Towards Google matrix of brain

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We apply the approach of the Google matrix, used in computer science and World Wide Web, to description of properties of neuronal networks. The Google matrix **G** is constructed on the basis of neuronal network of a brain model discussed in PNAS **105**, 3593 (2008). We show that the spectrum of eigenvalues of **G** has a gapless structure with long living relaxation modes. The PageRank of the network becomes delocalized for certain values of the Google damping factor α. The properties of other eigenstates are also analyzed. We discuss further parallels and similarities between the World Wide Web and neuronal networks.

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I INTRODUCTION

More than 50 years ago John von Neumann traced first parallels between architecture of the computer and the brain [1]. Since that time computers became an unavoidable element of the modern society forming a computer network connected by the World Wide Web (WWW). The WWW demonstrates a continuous growth approaching to 10^{11} web pages spread all over the world (see e.g. http://www.worldwidewebsize.com/). This number starts to become even larger than 10^{10} neurons in the brain. Each neuron can be viewed as an independent processing unit connected with about 10^4 other neurons by synaptic links (see e.g. [2–4]). About 20% of these links are unidirectional [5] and hence the brain can be viewed as a directed network of neuron links. At present, more and more experimental information about neurons and their links becomes available and the investigation of properties of neuronal networks attracts an active interest of many groups (see e.g. [6–13].

The WWW is also a directed network where a site j points to a site i but no necessary vice versa. The classification of web sites and information retrieval from such an enormous data base as the WWW becomes a formidable challenge of modern society where the search engines like Google are used by internet users in everyday life. An efficient way to classify and extract the information from WWW is based on the PageRank Algorithm (PRA), proposed by Brin and Page in 1998 [14], which forms the basis of the Google search engine. The PRA is based on the construction of the Google matrix which can be written as (see e.g. [15] for details):

$$\mathbf{G} = \alpha \mathbf{S} + (1-\alpha)\mathbf{E}/N \ . \tag{1}$$

Here the matrix **S** is constructed from the adjacency matrix **A** of directed network links between N nodes so that $S_{ij} = A_{ij}/\sum_k A_{kj}$ and the elements of columns with only zero elements are replaced by $1/N$. The second term in r.h.s. of (1) describes a finite probability $1-\alpha$ for WWW surfer to jump at random to any node so that the matrix elements $E_{ij} = 1$. This term with the Google damping factor α stabilizes the convergence of PRA introducing a gap between the maximal eigenvalue $\lambda = 1$ and other eigenvalues λ_i. As a result the first eigenvalue has $\lambda_1 = 1$ and the second one has $|\lambda_2| \le \alpha$. Usually the Google search uses the value $\alpha = 0.85$ [15]. By the construction $\sum_i G_{ij} = 1$ so that the asymmetric matrix **G** belongs to the class of Perron-Frobenium operators which naturally appear in the ergodic theory [16] and dynamical systems with Hamiltonian or dissipative dynamics [17].

The right eigenvector at $\lambda = 1$ is the PageRank vector with positive elements p_j and $\sum_j p_j = 1$. The classification of nodes in the decreasing order of p_j values is used to classify importance of WWW nodes as it is described in more detail in [15]. The PageRank can be efficiently obtained by a multiplication of a random vector by **G** which is of low cost since in average there are only about ten nonzero elements in a typical line of **G** of WWW. This procedure converges rapidly to the PageRank.

Fundamental investigations of the PageRank properties of the WWW have been performed in the computer science community (see e.g. [18–23]; involvement of physicists is visible, e.g. [24], but less pronounced). It was established that the PageRank is satisfactory characterized by an algebraic decay $p_j \sim 1/j^\beta$ with j being the ordering index and $\beta \approx 0.9$; the number of nodes with the PageRank p scales as $N_n \sim 1/p^\nu$ with the numerical value of the exponent $\nu = 1 + 1/\beta \approx 2.1$ [15, 18]. It is known that such type of algebraic dependencies appear in various types of scale-free networks [25]. The PageRank classification finds its applications not only for the WWW but also for the network of article citations in Physical Review as it is described in [26, 27]. This shows that the approach based on the Google matrix can be applied to vary different types of networks.

In this work we construct the Google matrix **G** for a model of brain analyzed in [11]. The properties of the spectrum and the eigenstates of **G** are described in the

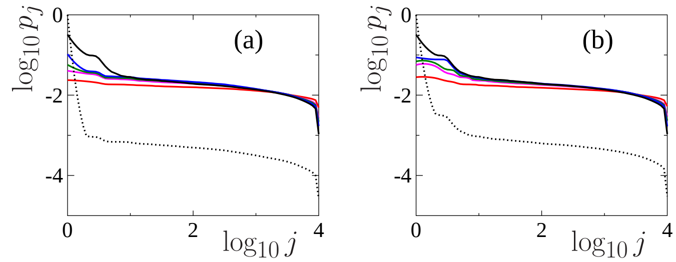

FIG. 2: (Color online) PageRank p_j for the Google matrix of brain model at $\alpha = 0.6, 0.85, 0.9, 0.95$ and 0.99 shown by red, magenta green, blue and black solid curves (full curves from bottom to top at $\log_{10} j = 0.3$). The dotted black curve corresponds to $\alpha = 0.999$ and demonstrates strong dependence of the PageRank on α in the vicinity of $\alpha = 1$. Panels (a) and (b) correspond to unweighted and weighted links. For panels (a) and (b) the values of PAR are $\xi = 8223.$ and $8314.$, $6295.$ and $6040.$, $5570.$ and $5046.$, $3283.$ and $3367.$, 28.4, 90.0, 1.09 and 1.19 for $\alpha = 0.6, 0.85, 0.9, 0.95, 0.99, 0.999$ respectively.

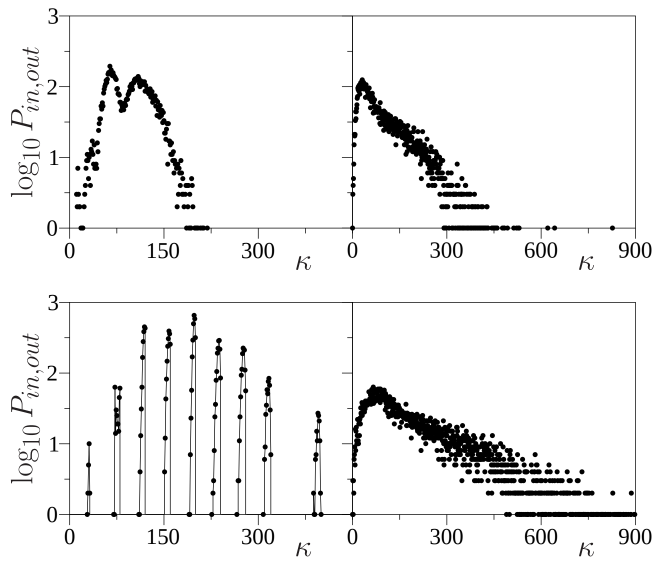

FIG. 1: Distribution of *ingoing* (left panels) and *outgoing* (right panels) links κ: P_{in} and P_{out} give number of nodes with κ ingoing and *outgoing* links respectively. Top panels: unweighted links; bottom panels: weighted links.

next Section II. The results are discussed in Section III.

II NUMERICAL RESULTS

To construct the Google matrix of brain we use a directed network of links between $N = 10^4$ neurons [28] generated from the brain model [11]. In total there are $N_l = 1960108$ links in the network. They form N_{out} *outgoing* links and N_{in} *ingoing* links ($N_l = N_{out} = N_{in}$), so that there are about 200 outlinks (or ingoing) per neuron. These numbers include multiple links between certain pairs of neurons; certain neurons have also links to themselves (there is one neuron linked only to itself). The number of weighted symmetric links is approximately 9.8%. Due to existence of multiple links between the same neurons we constructed two $\mathbf{G}$ matrices based on unweighted and weighted counting of links. In the first case all links from neuron j to neuron i are counted as one link, in the second case the weight of the link is proportional to the number of links from j to i. In both cases the sum of elements in one column is normalized to unity. The distributions of ingoing (P_{in}) and outgoing (P_{out}) links are shown in Fig. 1. The weighted distribution of ingoing links have a pronounced peaked structure corresponding to different regions of brain model considered in [11]. We note that the distribution of links is not of free-scale type.

The dependence of the PageRank on α is shown in Fig. 2. For $\alpha = 0.999$ almost all probability p_j is concentrated on one neuron. This is the only one neuron which is linked only to itself. With the increase of α up to 0.99 the main part of probability is concentrated mainly on about 10 neurons that approximately corresponds to the number of peaks in the distribution of weighted ingoing links in Fig. 1 (bottom left panel). At the same time the PageRank has a long tail at large j where the probability p_j is practically homogeneous. For $\alpha = 0.6$ the peak of probability at $1 \leq j \leq 10$ is washed out and the PageRank becomes completely delocalized. We note that a delocalization of the PageRank with α appears in the Ulam networks describing dynamical systems with dissipation [29, 30]. At the same time the WWW networks remain stable in respect to variation of α as it is discussed in [23, 31].

The spectrum λ_i and the right eigenvectors ψ_i of the Google matrix of brain are defined by the equation

$$\mathbf{G}\psi_i = \lambda_i \psi_i \ . \tag{2}$$

The spectrum of λ is complex and is shown in Fig. 3. The color of points is chosen to be proportional to the PArticipation Ratio (PAR) defined as $\xi = (\sum_j |\psi_i(j)|^2)^2 / \sum_j |\psi_i(j)|^4$. This quantity determines an effective number of sites populated by an eigenstate ψ_i, it is often used to characterize localization-delocalization transition in quantum solid-state systems with disorder (see e.g. [32]). The spectrum has eigenvalues with $|\lambda_i|$ being close to unity so that there is no gap in the spectrum of λ in the vicinity of $\lambda = 1$ (we remind that the second term in the r.h.s. of (1) transfers λ_i to $\alpha\lambda_i$ keeping only one $\lambda_1 = 1$ [15]). This is different from the spectrum of random scale-free networks characterized by a large gap in the spectrum of λ [33].

Compared to the spectra of the university WWW networks studied in [31] the spectrum of $\mathbf{G}$ in Fig. 3 is more flat being significantly compressed to the real axis. In

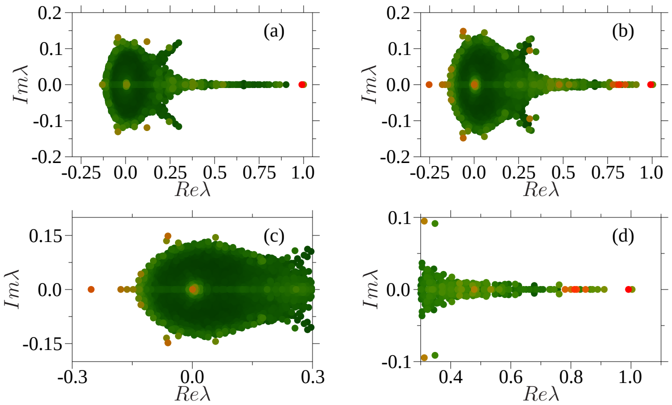

FIG. 3: (Color online) Spectrum of eigenvalues of the Google matrix $\mathbf{G}$ of brain at $\alpha = 0.99$ in the complex plain λ for (a) unweighted and (b) weighted links in the neuronal network. Panels (c) and (d) show zooms of data of panel (c). The color shows the degree of localization of eigenvectors of $\mathbf{G}$ being proportional to the value of PAR ξ and changing from one (red/light gray) to maximal value (dark green/black).

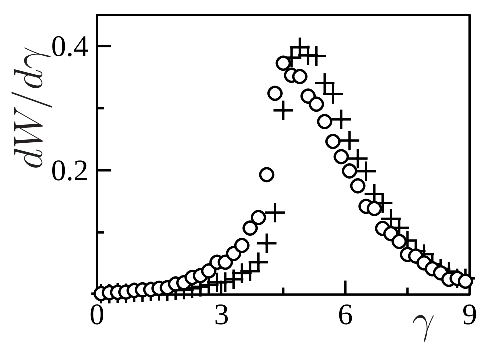

FIG. 4: Dependence of the density of states $dW/d\gamma$ of $\mathbf{G}$ on the relaxation rate γ for unweighted (pluses) and weighted (circles) links in the neuronal network.

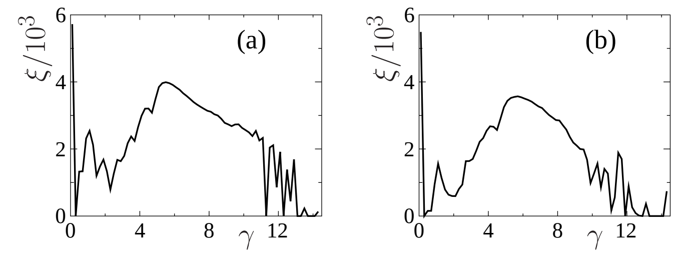

FIG. 5: Dependence of PAR ξ on relaxation rate γ at $\alpha = 0.85$ for (a) unweighted and (b) weighted links in the neuronal network.

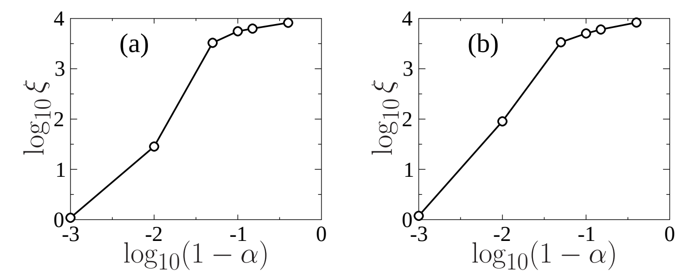

FIG. 6: Dependence of PAR ξ of the PageRank on parameter α for (a) unweighted and (b) weighted links in the neuronal network.

this respect our neuronal network has certain similarity with the spectra of vocabulary networks analyzed in [31] (see Fig. 1 there). At the same time the spectrum of $\mathbf{G}$ matrix of brain has visible structures in the eigenvalues distribution in the complex plane of λ while the vocabulary networks are characterized by structureless spectrum. The spectrum of Fig. 3 has global properties being similar to those of the Ulam networks considered in [29]. It is interesting to note that the spectra of unweighted and weighted networks of brain have similar structure. This supports the view of structural stability of the spectrum of $\mathbf{G}$ matrix.

It is useful to determine the relaxation rate of eigestates by the relation $\gamma = -2 \ln |\lambda|$. The dependence of density of states $dW/d\gamma$ on γ is shown in Fig. 4 (the density is normalized to unity so that $\int_0^\infty dW/d\gamma d\gamma = 1$ corresponds to $N = 10^4$ states). The distribution in γ has a pronounced peak at $\gamma \approx 5$, the density of states at small $\gamma < 1$ is relatively small (this is also seeing in Fig. 3). The comparison of unweighted and weighted links shows the stability of the density distribution in respect to such modification of links.

The dependence of the PAR ξ on γ is shown in Fig. 5 (we note that except of the PageRank ξ is independent of α due to the unity rank of matrix $\mathbf{E}$, see e.g. [15, 29]). The PageRank value of ξ at $\gamma = 0$ is very large being more than half of the total number of neurons $N = 10^4$. It is clear that this corresponds to a delocalized state. The eigenstates with $0 < \gamma < 2$ have relatively small $\xi \lesssim 10^3$ being close to a localized domain while eigenstates with $2 < \gamma < 10$ have $\xi > 10^3$ being delocalized on the main part of the network; the states with $\gamma > 10$ enter in the localized domain. For $\alpha > 0.99$ the PAR is close to $\xi \approx 1$. Taking as a criterion that the delocalization takes place when $\xi > N/2$ we obtain that the PageRank becomes delocalized at $\alpha_c \approx 0.9$ (see data of Figs. 2,6). The global dependence of the PAR ξ of the PageRank on parameter α is shown in Fig. 6 with a sharp delocalization of ξ for $\alpha < \alpha_c$. Of course, the above analysis should be considered as an approximate one since the localization properties should be studied in dependence on the system size N while we consider only one size of N.

III DISCUSSION

In this work we studied the properties of the Google matrix of a neuronal network of the brain model discussed in [11]. For this network of 10^4 neurons we found that the spectrum of the Google matrix has a gapless spectrum at $\alpha = 1$ demonstrating certain similarities with the spectra of university WWW networks and vocabulary networks studied in [31]. At the same time our neuronal network shows signs of delocalization transition of the PageRank at the Google damping factor $\alpha_c \approx 0.9$ which was absent in the networks studied in [23, 31]. A similar transition in α was detected in the Ulam networks generated by dissipative dynamical maps [29]. We attribute the appearance of such delocalization transition to a large number of links per neuron (200) which is by factor 10 larger than in the WWW networks (20).

Of course, our studies have certain limitations since we considered only a fixed size neuronal network and since this network is taken from a model system of brain analysed in [11]. Another weak point is that we do not consider the dynamical properties of the network which are probably more important for practical applications. Nevertheless, the spectral properties of $\mathbf{G}$ matrix can be rather useful. Indeed, the gapless spectrum of λ shows that long living excitations can exist in our neuronal network. Such relaxation modes with small rates γ can be the origin of long living oscillations found in numerical simulations [11]. It is quite possible that the properties of spectra of $\mathbf{G}$ can help to understand in a better way rapid relaxation processes and those with long relaxation times. We conjecture that the rapid relaxation modes correspond to relaxation of local groups of neurons while long living modes can represent relaxation of collective modes representing dynamics of human thoughts. The dynamics of such collective modes can contain significant elements of chaotic dynamics as it was discussed in the frame of the concept of creating chaos in [34].

It is possible that the brain effectively implements dynamics described by the evolution equation $d\psi/dt = \mathbf{G}\psi$ which without perturbations converges to the steady-state described by the PageRank (which may be linked with a sleeping phase). External perturbations give excitations of other eigenmodes of $\mathbf{G}$ discussed here. The evolution of these excitations will be significantly affected by the spectrum of $\mathbf{G}$.

Further development of the Google matrix approach to the brain looks to us to be rather promising. For example, a detection of isolated communities and personalized PageRank, represented by other types of matrix $\mathbf{E}$ in (1), is under active investigation in the computer science community (see e.g. [15, 23]). Such type of problems can find their applications for detection of specific quasi-isolated neuronal networks of brain. The usage of real neuronal networks, similar to those studied in [6–10, 13], in combination with the Google matrix approach can allow to discover new properties of processes in the brain. The development of parallels between the WWW and neuronal networks will give new progress of the ideas of John von Neumann.

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