Conservation laws for strings in the Abelian Sandpile Model

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The Abelian Sandpile generates complex and beautiful patterns and seems to display allometry. On the plane, beyond *patches*, patterns periodic in both dimensions, we remark the presence of structures periodic in one dimension, that we call *strings*. We classify completely their constituents in terms of their principal periodic vector $\boldsymbol{k}$, that we call *momentum*. We derive a simple relation between the momentum of a string and its density of particles, E, which is reminiscent of a dispersion relation, $E = |\boldsymbol{k}|^2$. Strings interact: they can merge and split and within these processes momentum is conserved, $\sum_a \boldsymbol{k}_a = \mathbf{0}$. We reveal the role of the modular group $\mathrm{SL}(2, \mathbb{Z})$ behind these laws.

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Introduction : Since the appearance of the masterpiece by D'Arcy Thompson [1], there have been many attempts to understand the complexity and variety of shapes appearing in Nature at macroscopic scales, in terms of the fundamental laws which govern the dynamics at microscopic level. Because of the second law of thermodynamics, the necessary Self-Organization can be studied only in non-equilibrium statistical mechanics.

In the context of a continuous evolution in a differential manifold, the definition of a shape implies a boundary and thus a discontinuity. This explains why catastrophe theory, the mathematical treatment of continuous action producing a discontinuous result, has been developed in strict connection to the problem of *Morphogenesis* [2]. More quantitative results have been obtained by the introduction of stochasticity, as for example in the diffusion-limited aggregation [3], where self-similar patterns with fractal scaling dimension [4] emerge which suggest a relation with scaling studies in non-equilibrium.

Cellular automata, that is, dynamical systems with discretized time, space and internal states, were originally introduced by Ulam and von Neumann in the 1940s, and then commonly used as a simplified description of phenomena like crystal growth, Navier-Stokes equations and transport processes [5]. They often exhibit intriguing patterns [6], and, in this regular discrete setting, *shapes* refer to sharply bounded regions in which periodic patterns appear. Despite very simple local evolution rules, very complex structures can be generated. The well known Conway's Game of Life can perform computations and can even emulate an universal Turing machine (see [6] also for a historical introduction on cellular automata).

In this Letter, we shall concentrate on a particularly simple cellular automaton, the Abelian Sandpile Model (ASM). Originally introduced as a statistical model of Self-Organized Criticality [7], because it shows scaling laws without any fine-tuning of an external control parameter, it has been shown afterwards to possess remarkable underlying algebraic properties [8, 9], and has been studied also in some deterministic approaches, exactly in connection with pattern formation [10]. The ASM has been shown to be able to produce *allometry*, that is a growth uniform and constant in all the parts of a pattern as to keep the whole shape substantially unchanged, and thus requires some coordination and communication between different parts [11]. This is at variance with diffusion-limited aggregation and other models of growing objects studied in physics literature so far, e.g. the Eden model, KPZ deposition and invasion-percolation [12], which are mainly models of aggregation, where growth occurs by accretion on the surface of the object, and inner parts do not evolve significantly.

In the sandpile, the regions of a configuration periodic in space, called *patches*, are the ingredients of pattern formation. In [11], a condition on the shape of patch interfaces has been established, and proven at a coarse-grained level.

We discuss how this result is strengthened by avoiding the coarsening, and describe the emerging fine-level structures, including linear interfaces and rigid domain walls with a residual one-dimensional translational invariance. These structures, that we shall call *strings*, are macroscopically extended in their periodic direction, while showing thickness in a full range of scales between the microscopic lattice spacing and the macroscopic volume size.

The model : While the main structural properties of the ASM can be discussed on arbitrary graphs [8], for the subject at hand here we shall need some extra ingredients (among which a notion of translation), that, for the sake of simplicity, suggest us to concentrate on the original realization on the square lattice [7], within a rectangular region $\Lambda \in \mathbb{Z}^2$.

We write $i \sim j$ if i and j are first neighbours. The configurations are vectors $z \equiv \{z_i\}_{i\in\Lambda} \in \mathbb{N}^\Lambda$ (z_i is the number of sand-grains at vertex i). Let $\bar{z} = 4$, the degree of vertices in the bulk, and say that a configuration z is *stable* if $z_i < \bar{z}$ for all $i \in \Lambda$. Otherwise, it is *unstable* on a non-empty set of sites, and undergoes a relaxation process whose elementary steps are called *topplings*: if i is unstable, we can decrease z_i by $\bar{z}$, and increase z_j by one, for all $j \sim i$. The sequence of topplings needed to produce a stable configuration is called an *avalanche*.

Avalanches always stop after a finite number of steps, which is to say that the diffusion is strictly *dissipative*. Indeed, the total amount of sand is preserved by topplings at sites far from the boundary of Λ, and strictly decreased by topplings at boundary sites. The stable configuration $\mathcal{R}(z)$ obtained from the relaxation of z, is univocally defined, as all valid stabilizing sequences of topplings only differ by permutations.

We call a stable configuration *recurrent* if it can be obtained through an avalanche involving all sites in Λ, and *transient* otherwise [8]. Recurrent configurations have a structure of Abelian group [9] under the operation $z \oplus w := \mathcal{R}(z + w)$. We have only a partial knowledge of the group identity for each Λ (see e.g. [9, 13]; recently a complete characterization has been achieved for a simplified directed lattice, the *F-lattice* [14]) nonetheless they are easily obtained on a computer and they provide a first example of the intriguing complex patches in which we are interested. The maximally-filled configuration $z_{\max}$, with $(z_{\max})_i = \bar{z} - 1 = 3$ for all i, is recurrent. More generally, for large Λ, recurrent configurations must have average density $\rho(z) = |\Lambda|^{-1} \sum_i z_i \geq 2 + o(1)$ (this bound is tight). So structures with density $\rho > 2$, $\rho = 2$ and $\rho < 2$ are said respectively *recurrent*, *marginal* and *transient*.

A *patch* is a region filled with a periodic pattern. The *density* ρ of a patch is the average of z_i within a unit tile. Neighbouring patches may have an interface, periodic in one dimension, along a vector which is principal for both patches. Let us suppose that in a deterministic protocol [11] we generate a region filled with polygonal patches, glued together with such a kind of interfaces. At a vertex where $\ell \geq 3$ patches meet, label cyclically with $\alpha = 1, \ldots, \ell$ these patches, call ρ_α the corresponding densities, and θ_α the angles of the interfaces between the patch α and $\alpha + 1$ (subscripts $\alpha = \ell + 1 \equiv 1$). These quantities are proven to satisfy the relation in $\mathbb{Q} + i\mathbb{Q}$

$$\sum_{\alpha=1}^{\ell} (\rho_{\alpha+1} - \rho_\alpha) \exp(2i\theta_\alpha) = 0 \qquad (1)$$

which has non-trivial solutions only for $\ell \geq 4$ [11].

A *string* is a one-dimensional periodic defect line, with periodicity integer vector $\boldsymbol{k} = (k_x, k_y) \in \mathbb{N}^2$, that we call *momentum*, in a background patch, periodic in both directions, and has $\boldsymbol{k}$ as a periodicity vector. The background on the two sides may possibly have a periodicity offset. See Fig. 1.

RESULTS : In this section we report about some results obtained by our investigations. These facts have explicit proofs, whose details will be reported elsewhere.

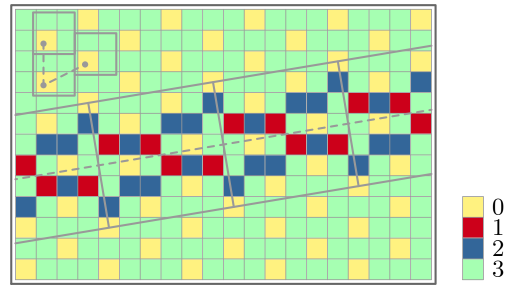

Figure 1. A string with momentum $(6, 1)$, in a background pattern with periodicities $V = \big((2, 1), (0, 2)\big)$. String and background unit cells are shown in gray. The density in the string tile is $\rho = (18 \cdot 3 + 8 \cdot 2 + 4 \cdot 1 + 7 \cdot 0)/(6^2 + 1^2) = 2$.

Strings in the maximally-filled background emerge in a simple protocol. Consider a rectangular region Λ, and the configuration $z_{\max}$. Add one grain of sand at some vertex j. The configuration after relaxation contains an inner rectangle, of strings $(1, 0)$ and $(0, 1)$, equidistant from the border of Λ and having j on its perimeter, and its corners are connected to the corners of Λ with strings $(1, 1)$ and $(-1, 1)$. There is one defect exactly at j, manifested as a single extra grain, w.r.t. the underlying periodic structure. See Fig. 2. This configuration is recurrent. Now remove this extra grain (the configuration is now transient), and repeat the game at some new vertex j', say in the region below the inner rectangle. In the configuration after relaxation appear also strings $(2, 1)$ and $(-2, 1)$, and once again we have a defect at j'. This procedure can be iterated, and, if Λ is large enough, strings with higher and higher momenta generated. Furthermore, given that the unit tiles of momenta are classified (see the following), this protocol is completely predictable, for arbitrary Λ, through a purely geometric construction.

In a given recurrent background, with translation vectors $V = (\boldsymbol{v}_1, \boldsymbol{v}_2)$, one and only one string of momentum $\boldsymbol{k} = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = \boldsymbol{m}V$ can be produced, if

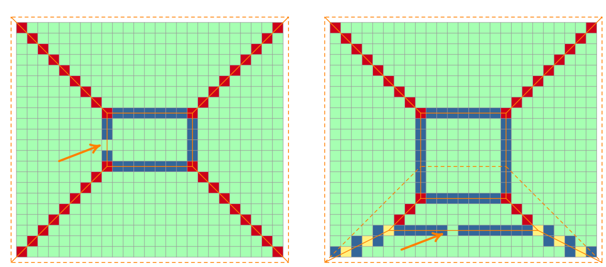

Figure 2. On the left, the configuration obtained after relaxation from $z_{\max}$ plus an extra grain of sand exactly at the vertex where a defect appears. On the right, the result after removing the defect and the addition of one more grain.

$\gcd(m_1, m_2) = 1$, and no strings of momentum $\boldsymbol{k}$ exists for $\boldsymbol{k}$ not of the form above. V and V' are equivalent descriptions of the background periodicity iff $V' = MV$, with $M \in \mathrm{SL}(2,\mathbb{Z})$. Accordingly, $\boldsymbol{m}' = \boldsymbol{m}M^{-1}$. And indeed, sets of $\boldsymbol{m} \in \mathbb{Z}^2$ with given gcd's are the only proper subsets invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$. The gcd constraint arises in the classification of the elementary strings, because, when $d = \gcd(k_x, k_y) > 1$, the corresponding periodic ribbon is just constituted of d parallel strings with momentum $\boldsymbol{k}/d$.

The unit tile of each string, as well as of each patch, is symmetric under 180-degree rotations. In particular, momenta $\boldsymbol{k}$ and $-\boldsymbol{k}$ describe the *same* string. The tile of a string of momentum $\boldsymbol{k}$ fits within a square having $\boldsymbol{k}$ as one of the sides, so that each string is a row of identically filled squares. This is a non-empty statement: the tile could have required rectangular boxes of larger aspect ratio, and even an aspect ratio depending on momentum and background.

A string of momentum $\boldsymbol{k}$ has an *energy* E, defined as the difference of sand-grains, in the framing unit box of side $\boldsymbol{k}$, w.r.t. $z_{\max}$. We have the relation $E = |\boldsymbol{k}|^2$, or, in other words, the unit tile has exactly marginal density, $\rho = 2$, irrespectively of the density of the surrounding background (as seen, e.g., in Fig. 1).

Two strings, respectively of momentum $\boldsymbol{p}$ and $\boldsymbol{q}$, can collapse in a single one of momentum $\boldsymbol{k}$ (see Fig. 3). In this process momentum is *conserved*: $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{k}$. More precisely, the strings join together in such a way that the square boxes surrounding the unit cells meet at an extended *scattering vertex*, a triangle of sides of lengths equal to $|\boldsymbol{k}|$, $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$, rotated by 90 degrees w.r.t. the corresponding momenta: given this geometrical construction, momentum conservation rephrases as the oriented perimeter of the triangle being a closed polygonal chain.

Local momentum conservation and the $\boldsymbol{k} \leftrightarrow -\boldsymbol{k}$ symmetry are reminiscent of equilibrium of tensions, in a planar network of tight material strings, from which the name.

On networks, this local conservation is extended to a global constraint. Choose an orthogonal frame (x, t), and orient momenta in the direction of increasing t. Then, sections at fixed t are all crossed by the same total momentum. Rigid extended domain walls between periodic patterns, satisfying similar local and global conservations, appear in certain tiling models [15], which remarkably show a Yang-Baxter integrable structure, where the corresponding strings are usefully interpreted as world-lines of particles in the (x, t)-frame. Note, however, that, at variance with these models, in the ASM we have an infinite tower of excitations, for a given background, and infinitely many different backgrounds too.

In the maximally-filled background, because of the (D_4 dihedral) symmetry, we can restrict without loss of generality to study strings of momentum $\boldsymbol{k}$ with both components positive. For each such $\boldsymbol{k}$ with $\gcd(k_x, k_y) = 1$, simple modularity reasonings show that there exists a *unique* ordered pair of momenta $\boldsymbol{p}$ and $\boldsymbol{q}$, with non-negative components, such that $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{k}$ and the matrix $\left(\begin{smallmatrix} p_x & p_y \\ q_x & q_y \end{smallmatrix}\right)$ is in $\mathrm{SL}(2,\mathbb{Z})$. We write in this case $\boldsymbol{k} \leftarrow (\boldsymbol{p}, \boldsymbol{q})$. For example, $(10, 3) \leftarrow \big((7, 2), (3, 1)\big)$. The endpoint of $\boldsymbol{p}$, starting from the top-left corner of the $\boldsymbol{k}$ framing box, is the (unique) lattice point which is nearest to the top-side of the box. This alternative definition generalizes to non-trivial backgrounds, and the $m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$ sublattice.

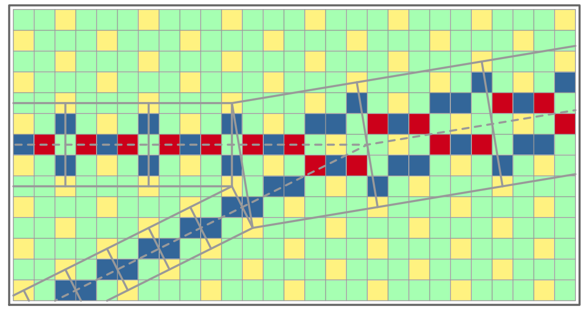

Figure 3. A scattering involving pseudo-propagators with momenta $(4, 0)$, $(2, 1)$ and $(6, 1)$, on the background pattern of Fig. 1 (also symbol code is as in Fig. 1).

We have found a simple algorithm to derive the string textures. The tile for $\boldsymbol{k} \leftarrow (\boldsymbol{p}, \boldsymbol{q})$ is essentially composed of four interlaced tiles, two for $\boldsymbol{p}$ and two for $\boldsymbol{q}$, adjacent to the four vertices of the $\boldsymbol{k}$ box. This opens a problem of consistency for the overlapping region, of side $\sim |\boldsymbol{p} - \boldsymbol{q}|$, which is solved by the fact that, for $\boldsymbol{k} \leftarrow (\boldsymbol{p}, \boldsymbol{q})$, then $\boldsymbol{p} \leftarrow (\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{q})$ if $|\boldsymbol{p}| > |\boldsymbol{q}|$, and $\boldsymbol{q} \leftarrow (\boldsymbol{p}, \boldsymbol{q} - \boldsymbol{p})$ otherwise, again by a property of $\mathrm{SL}(2,\mathbb{Z})$ (the reader can see in Fig. 3 the similarity of the tile of a composed string with the ones of its component). By iterating this procedure, starting with the strings of minimum momentum, we can understand the textures of the full catalog of strings, and, in particular, this makes the protocol for producing networks of strings completely predictable.

Let us go back to the problem of ℓ interfaces which meet at a given corner, but allow now, besides interfaces between patches, incident strings. Following the analysis of [11], we introduce the graph-vector $T = \{T_i\}$, where T_i is the number of topplings at i in the relaxation of the starting configuration, and study its characteristics in a region that, in the starting configuration, was uniformly filled with a patch. However, now we allow for toppling distributions which are piecewise both quadratic and linear (the linear term was neglected in [11], as subleading in the coarsening).

For any relevant direction α, allow for a patch interface, or a string, or both. Call $\tilde{E}^{(\alpha)}$ the difference for *unit length* (not for period), in the total number of grains of sand w.r.t. $z_{\max}$, due to presence of a string, i.e. $\tilde{E}^{(\alpha)} = E^{(\alpha)}/|\boldsymbol{k}^{(\alpha)}|$, or the contribution from a non-zero impact parameter in the interface. It can be shown,

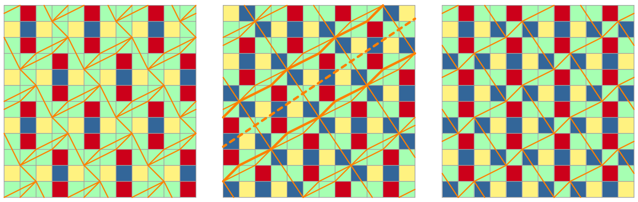

Figure 4. The recurrent, marginal and transient patches constructed from the propagator $\boldsymbol{k} = (3,2) \leftarrow \big((2,1),(1,1)\big)$ in the $z_{\max}$ background, having densities $\rho = 2$ and $2 \pm 1/12$ (symbol code is as in Fig. 1).

by reasonings as in [11], that the difference between the extrapolated toppling profile for two contiguous patches, at a polar coordinate (r,θ), must be of the form

$$T^{(\alpha+1)}_{r,\theta} - T^{(\alpha)}_{r,\theta} = \frac{r^2}{2}\,(\rho_{\alpha+1}-\rho_\alpha)\,\sin^2(\theta-\theta_\alpha) + r\,\tilde{E}^{(\alpha)}\sin(\theta-\theta_\alpha) + \mathcal{O}(1)\,. \tag{2}$$

Then, by summing over α and matching separately the quadratic and linear terms, we conclude that, for each θ,

$$\begin{cases} \sum_{\alpha=1}^{\ell} (\rho_{\alpha+1}-\rho_\alpha)\,\sin^2(\theta-\theta_\alpha) = 0 \\ \sum_{\alpha=1}^{\ell} \tilde{E}^{(\alpha)}\sin(\theta-\theta_\alpha) = 0 \end{cases} \tag{3}$$

so that, besides the anticipated equation (1) for patches alone, that was deduced in [11], we obtain

$$\sum_{\alpha=1}^{\ell} \tilde{E}^{(\alpha)}\,\exp(i\theta_\alpha) = 0 \tag{4}$$

which describes the string and interface-offset contributions.

In (1), the first non-trivial value for ℓ is 4 [11]. In our generalization, 4 is the minimal value for the number of patches *plus* the number of strings, and thus includes new possibilities: a *scattering* event, with three incident strings in a single background, as in Fig. 3, and the case of two strings and two patches, producing diagrams reminiscent of total *reflection* and *refraction* in optics, so that the specialization of (4) can be read as a *Snell's law* for ASM strings. For the case of three strings on a common background $\mathcal{B}$, we get

$$\sum_{\alpha=1}^{\ell} \frac{E^{(\alpha)}}{|\boldsymbol{k}^{(\alpha)}|^2}\,\boldsymbol{k}^{(\alpha)} = 0 \tag{5}$$

which shows that momentum conservation implies a dispersion relation of the form $E^{(\alpha)} = c_{\mathcal{B}}|\boldsymbol{k}^{(\alpha)}|^2$, and viceversa.

The classification of the strings preludes to a classification of the patches. To any string of momentum $\boldsymbol{k}$, univocally decomposed as $\boldsymbol{k} \leftarrow (\boldsymbol{p},\boldsymbol{q})$, we can associate three patches, respectively recurrent, marginal and transient, through a geometrical construction, involving $\boldsymbol{p}$ and $\boldsymbol{q}$, sketched in an example in Fig. 4.

Reflection and refraction events also appear, if $\boldsymbol{k} \leftarrow (\boldsymbol{p},\boldsymbol{q})$, in a single string of momentum $\boldsymbol{k}' = m\boldsymbol{p}+\boldsymbol{q}$ (for m a large integer), and in the scattering of $m\boldsymbol{p}+\boldsymbol{q}$ into $\boldsymbol{q}$ and m parallel $\boldsymbol{p}$ strings. Indeed, as a consequence of the recursive construction of the string textures, the string of momentum $\boldsymbol{k}'$ is forced to look as a strip-shaped patch of m-period width, of the marginal tile associated to $\boldsymbol{p}$, crossed by a 'soft' string, that reflects twice per period $\boldsymbol{k}'$, up to ultimately leaving the marginal patch, through a refraction, and propagates in the recurrent background.

We shall show elsewhere that the interplay between strings and patches, both at the level of classification, and of evolution in deterministic protocols, is the key-ingredient to clarify allometry in pattern formation for the ASM, and to design new protocols in which short-scale defects are totally absent. The resulting structure is a fractal, a Sierpiński triangoloid, where the theoretical formula (1) has infinitely many distinct realizations.

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