$ are plotted in Fig. 1.

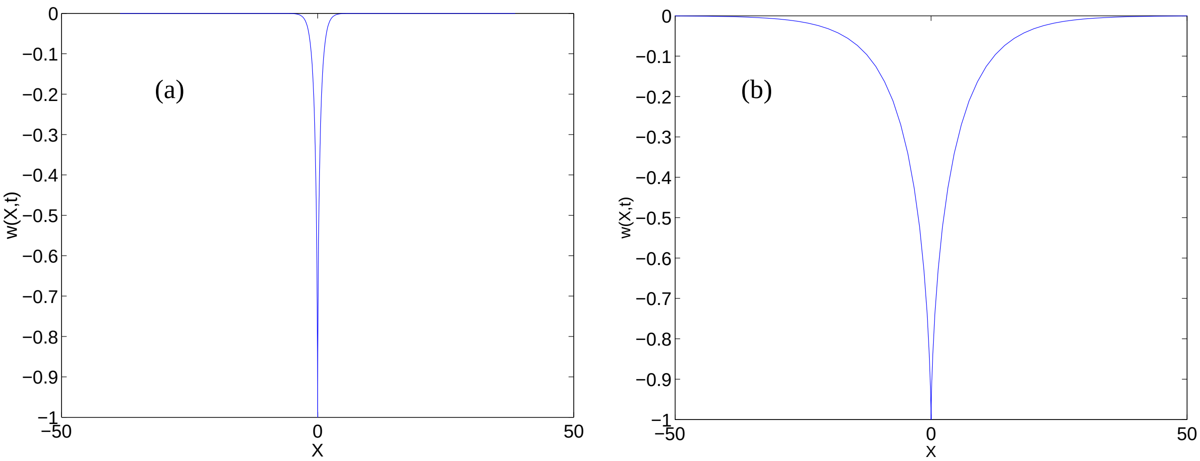

Figure 1. Plots for one-cuspon solution for $p_1 = \sqrt{2}$ and different κ: (a) $\kappa = 1.0$; (b) $\kappa = 0.1$.

The τ-function corresponding to the two-cuspon solution is

$$g = 1 + e^{\theta_1} + e^{\theta_2} + \left(\frac{p_1 - p_2}{p_1 - p_2}\right)^2 e^{\theta_1+\theta_2}\,,$$

with

$$\theta_i = 2p_i(y + \kappa T/p_i^2 + y_{i0}),\ i = 1, 2\,.$$

Here $a_{1,1}/a_{1,2} = -1$ and $a_{2,1}/a_{2,2} = 1$ are chosen to assure the regularity of the solution.

3. Integrable semi-discretization of the SCHE

Based on the link of the SCHE with the two-reduction of 2DTL equations clarified in the previous section, we attempt to construct the integrable semi-discrete analogue of the SCHE.

Consider a Casorati determinant

$$\tau_n(k) = \left|\psi_i^{(n+j-1)}(k)\right|_{1\le i,j\le N} = \begin{vmatrix} \psi_1^{(n)}(k) & \psi_1^{(n+1)}(k) & \cdots & \psi_1^{(n+N-1)}(k) \\ \psi_2^{(n)}(k) & \psi_2^{(n+1)}(k) & \cdots & \psi_2^{(n+N-1)}(k) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N^{(n)}(k) & \psi_N^{(n+1)}(k) & \cdots & \psi_N^{(n+N-1)}(k) \end{vmatrix},$$

with $\psi_i^{(n)}$ satisfies the following dispersion relations

$$\Delta_k \psi_i^{(n)} = \psi_i^{(n+1)}\,, \tag{20}$$

$$\partial_s \psi_i^{(n)} = \psi_i^{(n-1)}\,, \tag{21}$$

where Δ_k is defined as $\Delta_k\psi(k) = \frac{\psi(k)-\psi(k-1)}{a}$. In particular, we can choose $\psi_i^{(n)}$ as

$$\psi_i^{(n)}(k) = p_i^n(1-ap_i)^{-k}e^{\xi_i} + q_i^n(1-aq_i)^{-k}e^{\eta_i}\,,$$

$$\xi_i = \frac{1}{p_i}s + \xi_{i0}\,, \quad \eta_i = \frac{1}{q_i}s + \eta_{i0}\,,$$

which automatically satisfies the dispersion relations (20) and (21). The above Casorati determinant satisfies the bilinear form of the semi-discrete 2DTL equation (the Bäcklund transformation of the bilinear equation of the 2DTL equation) [17, 18]

$$\left(\frac{1}{a}D_s - 1\right)\tau_n(k+1)\cdot\tau_n(k) + \tau_{n+1}(k+1)\tau_{n-1}(k) = 0\,. \tag{22}$$

Applying a two-reduction condition $p_i = -q_i$, $i = 1, \cdots, N$, which implies $\tau_{n-1} \approx \tau_{n+1}$, we obtain

$$-\left(\frac{1}{a}D_s - 1\right) f_{k+1}\cdot f_k = g_{k+1}g_k\,, \tag{23}$$

$$-\left(\frac{1}{a}D_s - 1\right) g_{k+1}\cdot g_k = f_{k+1}g_k\,, \tag{24}$$

by letting $\tau_0(k) = f_k$, $\tau_1(k) = g_k$.

Letting $u_k = g_k/f_k$, Eqs.(23) and (24) are equivalent to

$$-\frac{1}{a}\left(\ln\frac{f_{k+1}}{f_k}\right)_s + 1 = u_{k+1}u_k\,, \tag{25}$$

$$-\frac{1}{a}\left(\ln\frac{g_{k+1}}{g_k}\right)_s + 1 = u_{k+1}^{-1}u_k^{-1}\,. \tag{26}$$

Subtracting Eq.(26) from Eq.(25), one obtains

$$\frac{u_{k+1}u_k}{a}\left(\ln\frac{u_{k+1}}{u_k}\right)_s + 1 = u_{k+1}^2u_k^2\,. \tag{27}$$

Introducing the discrete analogue of hodograph transformation

$$x_k = 2ka - 2(\ln g_k)_s\,,$$

and

$$\delta_k = x_{k+1} - x_k = 2a - 2\left(\ln\frac{g_{k+1}}{g_k}\right)_s\,.$$

It then follows from Eq.(26)

$$\delta_k = \frac{2a}{u_{k+1}u_k}\,,$$

or

$$\rho_{k+1}\rho_k = \frac{4a^2}{\delta_k^2}\,, \tag{28}$$

by assuming $\rho_k = u_k^2$.

Introducing the dependent variable transformation

$$w_k = -2(\ln g_k)_{ss}\,,$$

Eq.(27) becomes

$$\frac{1}{\delta_k}\left(\ln\frac{\rho_{k+1}}{\rho_k}\right)_s + 1 - \frac{4a^2}{\delta_k^2} = 0\,. \tag{29}$$

Differentiating Eq.(26) with respect to s, we have

$$\frac{1}{2a}(w_{k+1} - w_k) = -\frac{1}{u_{k+1}u_k}(\ln u_{k+1}u_k)_s = -\frac{1}{2u_{k+1}u_k}(\ln\rho_{k+1}\rho_k)_s\,,$$

or

$$(\ln \rho_{k+1}\rho_k)_s = -\frac{2}{\delta_k}(w_{k+1}-w_k)\,. \tag{30}$$

Eliminating ρ_k and ρ_{k+1} from Eqs.(29) and (30), we obtain

$$\frac{1}{\delta_k}(w_{k+1}-w_k)-\frac{1}{\delta_{k-1}}(w_k-w_{k-1}) = \frac{1}{2}(\delta_k+\delta_{k-1})-2a^2\left(\frac{1}{\delta_k}+\frac{1}{\delta_{k-1}}\right), \tag{31}$$

or

$$\Delta^2 w_k = \frac{1}{\delta_k}M\left(\delta_k-\frac{4a^2}{\delta_k}\right), \tag{32}$$

by defining a difference operator Δ and an average operator M as follows

$$\Delta F_k = \frac{F_{k+1}-F_k}{\delta_k}\,, \quad MF_k = \frac{F_{k+1}+F_k}{2}\,.$$

Furthermore, a substitution of Eq.(28) into Eq. (30) leads to

$$\frac{d\delta_k}{ds} = w_{k+1}-w_k\,. \tag{33}$$

Equations (31) and (33) constitute the semi-discrete analogue of the SCHE.

Next, let us show that in the continuous limit, $a \to 0$ ($\delta_k \to 0$), the proposed semi-discrete SCHE recovers the continuous SCHE. To this end, Eqs.(31) and (33) are rewritten as

$$\begin{cases} \dfrac{-2}{\delta_k+\delta_{k-1}}(\Delta w_k-\Delta w_{k-1})+1 = \dfrac{4a^2}{\delta_k\delta_{k-1}}\,, \\ \partial_s\delta_k = w_{k+1}-w_k\,. \end{cases}$$

By taking logarithmic derivative of the first equation, we get

$$\frac{\partial_s\left\{\dfrac{-2}{\delta_k+\delta_{k-1}}(\Delta w_k-\Delta w_{k-1})+1\right\}}{\dfrac{-2}{\delta_k+\delta_{k-1}}(\Delta w_k-\Delta w_{k-1})+1} = -\frac{\partial_s\delta_k}{\delta_k}-\frac{\partial_s\delta_{k-1}}{\delta_{k-1}}\,.$$

The dependent variable w is regarded as a function of x and t, where x is the space coordinate of the k-th lattice point and t is the time, defined by

$$x_k = x_0+\sum_{j=0}^{k-1}\delta_j\,, \qquad t = s\,.$$

In the continuous limit, $a \to 0$ ($\delta_k \to 0$), we have

$$\frac{\partial_s\delta_k}{\delta_k} = \frac{w_{k+1}-w_k}{\delta_k} \to w_x\,, \quad \frac{\partial_s\delta_{k-1}}{\delta_{k-1}} = \frac{w_k-w_{k-1}}{\delta_{k-1}} \to w_x\,,$$

$$\frac{2}{\delta_k+\delta_{k-1}}(\Delta w_k-\Delta w_{k-1}) \to w_{xx}\,,$$

$$\frac{\partial x_k}{\partial s} = \frac{\partial x_0}{\partial s}+\sum_{j=0}^{k-1}\frac{\partial\delta_j}{\partial s} = \frac{\partial x_0}{\partial s}+\sum_{j=0}^{k-1}(w_{j+1}-w_j) \to w\,,$$

$$\partial_s = \partial_t + \frac{\partial x}{\partial s}\partial_x \to \partial_t + w\partial_x\,,$$

where the origin of space coordinate x_0 is taken so that $\frac{\partial x_0}{\partial s}$ cancels w_0. Thus the above semi-discrete SCHE converges to

$$\frac{(\partial_t + w\partial_x)(-w_{xx}+1)}{-w_{xx}+1} = -2w_x\,,$$

or

$$(\partial_t + w\partial_x)w_{xx} = 2w_x\,(-w_{xx}+1)\,, \tag{34}$$

which is nothing but the SCHE (4).

In summary, the semi-discrete analogue of the SCHE and its determinant solution are given as follows:

The semi-discrete analogue of the SCHE

$$\begin{cases} \dfrac{1}{\delta_k}(w_{k+1}-w_k) - \dfrac{1}{\delta_{k-1}}(w_k - w_{k-1}) = \dfrac{1}{2}(\delta_k+\delta_{k-1}) - 2a^2\left(\dfrac{1}{\delta_k}+\dfrac{1}{\delta_{k-1}}\right)\,, \\ \dfrac{d\delta_k}{dt} = w_{k+1}-w_k\,. \end{cases} \tag{35}$$

The determinant solution of the semi-discrete SCHE

$$w_k = -2(\ln g_k)_{ss}\,,$$

$$\delta_k = x_{k+1}-x_k = 2a\frac{f_{k+1}f_k}{g_{k+1}g_k}\,,$$

$$\begin{cases} x_k = 2ka - 2(\ln g_k)_s\,, \\ t = s\,, \end{cases}$$

$$g_k = \left|\psi_i^{(j)}(k)\right|_{1\le i,j\le N}\,, \quad f_k = \left|\psi_i^{(j-1)}(k)\right|_{1\le i,j\le N}\,,$$

$$\psi_i^{(j)}(k) = a_{i,1}p_i^j(1-ap_i)^{-k}e^{p_i{}^{-1}s+\eta_{0i}} + a_{i,2}(-p_i)^j(1+ap_i)^{-k}e^{-p_i{}^{-1}s+\eta'_{0i}}\,. \tag{36}$$

Introducing new independent variables $X_k = x_k/\kappa$ and $T = t/\kappa$, we can include the parameter κ in the semi-discrete SCHE (35)

$$\begin{cases} \dfrac{1}{\delta_k}(w_{k+1}-w_k) - \dfrac{1}{\delta_{k-1}}(w_k - w_{k-1}) = \dfrac{1}{2\kappa^2}(\delta_k+\delta_{k-1}) - 2a^2\left(\dfrac{1}{\delta_k}+\dfrac{1}{\delta_{k-1}}\right)\,, \\ \dfrac{d\delta_k}{dT} = w_{k+1}-w_k\,, \end{cases} \tag{37}$$

where $\delta_k = X_{k+1}-X_k$ and $s = \kappa T$. This is the semi-discrete analogue of the SCHE (1).

The N-cuspon solution of the semi-discrete SCHE (37) with the parameter κ is given by

$$w_k = -2(\ln g_k)_{ss}\,,$$

$$\delta_k = X_{k+1}-X_k = \frac{2a}{\kappa}\frac{f_{k+1}f_k}{g_{k+1}g_k}\,,$$

$$\begin{cases} X_k = \frac{2ka}{\kappa} - \frac{2}{\kappa}(\ln g_k)_s\,, \\ T = \frac{s}{\kappa}\,, \end{cases}$$

$$g_k = \left|\psi_i^{(j)}(k)\right|_{1\le i,j\le N}\,, \quad f_k = \left|\psi_i^{(j-1)}(k)\right|_{1\le i,j\le N}\,,$$

$$\psi_i^{(j)}(k) = a_{i,1}p_i^j(1-ap_i)^{-k}e^{p_i{}^{-1}s+\eta_{0i}} + a_{i,2}(-p_i)^j(1+ap_i)^{-k}e^{-p_i{}^{-1}s+\eta'_{0i}}\,. \tag{38}$$

4. Full-discretization of the SCHE

In much the same way of finding the semi-discrete analogue of the SCHE, we seek for its full-discrete analogue and in the process we arrive at its N-cuspon solution.

Consider the following Casorati determinant

$$\tau_n(k,l) = \left|\psi_i^{(n+j-1)}(k,l)\right|_{1\le i,j\le N}, \tag{39}$$

where

$$\psi_i^{(n)}(k,l) = a_{i,1}p_i^n(1-ap_i)^{-k}\left(1-bp_i{}^{-1}\right)^{-l}e^{\xi_i} + a_{i,2}q_i^n(1-aq_i)^{-k}\left(1-bq_i{}^{-1}\right)^{-l}e^{\eta_i},$$

with

$$\xi_i = p_i{}^{-1}s + \xi_{i0}, \quad \eta_i = q_i{}^{-1}s + \eta_{i0}.$$

It is known that the above determinant satisfies bilinear equations [18]

$$\left(\frac{1}{a}D_s - 1\right)\tau_n(k+1,l)\cdot\tau_n(k,l) + \tau_{n+1}(k+1,l)\tau_{n-1}(k,l) = 0, \tag{40}$$

and

$$(bD_s-1)\tau_n(k,l+1)\cdot\tau_{n+1}(k,l) + \tau_n(k,l)\tau_{n+1}(k,l+1) = 0. \tag{41}$$

Here a,b are mesh sizes for space and time variables, respectively.

Applying the two-reduction $\tau_{n-1} = (\prod_{i=1}^N p_i^2)^{-1}\tau_{n+1}$, i.e., enforcing $p_i = -q_i$, $i = 1,\cdots,N$, and letting $\tau_0(k,l) = f_{k,l}$, $\tau_1(k,l) = g_{k,l}$, the above bilinear equations take the following form:

$$\left(\frac{1}{a}D_s - 1\right)f_{k+1,l}\cdot f_{k,l} + g_{k+1,l}g_{k,l} = 0, \tag{42}$$

$$\left(\frac{1}{a}D_s - 1\right)g_{k+1,l}\cdot g_{k,l} + f_{k+1,l}f_{k,l} = 0, \tag{43}$$

$$(bD_s-1)f_{k,l+1}\cdot g_{k,l} + f_{k,l}g_{k,l+1} = 0, \tag{44}$$

$$(bD_s-1)g_{k,l+1}\cdot f_{k,l} + g_{k,l}f_{k,l+1} = 0, \tag{45}$$

where the gauge transformation $\tau_n \to (\prod_{i=1}^N p_i)^n\tau_n$ is used. It is readily shown that the above equations are equivalent to

$$\frac{1}{a}\left(\ln\frac{f_{k+1,l}}{f_{k,l}}\right)_s = 1 - \frac{g_{k+1,l}g_{k,l}}{f_{k+1,l}f_{k,l}}, \tag{46}$$

$$\frac{1}{a}\left(\ln\frac{g_{k+1,l}}{g_{k,l}}\right)_s = 1 - \frac{f_{k+1,l}f_{k,l}}{g_{k+1,l}g_{k,l}}, \tag{47}$$

$$b\left(\ln\frac{f_{k,l+1}}{g_{k,l}}\right)_s = 1 - \frac{f_{k,l}g_{k,l+1}}{f_{k,l+1}g_{k,l}}, \tag{48}$$

$$b\left(\ln\frac{g_{k,l+1}}{f_{k,l}}\right)_s = 1 - \frac{g_{k,l}f_{k,l+1}}{g_{k,l+1}f_{k,l}}. \tag{49}$$

We introduce a dependent variable transformation

$$w_{k,l} = -2\left(\ln g_{k,l}\right)_{ss}, \tag{50}$$

and a discrete hodograph transformation

$$x_{k,l} = 2ka - 2(\ln g_{k,l})_s, \tag{51}$$

then the mesh

$$\delta_{k,l} = x_{k+1,l} - x_{k,l} = 2a - 2\left(\ln \frac{g_{k+1,l}}{g_{k,l}}\right)_s \tag{52}$$

is naturally defined. It then follows

$$\left(\ln \frac{g_{k+1,l}}{g_{k-1,l}}\right)_s = 2a - \frac{1}{2}\left(\delta_{k,l} + \delta_{k-1,l}\right) . \tag{53}$$

In view of Eq.(47), one obtains

$$\frac{f_{k+1,l}\, f_{k,l}}{g_{k+1,l}g_{k,l}} = \frac{\delta_{k,l}}{2a} . \tag{54}$$

A substitution into Eq.(46) yields

$$\left(\ln \frac{f_{k+1,l}}{f_{k,l}}\right)_s = a - \frac{2a^2}{\delta_{k,l}} , \tag{55}$$

it then follows

$$\left(\ln \frac{f_{k+1,l}}{f_{k-1,l}}\right)_s = 2a - 2a^2\left(\frac{1}{\delta_{k,l}} + \frac{1}{\delta_{k-1,l}}\right) . \tag{56}$$

Starting from an alternative form of Eq.(47)

$$2a - 2\left(\ln \frac{g_{k+1,l}}{g_{k,l}}\right)_s = 2a\frac{f_{k+1,l}\, f_{k,l}}{g_{k+1,l}g_{k,l}} , \tag{57}$$

we obtain

$$\frac{w_{k+1,l} - w_{k,l}}{\delta_{k,l}} = \frac{-2\left(\ln \frac{g_{k+1,l}}{g_{k,l}}\right)_{ss}}{2a - 2\left(\ln \frac{g_{k+1,l}}{g_{k,l}}\right)_s} = \left(\ln \frac{f_{k+1,l}\, f_{k,l}}{g_{k+1,l}g_{k,l}}\right)_s , \tag{58}$$

by taking logarithmic derivative with respect to s. A shift from k to $k-1$ gives

$$\frac{w_{k,l} - w_{k-1,l}}{\delta_{k-1,l}} = \left(\ln \frac{f_{k,l}\, f_{k-1,l}}{g_{k,l}g_{k-1,l}}\right)_s . \tag{59}$$

Subtracting Eq.(59) from Eq.(58), we obtain

$$\frac{w_{k+1,l} - w_{k,l}}{\delta_{k,l}} - \frac{w_{k,l} - w_{k-1,l}}{\delta_{k-1,l}} = \left(\ln \frac{f_{k+1,l}}{f_{k-1,l}}\right)_s - \left(\ln \frac{g_{k+1,l}}{g_{k-1,l}}\right)_s . \tag{60}$$

By using the relations (53) and (56), we finally arrive at

$$\frac{w_{k+1,l} - w_{k,l}}{\delta_{k,l}} - \frac{w_{k,l} - w_{k-1,l}}{\delta_{k-1,l}} - \frac{1}{2}\left(\delta_{k,l} + \delta_{k-1,l}\right) + 2a^2\left(\frac{1}{\delta_{k,l}} + \frac{1}{\delta_{k-1,l}}\right) = 0 . \tag{61}$$

Similar to Eq.(32), Eq.(61) constitutes the first equation of the full-discretization of the SCHE, which can be cast into a simpler form:

$$\Delta^2 w_{k,l} = \frac{1}{\delta_{k,l}} M\left(\delta_{k,l} - \frac{4a^2}{\delta_{k,l}}\right) . \tag{62}$$

Next, we seek for the second equation of the full-discretization. Recalling (46)–(49), one could obtain

$$\frac{x_{k+1,l+1} - x_{k,l+1}}{x_{k+1,l} - x_{k,l}} = \frac{2a - 2\left(\ln \frac{g_{k+1,l+1}}{g_{k,l+1}}\right)_s}{2a - 2\left(\ln \frac{g_{k+1,l}}{g_{k,l}}\right)_s} = \frac{\left(\ln \frac{g_{k+1,l+1}}{f_{k+1,l}}\right)_s - \frac{1}{b}}{\left(\ln \frac{f_{k,l+1}}{g_{k,l}}\right)_s - \frac{1}{b}} , \tag{63}$$

here a shift from l to $l+1$ in (47) and a shift from k to $k+1$ in (49) are employed.

From Eqs.(50), (55) and (58), one can find the following two relations

$$\left(\ln \frac{g_{k+1,l+1}}{f_{k+1,l}}\right)_s = -\frac{w_{k+1,l}-w_{k,l}-2a^2}{2\delta_{k,l}} + \frac{1}{4}\left(x_{k+1,l}+x_{k,l}-2x_{k+1,l+1}\right), \tag{64}$$

$$\left(\ln \frac{f_{k,l+1}}{g_{k,l}}\right)_s = \frac{w_{k+1,l+1}-w_{k,l+1}+2a^2}{2\delta_{k,l+1}} - \frac{1}{4}\left(x_{k+1,l+1}+x_{k,l+1}-2x_{k,l}\right), \tag{65}$$

after some tedious algebraic manipulations. Substituting these two relations into (63), we finally obtain the second equation of the fully discrete analogue of the SCHE

$$\begin{aligned}\frac{\delta_{k,l+1}-\delta_{k,l}}{b} &+ \frac{1}{4}\delta_{k,l+1}\left(x_{k+1,l+1}+x_{k,l+1}-2x_{k,l}\right)\\ &+ \frac{1}{4}\delta_{k,l}\left(x_{k+1,l}+x_{k,l}-2x_{k+1,l+1}\right)\\ &= \frac{1}{2}\left(w_{k+1,l+1}+w_{k+1,l}-w_{k,l+1}-w_{k,l}\right).\end{aligned} \tag{66}$$

Taking the continuous limit $b\to 0$ in time, we have

$$\frac{\delta_{k,l+1}-\delta_{k,l}}{b} \to \frac{d\delta_k}{ds},$$

$$\delta_{k,l+1}\left(x_{k+1,l+1}+x_{k,l+1}-2x_{k,l}\right) \to 0,$$

$$\delta_{k,l+1}\delta_{k,l}\left(x_{k+1,l}+x_{k,l}-2x_{k+1,l+1}\right) \to 0,$$

and

$$\frac{1}{2}\left(w_{k+1,l+1}+w_{k+1,l}-w_{k,l+1}-w_{k,l}\right) \to w_{k+1}-w_k.$$

Therefore, one recovers exactly the second equation of the semi-discrete SCHE (33).

In summary, the fully discrete analogue of the SCHE and its determinant solution are given as follows:

The fully discrete analogue of the SCHE

$$\left\{\begin{aligned}&\frac{w_{k+1,l}-w_{k,l}}{\delta_{k,l}} - \frac{w_{k,l}-w_{k-1,l}}{\delta_{k-1,l}} - \frac{1}{2}\left(\delta_{k,l}+\delta_{k-1,l}\right) + 2a^2\left(\frac{1}{\delta_{k,l}}+\frac{1}{\delta_{k-1,l}}\right) = 0,\\ &\frac{\delta_{k,l+1}-\delta_{k,l}}{b} + \frac{1}{4}\delta_{k,l+1}\left(x_{k+1,l+1}+x_{k,l+1}-2x_{k,l}\right)\\ &\qquad + \frac{1}{4}\delta_{k,l}\left(x_{k+1,l}+x_{k,l}-2x_{k+1,l+1}\right) = \frac{1}{2}\left(w_{k+1,l+1}+w_{k+1,l}-w_{k,l+1}-w_{k,l}\right).\end{aligned}\right. \tag{67}$$

The determinant solution of the fully discrete SCHE

$$w_{k,l} = -2(\ln g_{k,l})_{ss} = -2\frac{\overline{h}_{k,l}g_{k,l}-h_{k,l}^2}{g_{k,l}^2},$$

$$x_{k,l} = 2ka - 2(\ln g_{k,l})_s = 2ka - 2\frac{h_{k,l}}{g_{k,l}},$$

$$\delta_{k,l} = x_{k+1,l}-x_{k,l} = 2a\frac{f_{k+1,l}f_{k,l}}{g_{k+1,l}g_{k,l}},$$

$$g_{k,l} = \left|\psi_i^{(j)}(k,l)\right|_{1\le i,j\le N}, \quad f_{k,l} = \left|\psi_i^{(j-1)}(k,l)\right|_{1\le i,j\le N},$$

$$h_{k,l} = \frac{\partial g_{k,l}}{\partial s} = \begin{vmatrix} \psi_1^{(0)}(k,l) & \psi_1^{(2)}(k,l) & \psi_1^{(3)}(k,l) & \cdots & \psi_1^{(N)}(k,l) \\ \psi_2^{(0)}(k,l) & \psi_2^{(2)}(k,l) & \psi_2^{(3)}(k,l) & \cdots & \psi_2^{(N)}(k,l) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_N^{(0)}(k,l) & \psi_N^{(2)}(k,l) & \psi_N^{(3)}(k,l) & \cdots & \psi_N^{(N)}(k,l) \end{vmatrix},$$

$$\overline{h}_{k,l} = \frac{\partial^2 g_{k,l}}{\partial s^2} = \begin{vmatrix} \psi_1^{(-1)}(k,l) & \psi_1^{(2)}(k,l) & \psi_1^{(3)}(k,l) & \cdots & \psi_1^{(N)}(k,l) \\ \psi_2^{(-1)}(k,l) & \psi_2^{(2)}(k,l) & \psi_2^{(3)}(k,l) & \cdots & \psi_2^{(N)}(k,l) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_N^{(-1)}(k,l) & \psi_N^{(2)}(k,l) & \psi_N^{(3)}(k,l) & \cdots & \psi_N^{(N)}(k,l) \end{vmatrix}$$
$$+ \begin{vmatrix} \psi_1^{(0)}(k,l) & \psi_1^{(1)}(k,l) & \psi_1^{(3)}(k,l) & \cdots & \psi_1^{(N)}(k,l) \\ \psi_2^{(0)}(k,l) & \psi_2^{(1)}(k,l) & \psi_2^{(3)}(k,l) & \cdots & \psi_2^{(N)}(k,l) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_N^{(0)}(k,l) & \psi_N^{(1)}(k,l) & \psi_N^{(3)}(k,l) & \cdots & \psi_N^{(N)}(k,l) \end{vmatrix},$$

$$\psi_i^{(j)}(k,l) = a_{i,1}p_i^j(1-ap_i)^{-k}\left(1-bp_i{}^{-1}\right)^{-l}e^{\xi_i} + a_{i,2}(-p_i)^j(1+ap_i)^{-k}\left(1+bp_i{}^{-1}\right)^{-l}e^{\eta_i},$$
$$\xi_i = p_i{}^{-1}s + \xi_{i0}, \quad \eta_i = -p_i{}^{-1}s + \eta_{i0}. \tag{68}$$

Note that s is an auxiliary parameter. By virtue of s, $h_{k,l}$ and $\overline{h}_{k,l}$ can be expressed as $h_{k,l} = \partial_s g_{k,l}$ and $\overline{h}_{k,l} = \partial_s^2 g_{k,l}$, respectively, because the auxiliary parameter s works on elements of the above determinant by $\partial_s \psi_i^{(n)}(k,l) = \psi_i^{(n-1)}(k,l)$.

Introducing new independent variables $X_{k,l} = x_{k,l}/\kappa$ and $\tilde{b} = b/\kappa$, we can include the parameter κ in the full-discrete SCHE (67):

$$\begin{cases} \dfrac{w_{k+1,l}-w_{k,l}}{\delta_{k,l}} - \dfrac{w_{k,l}-w_{k-1,l}}{\delta_{k-1,l}} - \dfrac{1}{2\kappa^2}(\delta_{k,l}+\delta_{k-1,l}) + 2a^2\left(\dfrac{1}{\delta_{k,l}}+\dfrac{1}{\delta_{k-1,l}}\right) = 0, \\ \dfrac{\delta_{k,l+1}-\delta_{k,l}}{\tilde{b}} + \dfrac{1}{4\kappa^2}\delta_{k,l+1}(X_{k+1,l+1}+X_{k,l+1}-2X_{k,l}) \\ \qquad + \dfrac{1}{4\kappa^2}\delta_{k,l}(X_{k+1,l}+X_{k,l}-2X_{k+1,l+1}) = \dfrac{1}{2}(w_{k+1,l+1}+w_{k+1,l}-w_{k,l+1}-w_{k,l}). \end{cases} \tag{69}$$

Similarly, the N-cuspon solution of the full-discrete SCHE (69) with the parameter κ is given as follows:

$$w_{k,l} = -2(\ln g_{k,l})_{ss} = -2\frac{\overline{h}_{k,l}g_{k,l} - h_{k,l}^2}{g_{k,l}^2},$$

$$X_{k,l} = \frac{2ka}{\kappa} - \frac{2}{\kappa}(\ln g_{k,l})_s = \frac{2ka}{\kappa} - \frac{2}{\kappa}\frac{h_{k,l}}{g_{k,l}},$$

$$\delta_{k,l} = X_{k+1,l} - X_{k,l} = \frac{2a}{\kappa}\frac{f_{k+1,l}f_{k,l}}{g_{k+1,l}g_{k,l}},$$

$$g_{k,l} = \left|\psi_i^{(j)}(k,l)\right|_{1\le i,j\le N}, \quad f_{k,l} = \left|\psi_i^{(j-1)}(k,l)\right|_{1\le i,j\le N},$$

$$h_{k,l}=\frac{\partial g_{k,l}}{\partial s}=\frac{1}{\kappa}\begin{vmatrix}\psi_1^{(0)}(k,l) & \psi_1^{(2)}(k,l) & \psi_1^{(3)}(k,l) & \cdots & \psi_1^{(N)}(k,l)\\ \psi_2^{(0)}(k,l) & \psi_2^{(2)}(k,l) & \psi_2^{(3)}(k,l) & \cdots & \psi_2^{(N)}(k,l)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \psi_N^{(0)}(k,l) & \psi_N^{(2)}(k,l) & \psi_N^{(3)}(k,l) & \cdots & \psi_N^{(N)}(k,l)\end{vmatrix},$$

$$\overline{h}_{k,l}=\frac{\partial^2 g_{k,l}}{\partial s^2}=\frac{1}{\kappa^2}\begin{vmatrix}\psi_1^{(-1)}(k,l) & \psi_1^{(2)}(k,l) & \psi_1^{(3)}(k,l) & \cdots & \psi_1^{(N)}(k,l)\\ \psi_2^{(-1)}(k,l) & \psi_2^{(2)}(k,l) & \psi_2^{(3)}(k,l) & \cdots & \psi_2^{(N)}(k,l)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \psi_N^{(-1)}(k,l) & \psi_N^{(2)}(k,l) & \psi_N^{(3)}(k,l) & \cdots & \psi_N^{(N)}(k,l)\end{vmatrix}$$

$$+\frac{1}{\kappa^2}\begin{vmatrix}\psi_1^{(0)}(k,l) & \psi_1^{(1)}(k,l) & \psi_1^{(3)}(k,l) & \cdots & \psi_1^{(N)}(k,l)\\ \psi_2^{(0)}(k,l) & \psi_2^{(1)}(k,l) & \psi_2^{(3)}(k,l) & \cdots & \psi_2^{(N)}(k,l)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \psi_N^{(0)}(k,l) & \psi_N^{(1)}(k,l) & \psi_N^{(3)}(k,l) & \cdots & \psi_N^{(N)}(k,l)\end{vmatrix},$$

$$\psi_i^{(j)}(k,l)=a_{i,1}p_i^j(1-ap_i)^{-k}\left(1-bp_i{}^{-1}\right)^{-l}e^{\xi_i}+a_{i,2}(-p_i)^j(1+ap_i)^{-k}\left(1+bp_i{}^{-1}\right)^{-l}e^{\eta_i},$$

$$\xi_i=p_i{}^{-1}s+\xi_{i0},\quad \eta_i=-p_i{}^{-1}s+\eta_{i0}. \tag{70}$$

5. Concluding remarks

In the present paper, bilinear equations and the determinant solution of the SCHE are obtained from the two-reduction of 2DTL equations. Based on this fact, integrable semi-and full-discrete analogues of the SCHE are constructed. The N-soliton solutions of both continuous and discrete SCHEs are formulated in the form of the Casorati determinant. Note that the short pulse equation was also obtained from the two-reduction of the 2DTL equation [19].

Finally, we remark that the present paper is one of our series of work in an attempt of obtaining integrable discrete analogues for a class of integrable nonlienar PDEs whose solutions possess singularities such as peakon, cuspon or loop soliton solutions. New discrete integrable systems obtained in this paper, along with the semi-discrete analogue for the Camassa-Holm equation [15] and the semi-discrete and fully discrete analogues of the short pulse equation [19] deserves further study in the future.

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