Bäcklund Transformations as exact integrable time-discretizations for the trigonometric Gaudin model

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Abstract

We construct a two-parameter family of Bäcklund transformations for the trigonometric classical Gaudin magnet. The approach follows closely the one introduced by E.Sklyanin and V.Kuznetsov (1998,1999) in a number of seminal papers, and takes advantage of the intimate relation between the trigonometric and the rational case. As in the paper by A.Hone, V.Kuznetsov and one of the authors (O.R.) (2001) the Bäcklund transformations are presented as explicit symplectic maps, starting from their Lax representation. The (expected) connection with the XXZ Heisenberg chain is established and the rational case is recovered in a suitable limit. It is shown how to obtain a "physical" transformation mapping real variables into real variables. The interpolating Hamiltonian flow is derived and some numerical iterations of the map are presented.

KEYWORDS: Bäcklund Transformations, Integrable maps, Gaudin systems, Lax representation, r-matrix.

1 Introduction

Bäcklund transformations are nowadays a widespread useful tool related to the theory of nonlinear differential equations. The first historical evidence of their mathematical significance was given by Bianchi [3] and Bäcklund [2] on their works on surfaces of constant curvature. A simple approach to understand their importance can be to regard them as a mechanism allowing to endow a given nonlinear differential equation with a nonlinear superposition principle yielding an infinite set of solutions through a merely *algebraic procedure* [17][1],[10]. Backlund transformations are indeed parametric families of difference equations encoding the whole set of symmetries of a given integrable dynamical system. For finite-dimensional integrable systems the technique of Bäcklund transformations leads to the construction of integrable Poisson maps that discretize a family of continuous flows [24],[23],[22],[20],[8],[7]. Actually in the last two decades numerous results have appeared in the field of exact discretization of many-body integrable systems employing the Bäcklund transformations tools [15][22][14][7][8][13][20]. For the *rational* Gaudin model such discretization has been obtained ten years ago in [6]; afterwards, these results have been used for constructing an integrable discretization of classical dynamical systems (as the Lagrange top) connected to Gaudin model through Inönu Wigner contractions [11][9][12].

The aim of the present work is to construct Bäcklund transformations for the Gaudin model in the partially anisotropic ("xxz") case, i.e. for the *trigonometric* Gaudin model. We point out that partial results on this issue have already been given in [16].

The paper is organized as follows.
In Section (2) we review the main features of the trigonometric Gaudin model from the point of view of its integrability structure. For the sake of completeness, in Section (3) we briefly recall the preliminary results on Bäcklund Transformations (BTs) for trigonometric Gaudin given in [16]. In Section (4) the *explicit form* of BTs is given; it is shown that they are indeed a trigonometric generalization of the rational ones (see [6]) which can be recovered in a suitable ("small angle") limit. The simplecticity of the transformations is also discussed in the same Section and the proof allows us to elucidate the (expected) link between the Darboux-dressing matrix and the elementary Lax matrix for the xxz Heisenberg magnet on the lattice. We end the Section by mentioning an open question, namely the construction of an explicit generating function for these Bäcklund transformations. In Section (5) we will show how our map can lead, with an appropriate choice of Bäcklund parameters, to physical transformations, i.e. transformations from real variables to real variables. In the last Section we show how a suitable continuous limit yields the interpolating Hamiltonian flow and finally present numerical examples of iteration of the map.

2 Gaudin magnet in the trigonometric case

In this section we briefly outline the integrability structure underlying the trigonometric Gaudin magnet. The Lax matrix of the model is given by the expression:

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} \tag{1}$$

$$A(\lambda) = \sum_{j=1}^{N} \cot(\lambda - \lambda_j) s_j^3, \qquad B(\lambda) = \sum_{j=1}^{N} \frac{s_j^-}{\sin(\lambda - \lambda_j)}, \qquad C(\lambda) = \sum_{j=1}^{N} \frac{s_j^+}{\sin(\lambda - \lambda_j)}. \tag{2}$$

In (1) and (58) $\lambda \in \mathbb{C}$ is the spectral parameter, λ_j are arbitrary real parameters of the model, while $\left(s_j^+, s_j^-, s_j^3\right)$, $j = 1, \dots, N$, are the dynamical variables of the system obeying to $\oplus^N sl(2)$ algebra, i.e.

$$\left\{s_j^3, s_j^{\pm}\right\} = \mp i \delta_{jk} s_k^{\pm}, \qquad \left\{s_j^+, s_j^-\right\} = -2i\delta_{jk} s_k^3, \tag{3}$$

By fixing the N Casimirs $s_j^2 = \left(s_j^3\right)^2 + s_j^+ s_j^-$ one obtains a symplectic manifold given by the direct sum of the correspondent N two-spheres.
Reformulating the Poisson structure in terms of the r-matrix formalism amounts to state that the Lax matrix satisfies the *linear* r-matrix Poisson algebra:

$$\left\{L(\lambda) \otimes I, I \otimes L(\mu)\right\} = \left[r_t(\lambda - \mu), L(\lambda) \otimes I + I \otimes L(\mu)\right], \tag{4}$$

where $r_t(\lambda)$ stands for the trigonometric r matrix [4]:

$$r_t(\lambda) = \frac{i}{\sin(\lambda)} \begin{pmatrix} \cos(\lambda) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(\lambda) \end{pmatrix}, \tag{5}$$

Equation (4) entails the following Poisson brackets for the functions (2):

$$\begin{aligned} \{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} &= \{C(\lambda), C(\mu)\} = 0, \\ \{A(\lambda), B(\mu)\} &= i\frac{\cos(\lambda - \mu)B(\mu) - B(\lambda)}{\sin(\lambda - \mu)}, \\ \{A(\lambda), C(\mu)\} &= i\frac{C(\lambda) - \cos(\lambda - \mu)C(\mu)}{\sin(\lambda - \mu)}, \\ \{B(\lambda), C(\mu)\} &= i\frac{2(A(\mu) - A(\lambda))}{\sin(\lambda - \mu)}. \end{aligned} \tag{6}$$

By taking the residues at the simple poles of the determinant of the Lax matrix we get a set of N integrals of motion for our model (the poles of order two give the Casimirs):

$$H_j = \sum_{k \neq j}^{N} \frac{2cos(\lambda_j - \lambda_k) s_j^3 s_k^3 + s_j^+ s_k^- + s_j^- s_k^+}{sin(\lambda_j - \lambda_k)} \tag{7}$$

Only $N-1$ among the above functions are independent as $\sum_{j=1}^{n} H_j = 0$; the N^{th} integral is given by the projection of the total spin on the z axis:

$$H_0 = \sum_{j=1}^{N} s_j^3 \doteq J^3 \tag{8}$$

The Hamiltonians H_n are in involution for the Poisson bracket (3):

$$\{H_i, H_j\} = 0 \quad i, j = 0, \ldots, N-1 \tag{9}$$

The corresponding Hamiltonian flows are then given by:

$$\frac{ds_j^3}{dt_i} = \{H_i, s_j^3\} \qquad \frac{ds_j^{\pm}}{dt_n} = \{H_i, s_j^{\pm}\} \tag{10}$$

3 A first approach to Darboux-dressing matrix

For clarity and completeness in this Section we recall the results already appeared in [16]. The leading observation is that by performing the "uniformization" mapping:

$$\lambda \to z \doteq e^{i\lambda}$$

the N-sites trigonometric Lax matrix takes a rational form in z that resembles the $2N$-sites rational Lax matrix plus an additional reflection symmetry (see also [5]); in fact, by performing the substitution (3), the Lax matrix (1) becomes:

$$L(z) = iJ^3 + \sum_{j=1}^{N} \left(\frac{L_1^j}{z - z_j} - \sigma_3 \frac{L_1^j}{z + z_j} \sigma_3 \right), \tag{11}$$

Where the matrices $L_1^j, \quad j = 1, \ldots, N$, are given by:

$$L_1^j = iz_j \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix}$$

So, equation (11) entails the following involution on $L(z)$:

$$L(z) = \sigma_3 L(-z) \sigma_3, \tag{12}$$

The dressed Lax matrix, say $\tilde{L}$, has to yield the same spectral invariants as L, so the two matrices are related by a similarity transformation:

$$L(z) \to D(z) L(z) D^{-1}(z) \equiv \tilde{L}(z) \tag{13}$$

But L and $\tilde{L}$ have to enjoy the same reflection symmetry (12) too: to preserve this involution the Darboux dressing matrix D has to share with L the property (12); the elementary dressing matrix D is then obtained by requiring the existence of only one pair of opposite poles for D in the complex plane of the spectral parameter. We will show in the next Section that, thanks to this constraint, once returned to the real line, one recovers the form of the Lax matrix for the elementary xxz Heisenberg spin chain: on the other hand, this is quite natural if one just recalls that for the rational Gaudin model the elementary Darboux-dressing matrix is given by the Lax matrix for the elementary xxx Heisenberg spin chain. The previous observations lead to the following Darboux matrix:

$$D(z) = D_\infty + \frac{D_1}{z-\xi} - \sigma^3 \frac{D_1}{z+\xi} \sigma^3 \tag{14}$$

By taking the limit $z \to \infty$ in (13) it is readily seen that D_∞ has to be a diagonal matrix. In order to ensure that L and $\tilde{L}$ have the same rational structure in z, we rewrite equation (12) in the form:

$$\tilde{L}(z)D(z) = D(z)L(z) \tag{15}$$

Now it is clear that both sides have the same residues at the poles $z = z_j$, $z = \xi_j$ (it is unnecessary to look at the poles in $z = -z_j$ and $z = -\xi_j$ because of the symmetry (12)), so that the following set of equations have to be satisfied:

$$\tilde{L}_1^{(j)} D(z_j) = D(z_j) L_1^{(j)}, \tag{16}$$

$$\tilde{L}(\xi) D_1 = D_1 L(\xi). \tag{17}$$

In principle, equations (16), (17) yield a Darboux matrix depending *both* on the old (untilded) variables and the new (tilded) ones, implying in turn an implicit relationship between the same variables. To get an explicit relationship one has to resort to the so-called spectrality property [8] [7]. To this aim we need to force the determinant of the Darboux matrix $D(z)$ to have, besides the pair of poles at $z = \pm\xi$, a pair of opposite *nondynamical* zeroes, say at $z = \pm\eta$, and to allow the matrix D_1 to be proportional to a projector [16]. Again by symmetry it suffices to consider just one of these zeroes. If η is a zero of $\det D(z)$, then $D(\eta)$ is a rank one matrix, possessing a one dimensional kernel $|K(\eta)\rangle$; the equation (15) :

$$\tilde{L}(\eta)D(\eta) = D(\eta)L(\eta). \tag{18}$$

entails

$$D(\eta)L(\eta)|K(\eta)\rangle = 0 \tag{19}$$

And this equation in turn allows to infer that $|K(\eta)\rangle$ is an eigenvector for the Lax matrix $L(\eta)$:

$$L(\eta)|K(\eta)\rangle = \mu(\eta)|K(\eta)\rangle, \tag{20}$$

This relations gives a direct link between the parameters appearing in the dressing matrix D and the *old* dynamical variables in L. Because of (17) we have another one dimensional kernel $|K(\xi)\rangle$ of D_1, such that:

$$L(\xi)|K(\xi)\rangle = \mu(\xi)|K(\xi)\rangle. \tag{21}$$

In [16] we have shown how the two spectrality conditions (20), (21) enable to write D in terms of the old dynamical variables and of the two Bäcklund parameters ξ and η. The explicit expression of the Darboux dressing matrix is given by:

$$D(z) = \frac{\beta z}{z^2-\xi^2}\begin{pmatrix} \frac{z(p(\eta)\eta-p(\xi)\xi)}{\gamma}+\frac{(p(\xi)\eta-p(\eta)\xi)\eta\xi}{\gamma z} & \frac{\xi^2-\eta^2}{\gamma} \\ \frac{\gamma p(\xi)p(\eta)\left(\xi^2-\eta^2\right)}{\eta\xi} & \frac{\gamma(p(\eta)\eta-p(\xi)\xi)}{z}+\frac{\gamma z(p(\xi)\eta-p(\eta)\xi)}{\eta\xi}\end{pmatrix}. \tag{22}$$

In this expression β is an inessential (with respect to the form of the BT) global multiplicative factor, γ is an undeterminate parameter that, when passing to the real line, we will fix in order to recover the form of the Lax matrix for the discrete xxz Heisenberg spin chain, the functions $p(\eta)$ and $p(\xi)$ characterize completely the kernels of $D(\eta)$ and $D(\xi)$. In fact we have the following formulas [16]:

$$|K(\xi)\rangle = \begin{pmatrix} 1 \\ p(\xi)\end{pmatrix} \qquad |K(\eta)\rangle = \begin{pmatrix} 1 \\ p(\eta)\end{pmatrix} \tag{23}$$

As $|K(\xi)\rangle$ and $|K(\eta)\rangle$ are respectively eigenvectors of $L(\xi)$ and $L(\eta)$, $p(\xi)$ and $p(\eta)$ must satisfy:

$$p(\xi) = \frac{\mu(\xi)-A(\xi)}{B(\xi)}, \qquad p(\eta) = \frac{\mu(\eta)-A(\eta)}{B(\eta)} \tag{24}$$

with $A(z)$, $B(z)$, $C(z)$ given by (2) and $\mu^2(z) = A^2(z)+B(z)C(z)$.

4 Explicit map and an equivalent approach to Darboux-dressing matrix

The matrix (22) contains just one set of dynamical variables so that the relation (13) gives now an explicit map between the variables $\left(\tilde{s}_j^+, \tilde{s}_j^-, \tilde{s}_j^3\right)$ and $\left(s_j^+, s_j^-, s_j^3\right)$. The map is easily found by (16); it reads:

$$\begin{aligned}\tilde{s}_k^3 = & \frac{p(\xi)p(\eta)\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\eta^2\right)p(\xi)\xi-\left({z_k}^2-\xi^2\right)p(\eta)\eta\right)s_k^- z_k}{\Delta_k}+ \\ & \frac{\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\xi^2\right)p(\xi)\eta-p(\eta)\xi\left({z_k}^2-\eta^2\right)\right)s_k^+ z_k}{\Delta_k}+ \\ & \frac{s_k^3\Big[p(\xi)p(\eta)\left(\left(\xi^2+{z_k}^2\right)\left(\eta^2+{z_k}^2\right)-\left(\eta^2+\xi^2\right)-8\eta^2\xi^2{z_k}^2\right)}{\Delta_k}+ \\ & -\frac{\left(\eta\xi\left(\xi^2-{z_k}^2\right)\left(\eta^2-{z_k}^2\right)\left(p(\xi)^2+p(\eta)^2\right)\right)\Big]}{\Delta_k}\end{aligned} \tag{25a}$$

$$\tilde{s}_k^+ = -\frac{\gamma^2 p(\xi)^2 p(\eta)^2 \left(\eta^2-\xi^2\right)^2 s_k^- z_k^2}{\xi\eta\Delta_k} + \frac{\gamma^2\left(\left({z_k}^2-\xi^2\right)p(\xi)\eta - p(\eta)\xi\left({z_k}^2-\eta^2\right)\right)^2 s_k^+}{\eta\xi\Delta_k} + \frac{2\gamma^2 p(\xi)p(\eta)\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\xi^2\right)p(\xi)\eta - p(\eta)\xi\left({z_k}^2-\eta^2\right)\right)s_k^3 z_k}{\eta\xi\Delta_k} \tag{25b}$$

$$\tilde{s}_k^- = -\frac{\left(\eta^2-\xi^2\right)^2 s_k^+ z_k^2 \xi\eta}{\gamma^2\Delta_k} + \frac{\left(\left({z_k}^2-\eta^2\right)p(\xi)\xi - \left({z_k}^2-\xi^2\right)p(\eta)\eta\right)^2 s_k^- \xi\eta}{\gamma^2\Delta_k} + \frac{2\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\eta^2\right)p(\xi)\xi - \left({z_k}^2-\xi^2\right)p(\eta)\eta\right)s_k^3 z_k\xi\eta}{\gamma^2\Delta_k} \tag{25c}$$

where Δ_k is proportional to the determinant of $D(z_k)$, i.e.

$$\Delta_k = (z_k^2-\xi^2)(z_k^2-\eta^2)(p(\xi)\eta - p(\eta)\xi)(p(\eta)\eta - p(\xi)\xi) \tag{26}$$

Formulas (25a), (25b), (25c) define a two-parameter Bäcklund transformation, the parameters being ξ and η: as we will show in the next section, it is a crucial point to have a *two*-parameter family of transformations when looking for a physical map from real variables to real variables. As outlined in the previous Section, we now show that indeed, by posing:

$$b = i\sqrt{\eta\xi} \tag{27}$$

the expression (22) of the dressing matrix goes into the expression of the elementary Lax matrix for the classical, partially anisotropic, Heisenberg spin chain on the lattice [4].

Obviously two matrices differing only for a global multiplicative factor give rise to the same transformation. So we omit the term $\frac{\beta z}{z^2-\xi^2}$ in (22), and, after returning to the real line and taking into account (27), we write for the diagonal part D_d of (22):

$$D_d = \frac{i}{2}\Big((p(\xi)-p(\eta))(v-w)\mathbb{1} + (p(\xi)+p(\eta))(v+w)\sigma^3\Big) \tag{28}$$

where $v(\xi,\eta)$ and $w(\xi,\eta)$ are given by:

$$v(\xi,\eta) = \frac{z\xi}{\sqrt{\eta\xi}} - \frac{\eta\sqrt{\eta\xi}}{z} \qquad w(\xi,\eta) = \frac{\xi\sqrt{\eta\xi}}{z} - \frac{z\eta}{\sqrt{\eta\xi}} = -v(\eta,\xi) \tag{29}$$

Passing to the real line we substitute:

$$\xi \to e^{i\zeta_1} \qquad \eta \to e^{i\zeta_2} \qquad z \to e^{i\lambda} \tag{30}$$

and take a suitable redefinition of the Bäcklund parameters to clarify the structure of the D matrix:

$$\lambda_0 \doteq \frac{\zeta_1+\zeta_2}{2} \qquad \mu \doteq \frac{\zeta_1-\zeta_2}{2} \tag{31}$$

With these positions it is simple to find that $v - w = 4ie^{i\lambda_0} sin(\lambda - \lambda_0)cos(\mu)$ and $v + w = 4ie^{i\lambda_0} cos(\lambda - \lambda_0)sin(\mu)$. So, considering equation (28) jointly with the anti-diagonal part of (22), the dressing matrix can be written as:

$$\begin{aligned} D(\lambda) =& \alpha\Big[sin(\lambda-\lambda_0)\mathbb{1} + \frac{p(\zeta_1)+p(\zeta_2)}{p(\zeta_1)-p(\zeta_2)} tan(\mu)cos(\lambda-\lambda_0)\sigma^3 + \\ &+ \frac{2sin(\mu)}{p(\zeta_2)-p(\zeta_1)} \begin{pmatrix} 0 & 1 \\ -p(\zeta_1)p(\zeta_2) & 0 \end{pmatrix}\Big] \end{aligned} \tag{32}$$

where α is the global factor $2e^{i\lambda_0}(p(\zeta_2) - p(\zeta_1))$. A last change of variables allows to identify, as we will see, the dressing matrix with the elementary Lax matrix of the classical xxz Heisenberg spin chain on the lattice, and furthermore to recover the form of the Darboux matrix for the *rational* Gaudin model [6][18] in the limit of *small angles*. By letting:

$$p(\zeta_1) = -Q \qquad p(\zeta_2) = \frac{2sin(\mu)}{P} - Q \tag{33}$$

equation (32) becomes:

$$D(u) = \alpha \begin{pmatrix} sin(\lambda-\lambda_0-\mu) + PQ\,cos(\lambda-\lambda_0) & P\,cos(\mu) \\ Q\,sin(2\mu) - PQ^2 cos(\mu) & sin(\lambda-\lambda_0+\mu) - PQ\,cos(\lambda-\lambda_0) \end{pmatrix} \tag{34}$$

Obviously now we can repeat the argument made before about spectrality; indeed now $D\big|_{\lambda=\lambda_0+\mu}$ and $D\big|_{\lambda=\lambda_0-\mu}$ are rank one matrices. So if Ω_+ and Ω_- are respectively the kernels of $D(\lambda_0+\mu)$ and $D(\lambda_0-\mu)$ one has again that Ω_+ and Ω_- are eigenvectors of $L(\lambda_0+\mu)$ and $L(\lambda_0-\mu)$ with eigenvalues γ_+ and γ_- where

$$\gamma_\pm^2 = A^2(\lambda) + B(\lambda)C(\lambda)\Big\|_{u=\lambda_0\pm\mu}$$

The two kernels are given by:

$$\Omega_+ = \begin{pmatrix} 1 \\ -Q \end{pmatrix} \qquad \Omega_- = \begin{pmatrix} P \\ 2\,sin(\mu) - PQ \end{pmatrix} \tag{35}$$

and the eigenvectors relations yields the following expression of P and Q in terms of the old variables only:

$$Q = Q(\lambda_0+\mu) = \frac{A(\lambda)-\gamma(\lambda)}{B(\lambda)}\Big\|_{\lambda=\lambda_0+\mu} \qquad \frac{1}{P} = \frac{Q(\lambda_0+\mu) - Q(\lambda_0-\mu)}{2\,sin(\mu)} \tag{36}$$

The explicit map can be found by equating the residues at the poles $\lambda = \lambda_k$ in (15), that is by the relation:

$$\tilde{L}_k D_k = D_k L_k \tag{37}$$

where

$$L_k = \begin{pmatrix} s_k^3 & s_k^- \\ s_k^+ & -s_k^3 \end{pmatrix} \qquad D_k = D(\lambda = \lambda_k) \tag{38}$$

or by performing the needed changes of variables in (25a), (25b), (25c). Anyway now the map reads:

$$\begin{aligned}\tilde{s}_k^3 =& \frac{2\,cos^2(\mu) - (cos^2(\mu) + cos^2(\delta_0^k))(1 - 2PQ\,sin(\mu) + P^2Q^2)}{\Delta_k} s_k^3 + \\ &+ \frac{P\,cos(\mu)(sin(\delta_+^k) - PQ\,cos(\delta_0^k))}{\Delta_k} s_k^+ + \\ &- \frac{Q\,cos(\mu)(2\,sin(\mu) - PQ)(sin(\delta_-^k) + PQ\,cos(\delta_0^k))}{\Delta_k} s_k^-\end{aligned} \tag{39a}$$

$$\begin{aligned}\tilde{s}_k^+ =& \frac{(sin(\delta_+^k) - PQ\,cos(\delta_0^k))^2}{\Delta_k} s_k^+ - \frac{(Q^2 cos^2(\mu)(2\,sin(\mu) - PQ))^2}{\Delta_k} s_k^- + \\ &+ \frac{2Q\,cos(\mu)(2\,sin(\mu) - PQ)(sin(\delta_+^k) - PQ\,cos(\delta_0^k))}{\Delta_k} s_k^3\end{aligned} \tag{39b}$$

$$\begin{aligned}\tilde{s}_k^- =& \frac{(sin(\delta_-^k) + PQ\,cos(\delta_0^k))^2}{\Delta_k} s_k^- - \frac{P^2\,cos^2(\mu)}{\Delta_k} s_k^+ + \\ &- \frac{2P\,cos(\mu)(sin(\delta_-^k) + PQ\,cos(\delta_0^k))}{\Delta_k} s_k^3\end{aligned} \tag{39c}$$

where for typesetting brevity we have put:

$$\begin{cases} \delta_0^k = \lambda_k - \lambda_0 \\ \delta_\pm^k = \lambda_k - \lambda_0 \pm \mu \end{cases} \tag{40}$$

and we have denoted by Δ_k the determinant of $D(\lambda_k)$, that is:

$$\Delta_k := sin(\lambda_k - \lambda_0 - \mu) sin(\lambda_k - \lambda_0 + \mu)(1 - 2PQ\,sin(\mu) + P^2Q^2)$$

At this point we can show that for "small" λ_0 and μ one obtains, at first order, the Bäcklund for the rational Gaudin model, independently found by Sklyanin [18] on one hand and Hone, Kuznetsov and Ragnisco [6] on the other, as the composition of two one-parameter Bäcklunds. So let us take $\lambda_0 \to h\lambda_0$, $\mu \to h\mu$ and $\lambda \to h\lambda$ where h is the expansion parameter. One has:

$$cot(\lambda - \lambda_k) \to \frac{1}{h(\lambda - \lambda_k)} + O(h) \qquad \frac{1}{sin(\lambda - \lambda_k)} \to \frac{1}{h(\lambda - \lambda_k)} + O(h),$$

so that $Q \to q^r + O(h^2)$ where the superscript r is for "rational" . Thus, q^r coincides with the variable q that one finds with the rational r-matrix [6]. For the variable P one has:

$$P \to h(p^r + O(h^2)) \qquad \text{where} \qquad p^r = \frac{2\mu}{q^r(\lambda_0 + \mu) - q^r(\lambda_0 - \mu)}$$

With these expressions it is straightforward to see that the matrix (34) goes into:

$$D(\lambda) \to h \begin{pmatrix} \lambda - \lambda_0 - \mu + p^r q^r & p^r \\ q^r(2\mu - p^r q^r) & \lambda - \lambda_0 + \mu - p^r q^r \end{pmatrix} + O(h^3) \tag{41}$$

Or, needless to say:

$$D(\lambda) \to hD^r(\lambda) + O(h^3)$$

The limit of "small angles" in (25a), (25b), (25c) obviously leads to the rational map of [6].

4.1 Symplecticity

In this subsection we face the question of the simplecticity of our map; the correspondence with the rational Bäcklund in the limit of "small angles" shows that the transformations are surely canonical at this order. Indeed, as our map is explicit, we could check by brute-foce calculations whether the Poisson structure (3) is preserved by tilded variables. However we will follow a finer argument due to Sklyanin [19] . Suppose that $D(\lambda)$ solves the *quadratic* Poisson bracket, that is

$$\{D^1(\lambda), D^2(\tau)\} = [r_t(\lambda-\tau), D^1(\lambda)\otimes D^2(\tau)] \tag{42}$$

and consider the relation

$$\tilde{L(\lambda)}\tilde{D}(\lambda-\lambda_0) = D(\lambda-\lambda_0)L(\lambda) \tag{43}$$

in an extended phase space, where the variables of D commutes with those of L. Note that in (43) we have used tilded variables also for $D(\lambda)$ (in its l.h.s.) because (43) is indeed the Bäcklund transformation in this extended phase space, whose coordinates are $(s_j^3, s_j^\pm, P, Q)$, so that we have also a $\tilde{P}$ and a $\tilde{Q}$. The key observation is that if both L and D have the same Poisson structure, given by equation (42), then also LD and DL have the same Poisson structure because in this extended space the variables of D and the variables of L Poisson commute. This means that the transformation (43) defines a "canonical" transformation. Sklyanin showed [19] that if one now restricts the variables on the constraint manifold $\tilde{P}=P$ and $\tilde{Q}=Q$ the symplecticity is preserved; this constraint however gives a dependence of P and Q on the variable of L that for consistency must be the same as the one given by the equation (43) on this constrained manifold. But there (43) is just given by the usual BT:

$$\tilde{L}(\lambda)D(\lambda-\lambda_0) = D(\lambda-\lambda_0)L(\lambda)$$

so that the BT preserves the spectrum of $L(\lambda)$ and is canonical. What remains to show is that indeed (42) is fullfilled by our $D(\lambda)$. Obviously $D(\lambda)$ cannot have this Poisson structure for any Poisson bracket between P and Q. In the rational case the Darboux matrix has the Poisson structure imposed by the rational r-matrix provided P and Q are canonically conjugated in the extended space [19] (and this is why they were called P and Q); in the trigonometric case P and Q are no longer canonically conjugated but obviously one recovers this property at order h in the "small angle" limit.

First note that $D(\lambda)$ can be conveniently written as:

$$D(\lambda) = \alpha\, cos(\mu)\Big[sin(\lambda)\mathbb{1} + a\, cos(\lambda)\sigma^3 + \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\Big] \tag{44}$$

where the coefficients a, b, c are given by:

$$a = \frac{PQ - sin(\mu)}{cos(\mu)}, \quad b = P, \quad c = 2Q\, sin(\mu) - PQ^2 \tag{45}$$

Inserting (44) in (42) we have the following constraints on the coefficients α, a, b, c:

$$\{\alpha, \alpha a\} = 0 \quad \longrightarrow \quad \alpha = \alpha(PQ) \tag{46}$$

$$\{\alpha, \alpha b\} = -\alpha^2 ab \quad \longrightarrow \quad \{\alpha, P\} = \alpha P \frac{sin(\mu) - PQ}{cos(\mu)} \tag{47}$$

$$\{\alpha, \alpha c\} = \alpha^2 ac \quad \longrightarrow \quad \{\alpha, Q\} = -\alpha Q \frac{sin(\mu) - PQ}{cos(\mu)} \tag{48}$$

All remaining relations, namely

$$\{\alpha b, \alpha c\} = 2\alpha^2 a \quad \{\alpha a, \alpha b\} = \alpha^2 b \quad \{\alpha a, \alpha c\} = -\alpha^2 c \tag{49}$$

give the same constraint, i.e.:

$$\{Q, P\} = \frac{1 + P^2Q^2 - 2PQsin(\mu)}{cos(\mu)} \tag{50}$$

This expression can be used to find, after a simple integration,

$$\alpha(PQ) = \frac{k}{\sqrt{(1 + P^2Q^2 - 2PQsin(\mu))}}$$

so that the Darboux matrix (34) is fixed (up to a constant multiplicative factor k). As previously pointed out, it turns out that the Darboux-dressing matrix (34) is formally equivalent to the elementary Lax matrix for the classical xxz Heisenberg spin chain on the lattice [4]. Moreover it has also the same (quadratic) Poisson bracket. This suggests that indeed $D(\lambda)$ can be recast in the form (see [4]):

$$D(\lambda) = \mathscr{S}_0 1 + \frac{i}{sin(\lambda)}\left(\mathscr{S}_1\sigma^1 + \mathscr{S}_2\sigma^2 + cos(\lambda)\mathscr{S}_3\sigma^3\right) \tag{51}$$

where the σ^i are the Pauli matrices and the variables $\mathscr{S}_i$ satisfies the following Poisson bracket:

$$\begin{array}{c} \{\mathscr{S}_i, \mathscr{S}_0\} = J_{jk}\mathscr{S}_j\mathscr{S}_k \\ \{\mathscr{S}_i, \mathscr{S}_j\} = -\mathscr{S}_0\mathscr{S}_k \end{array} \tag{52}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and J_{jk} is antisymmetric with $J_{12} = 0, J_{13} = J_{23} = 1$. Indeed it is straightforward to show that the link between the two representations (44) and (51), up to a factor (44) $cos(\mu)sin(\lambda)$ that does not affect neither (15) nor the Poisson bracket (42), is given by :

$$\alpha = \mathscr{S}_0 \quad -\frac{i\alpha}{2}(b + c) = \mathscr{S}_1 \quad \frac{\alpha}{2}(b - c) = \mathscr{S}_2 \quad -ia\alpha = \mathscr{S}_3 \tag{53}$$

and the Poisson brackets (46), (47), (48), (49) correspond to those given in (52).
An open question regards the generating function of our BT. So far we have not been able to write it in any closed form; in our opinion the question is harder than in the rational case (where the generating function is known from [6]): in fact the rational map corresponding to (25a), (25b), (25c) can be written as the composition of two simpler *one*-parameter Bäcklund transformations, and this entails the same property to hold for the generating function; in the trigonometric case a factorization of the Bäcklund transformations cannot preserve the symmetry (12): so probably one should look for symmetry-violating generating functions such that their composition enables symmetry to be restored.

5 Physical Bäcklund transformations

The transformations we have found do not map, in general, real variables into real variables. A sufficient condition to ensure this property is given by:

$$\zeta_1 = \bar{\zeta}_2 \tag{54}$$

which amounts te require that λ_0 and μ in (39a), (39b), (39c) be, respectively, real and imaginary numbers.
Indeed we claim that, if (54) holds, starting from a physical solution of the dynamical equations, we can find a new physical solution with two real parameters. Let us prove the assertion. Bäcklund transformation are obtained by (37); starting from a real solution means starting from an Hermitian L_k. Thus, if the transformed matrix $\tilde{L}_k$ has to be Hermitian too, the Darboux matrix has to be proportional to a unitary matrix. We will show that indeed this is the case, by choosing $\zeta_1 = \bar{\zeta}_2$ and $\gamma(\zeta_1) = -\bar{\gamma}(\zeta_2)$. Note that the condition on the γ's specifies their relative sign (the sheet on the Riemann surface) , inessential for the spectrality property. Hereafter in this section we pose:

$$\zeta_1 = \lambda_0 + i\mu \qquad \zeta_2 = \lambda_0 - i\mu \qquad (\lambda_0, \mu) \in \mathbb{R}^2 \tag{55}$$

The Darboux matrix at $\lambda = \lambda_k$ can be rewritten as:

$$D_k = \begin{pmatrix} sin(v_k - i\mu) + PQ\, cos(v_k) & P\, cos(i\mu) \\ Q\, cos(i\mu)\,(2\, sin(i\mu) - PQ) & sin(v_k + i\mu) - PQ\, cos(v_k) \end{pmatrix} \tag{56}$$

where $v_k \equiv \lambda_k - \lambda_0$ (we are assuming that the parameters λ_k of the model are real) We recall that in (56):

$$Q = Q(\zeta_1) = \frac{A(\zeta_1) - \gamma(\zeta_1)}{B(\zeta_1)} = -\frac{C(\zeta_1)}{A(\zeta_1) + \gamma(\zeta_1)}; \qquad P = \frac{2\, sin(i\mu)}{Q(\zeta_1) - Q(\zeta_2)}. \tag{57}$$

Furthermore it is a simple matter to show that

$$A(\zeta_1) = \bar{A}(\zeta_2); \quad B(\zeta_1) = \bar{C}(\zeta_2); \quad C(\zeta_1) = \bar{B}(\zeta_2). \tag{58}$$

If the off-diagonal terms of $D_k D_k^\dagger$ has to be zero, then the following equation has to be fullfilled:

$$P(sin(v_k - i\mu) - \bar{P}\bar{Q}\, cos(v_k)) = \bar{Q}(2\, sin(i\mu) + \bar{P}\bar{Q})(sin(v_k - i\mu) + PQ\, cos(v_k)) \tag{59}$$

Using relations (57) and rearranging the terms,the previous equation becomes:

$$\begin{aligned} &(\frac{1}{\bar{Q}(\zeta_1)} - \frac{1}{\bar{Q}(\zeta_2)})cos(i\mu)sin(v_k) + (\frac{1}{\bar{Q}(\zeta_1)} + \frac{1}{\bar{Q}(\zeta_2)})cos(v_k)sin(i\mu) = \\ &= (Q(\zeta_1) - Q(\zeta_2))cos(i\mu)sin(v_k) + cos(v_k)sin(i\mu)(q(\zeta_1) - q(\zeta_2)) \end{aligned} \tag{60}$$

Note that the relations (58) gives $\gamma^2(\zeta_1) = \overline{\gamma^2(\zeta_2)}$. By choosing an opposite sign for the two γ's, then the relation $\bar{Q}(\zeta_1) = -\frac{1}{Q(\zeta_2)}$ holds. With this constraint the equation (60) holds too. Moreover the choice $\gamma(\zeta_1) = -\bar{\gamma}(\zeta_2)$ makes the diagonal terms in $D_k D_k^\dagger$ equal. This shows that, under the given assumptions, D_k is an unitary matrix.

6 Interpolating Hamiltonian flow

The Bäcklund transformation can be seen as a time discretization of a one-parameter (λ_0) family of hamiltonian flows with the difference $i(\zeta_2 - \zeta_1)$ playing the role of the time-step. If it goes to zero we recover the correspondent continuous flows. To clarify this point, let us set:

$$\zeta_1 = \lambda_0 + i\frac{\epsilon}{2} \qquad \zeta_2 = \lambda_0 - i\frac{\epsilon}{2} \tag{61}$$

and take the limit $\epsilon \to 0$.
We have:

$$Q = \frac{A(\lambda_0) - \gamma(\lambda_0)}{B(\lambda_0)} + O(\epsilon) \equiv Q_0 + O(\epsilon) \tag{62}$$

$$P = -\frac{i\epsilon}{2}\frac{B(\lambda_0)}{\gamma(\lambda_0)} + O(\epsilon^2) \equiv \frac{i\epsilon}{2}P_0 + O(\epsilon^2) \tag{63}$$

and for the dressing matrix we can write:

$$\begin{aligned} D(\lambda) &= k\, sin(\lambda-\lambda_0)\mathbb{1} + \\ &+\frac{i\epsilon k}{2}\begin{pmatrix} cos(\lambda-\lambda_0)(P_0Q_0-1) & P_0 \\ Q_0(2-P_0Q_0) & cos(\lambda-\lambda_0)(1-P_0Q_0) \end{pmatrix} + O(\epsilon^2) \end{aligned} \tag{64}$$

Reorganizing the terms with the help of P_0 and Q_0 given in the equations (62) and (63) we arrive at the expression:

$$\begin{aligned} D(\lambda) &= k\, sin(\lambda-\lambda_0)\mathbb{1} + \\ &-\frac{i\epsilon k}{2\gamma(\lambda_0)}\begin{pmatrix} A(\lambda_0)cos(\lambda-\lambda_0) & B(\lambda_0) \\ C(\lambda_0) & -A(\lambda_0)cos(\lambda-\lambda_0) \end{pmatrix} + O(\epsilon^2) \end{aligned} \tag{65}$$

It is now straightforward to show that in the limit $\epsilon \to 0$ the equation of the map $\tilde{L}D = DL$ turns into the Lax equation for a continuous flow:

$$\dot{L}(\lambda) = [L(\lambda), M(\lambda,\lambda_0)] \tag{66}$$

where the time derivative is given by:

$$\dot{L} = \lim_{\epsilon\to 0}\frac{\tilde{L} - L}{\epsilon} \tag{67}$$

and the matrix $M(\lambda,\lambda_0)$ has the form

$$\frac{i}{2\gamma(\lambda_0)}\begin{pmatrix} A(\lambda_0)cot(\lambda-\lambda_0) & \frac{B(\lambda_0)}{sin(\lambda-\lambda_0)} \\ \frac{C(\lambda_0)}{sin(\lambda-\lambda 0)} & -A(\lambda_0)cot(\lambda-\lambda_0) \end{pmatrix} \tag{68}$$

The system (66) can be cast in Hamiltonian form:

$$\dot{L}(\lambda) = \{H(\lambda_0), L(\lambda)\} \tag{69}$$

with the Hamilton's function given by:

$$H(\lambda_0) = \gamma(\lambda_0) = \sqrt{A^2(\lambda_0) + B(\lambda_0)C(\lambda_0)} \tag{70}$$

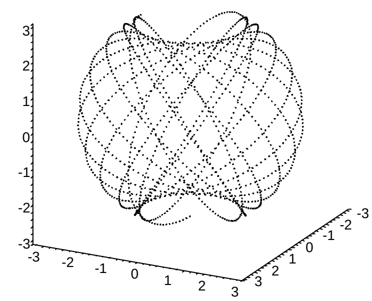

Figure 1: input parameters: $s_1^+ = 2+i$, $s_1^- = 2-i$, $s_1^3 = -2$, $s_2^+ = 50+40i$, $s_2^- = 50-40i$, $s_2^3 = 70$, $\lambda_1 = \pi/110$, $\lambda_2 = 7\pi/3$, $\lambda_0 = 0.1$, $\mu = -0.002i$

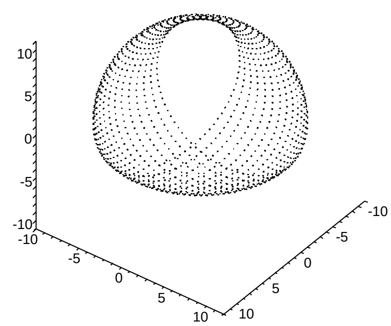

Figure 2: input parameters: $s_1^+ = 0.2 + 10i$, $s_1^- = 0.2 - 10i$, $s_1^3 = -1$, $s_2^+ = 10 - 30i$, $s_2^- = 10 + 30i$, $s_2^3 = 100$, $\lambda_1 = \pi$, $\lambda_2 = 7\pi/3$, $\lambda_0 = 0.1$, $\mu = -0.004i$

6.1 Numerical integration

The figures report an example of numerical integration of the map (39a), (39b), (39c). For simplicity we take $N = 2$. The computations shows the first 1500 iterations of the map: the plotted variables are the physical one (s_1^x, s_1^y, s_1^z). Only one of the two spins is shown, that labeled by "1" subscript. The figure is obtained with a $^{\text{TM}}$Maple code.

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