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On Fermi-Bose symmetry in Conformal Field Theory*

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Abstract

The minimal models of CFT possess a few algebraic structures. Besides the Virasoro Algebra which is an explicit symmetry of these models they have also Quantum Group structure and Fermi-Bose duality. We discuss the relationship between these things.

1. Introduction

The minimal models of conformal field theory possess not only an explicit symmetry which is the Virasoro algebra. Due to the fact that the models satisfy simultaneously other axioms of quantum field theory such as analiticity, operator fujion algebra et cet. they get also another remarkable algebraic structure [1]. One of manifistations of this structure is the famous Rogers-Ramanujian (R-R) Identity and its generalizations which were discovered by McCoy, Melzer, Klassen and Kedem (KKMM) [2] a few years ago. The characters of the minimal models of conformal field theory satisfy to these identities. Here we discuss the relationship between these things.

2. The Identities for Characters of CFT

The Rogers-Ramanujian Identity itself is the following equality

$$\frac{1}{\prod_{n=1}^{\infty}(1-t^n)}\sum_{k=-\infty}^{\infty}[t^{k(10k+1)}-t^{(2k+1)(5k+2)}]=\sum_{m=1}^{\infty}\frac{t^{m^2}}{(1-t)\dots(1-t^m)} \qquad (1)$$

The function which is the subject of this equation is nothing but a character of M(2,5) minimal model of CFT.

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The simplest generalization of the R-R identity has the form

$$\frac{1}{\prod_{n=1}^{\infty}(1-t^n)} \sum_{k=-\infty}^{\infty} [t^{k(12k+1)} - t^{(4k+1)(3k+1)}] = \sum_{\substack{m-even \\ m>0}}^{\infty} \frac{t^{m^2/2}}{(1-t)\dots(1-t^m)} \quad (2)$$

and is just a character of M(3,4),that is Ising model.

To write down the generic formula for characters of minimal models let us remind some facts on CFT [3]. The symmetry of CFT is Virasoro algebra whose generating elements L_n satisfy the following relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m} \quad (3)$$

For the minimal models M_p the Virasoro central charge

$$c = 1 - \frac{6}{p(p+1)} \quad (4)$$

The chiral spase of states $\mathcal{H}_{chiral}$ of M_p can be written as

$$\mathcal{H}_{chiral} = \oplus[r,s]$$

Here [r,s] is the irreducible representation of the Virasoro algebra with the highest weight

$$\Delta = \Delta_{rs} = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)} \quad (5)$$

where $1 \le r < p, 1 \le s < p+1$ The representation [r,s] is generated by an action of the operators L_n with $n < 0$ on the vacuum vector $|rs\rangle$ which satisfies

$$L_0|rs\rangle = \Delta_{rs}|rs\rangle; \qquad L_n|rs\rangle = 0 \quad n > 0 \quad (6)$$

The character of the irreducible representation [r,s] is defined as the generating function of the number of states in [r.s] with the given value L_0

$$\chi^p_{rs}(t) \stackrel{\text{def}}{=} \text{tr}_{[r,s]} t^{L_0 - \Delta_{rs}} \quad (7)$$

It follows from the analysis of the structure [r,s] by Feigin and Fuks that

$$\chi^p_{rs}(t) = \frac{1}{\prod_{n=1}^{\infty}(1-t^n)} \sum_{k=-\infty}^{\infty} [t^{\Delta_{r+2pk,s} - \Delta_{rs}} - t^{\Delta_{r+2pk,-s} - \Delta_{rs}}] \quad (8)$$

It was dicovered by Kedem,Klassen,McCoy and Melzer [2] that there exists another representation for the same function

$$\chi^p_{rs}(t) = \sum_{m_1 \ge 0} \dots \sum_{m_{p-2} \ge 0} t^{m_a C_{ab} m_b} \prod_{a=2}^{p-2} \left[\begin{array}{c} m_{a-1} + m_{a+1} \\ m_a \end{array} \right]_t \quad (9)$$

where

$$\left[\begin{array}{c} N \\ m \end{array} \right]_t \equiv \frac{(t)_N}{(t)_m (t)_{N-m}} \tag{10}$$

and

$$(t)_m \equiv (1-t)\dots(1-t^m) \tag{11}$$

The Cartan matrix

$$C_{ab} = \delta_{a,b-1} + \delta_{a,b+1} \quad 1 \le a, \quad b \le p-2 \tag{12}$$

Comparing of the expressions (8) and (9) for the same character $\chi^p_{rs}(t)$ we arrive to the generalized R-R type identities.The left side of the identities is called bosonic while the right one is usually refered to as fermionic. The reason of these denominations will be explaned later.

These identities were proved numerically and combinatorically. The problem of algebraic and physical understanding of them is still open.

3. Minimal models of integrable lattice theory.

We consider the one-dimensional XXZ chain with free boundary conditions [5]

$$\begin{aligned} H_{xxz} &= \sum_{n=1}^{N-1} \Big[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{q+q^{-1}}{4} \sigma_n^z \sigma_{n+1}^z \\ &\qquad + \frac{q-q^{-1}}{4} (\sigma_n^z - \sigma_{n+1}^z) \Big] \qquad (13) \\ \sigma_n^\pm &= 1 \otimes \cdots \otimes \sigma^\pm \otimes \cdots \otimes 1, \\ \sigma_n^z &= 1 \otimes \cdots \otimes \sigma^z \otimes \cdots \otimes 1, \\ \sigma^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

It was observed in [5, 7] that the eigenvalues of $2L$-site XXZ Hamiltonian (13) with the couplings $\eta = \pi/4$ and $\eta = \pi/6$ (we set $q \equiv e^{\imath\eta}$) exactly coincide with some of the eigenenergies of L-site self-dual quantum Ising and 3-Potts models with free ends (see also [8]).

The remarkable properties of the model are connected with its $U_q(sl(2))$ symmetry found by Pasqier and Saleur [6]. Namely, Hamiltonian H_{xxz} commutes with the generators X, Y, H of this quantum algebra that are defined as

$$\begin{aligned} &X = \sum_{n=1}^{N} q^{(\sigma_1^z + \ldots + \sigma_{n-1}^z)/2} \sigma_n^+ q^{-(\sigma_{n+1}^z + \ldots + \sigma_N^z)/2}, \\ &X \to Y, \quad \sigma^+ \to \sigma^-, \\ &H = \sum_{n=1}^{N} \frac{\sigma_n^z}{2} \end{aligned}$$

and satisfy the relations

$$[H,X]=X, \qquad [H,Y]=-Y, \qquad [X,Y]=\frac{q^{2H}-q^{-2H}}{q-q^{-1}}. \tag{14}$$

Because of the quantum group symmetry, the spectrum of the Hamiltonian can be classified according to the representation theory of the algebra. For general q, its representations are equivalent to those of the ordinary $U(sl(2))$ algebra, and the configuration space $(C^2)^N$ of the spin chain can be split into a direct sum of irreducible highest-weight representations ρ_j (j is the highest weight), which are in one-to-one correspondence with the ordinary $sl(2)$ representations. In the $N=4$ case, for example, $(C^2)^4$ can be decomposed as $\rho_2+3\rho_1+2\rho_0$.

We concentrate on the case $q^{p+1}=-1$ [10, 6]. In this case, the generators X and Y are nilpotent in the state space of the model,

$$X^{p+1}=0, \qquad Y^{p+1}=0. \tag{15}$$

As a consequence, we obtain a very different picture of the decomposition of the configuration space $(C^2)^N$.

For example, if $q^4=-1$ and we try to decompose $(C^2)^4$, we find that $(C^2)^N$ now decomposes into the sum of one "bad" eight-dimensional representation (ρ_2,ρ_1) of type I and four other "good" representations $(2\rho_1+2\rho_0)$ of type II [6]. The type II representations are isomorphic to the ordinary $U(sl(2))$ ones. The type I representation (ρ_2,ρ_1) can be considered a result of gluing two representations ρ_2 and ρ_1. This (ρ_2,ρ_1) representation is indecomposable, but because it contains a three-dimensional invariant subspace, it is not irreducible.

In the general $q^{p+1}=-1$ case [10, 6], the configuration space splits into the sum of "bad" type I representations with the highest weights $S_z\geq p/2$ and "good" type II representations with the highest weights $S_z<p/2$ that are simultaneously not subspaces of some "bad" ones. The highest-weight vectors v_j of the good representations can be characterized [6] by the condition

$$v_j\in V_p\equiv \operatorname{Ker}X/\operatorname{Im}X^p. \tag{16}$$

Because of the $U_q(sl(2))$ invariance of H_{xxz}, we can restrict its action on the space V_p. We call the result of this quantum group reduction the minimal model of the integrable lattice theory ($LM(p,p+1)$) because its thermodynamic limit is $M(p,p+1)$, the ordinary minimal model of CFT with the Virasoro central charge $c=1-\frac{6}{p(p+1)}$ [5, 6, 9].

4. Quantum group reduction and truncation of fusion functional relations [21]

Alcaraz et al. [5] solved XXZ model (13) using the coordinate Bethe ansatz method. Sklyanin constructed the family of transfer-matrices $T(u)$ [11, 12, 14] commuting between themselves and with H_{xxz}. It was shown in [13] that the Sklyanin transfer matrix $T(u)$ commutes with the quantum group $U_q(sl(2))$. Therefore, the action of $T(u)$ as well as

H_{xxz} can be restricted on the space V_p if $q^{p+1} = -1$. To perform this quantum group reduction for $T(u)$, we use the Baxter T–Q equation [9]

$$t(u)Q(u) = \phi(u+\eta/2)Q(u-\eta) + \phi(u-\eta/2)Q(u+\eta), \tag{17}$$

where $\phi(u) = \sin 2u \sin^{2N} u$, $t(u) = \sin 2u\, T(u)$, and $Q(u)$ are eigenvalues of the Baxter auxilary matrix $\hat{Q}(u)$ commuting with $\hat{T}(u)$.

The eigenvalue $Q(u)$ for the eigenvector with $M = N/2 - S_z$ reversed spins has the form

$$Q(u) = \prod_{m=1}^{M} \sin(u-u_m)\sin(u+u_m). \tag{18}$$

Equation (17) is equivalent to the Bethe ansatz equations [14]

$$\left[\frac{\sin(u_k+\eta/2)}{\sin(u_k-\eta/2)}\right]^{2N} = \prod_{m\neq k}^{M} \frac{\sin(u_k-u_m+\eta)\sin(u_k+u_m+\eta)}{\sin(u_k-u_m-\eta)\sin(u_k+u_m-\eta)} \tag{19}$$

for model (13) provided that $T(u)$ has no poles.

Baxter equation (17) can be considered a discrete version of a second-order differential equation [1]. We can therefore seek its second (linearly independent) solution $P(u)$ with the same eigenvalue $t(u)$ as in (17),

$$t(u)P(u) = \phi(u+\eta/2)P(u-\eta) + \phi(u-\eta/2)P(u+\eta). \tag{20}$$

It follows from (17) and (20) that

$$\frac{Q(u+\eta)P(u) - P(u+\eta)Q(u)}{\phi(u+\eta/2)} = \frac{Q(u)P(u-\eta) - P(u)Q(u-\eta)}{\phi(u-\eta/2)}. \tag{21}$$

If η/π is irrational, then both parts of (21) are equal to a constant, and we can choose this constant to be 1, which means just a normalization of $P(u)$. We thus obtain the "quantum Wronskian" condition [1]

$$Q(u+\eta/2)P(u-\eta/2) - P(u+\eta/2)Q(u-\eta/2) = \phi(u). \tag{22}$$

If η is a rational part of π, the expressions in (21) could be equal to a periodic function $f(u)$ such that $f(u+\eta) = f(u)$, but in this case, we assume that $f(u)$ is also equal to 1.

Inserting (22) in (17) or (20), we obtain

$$t(u) = Q(u+\eta)P(u-\eta) - P(u+\eta)Q(u-\eta). \tag{23}$$

Dividing (22) by $Q(u+\eta/2)Q(u-\eta/2)$, we obtain

$$\frac{P(u+\eta/2)}{Q(u+\eta/2)} - \frac{P(u-\eta/2)}{Q(u-\eta/2)} = \frac{\phi(u)}{Q(u+\eta/2)Q(u-\eta/2)}. \tag{24}$$

The r.h.s. of (24) is a fraction of two trigonometric polynomials. Therefore, it can be uniquely expressed as the sum

$$\frac{\phi(u)}{Q(u+\eta/2)Q(u-\eta/2)} = R(u) + \frac{A(u+\eta/2)}{Q(u+\eta/2)} - \frac{B(u-\eta/2)}{Q(u-\eta/2)}, \tag{25}$$

where $R(u+\pi) = R(u)$ is a trigonometric polynomial with $\deg R(u) = 2N+2-4M$ and $A(u+\pi) = A(u)$ and $B(u+\pi) = B(u)$ are some trigonometric polynomials with degrees less than $\deg Q(u) = 2M$. Then (23) can be rewritten as

$$\begin{aligned} \frac{t(u)}{Q(u+\eta)Q(u-\eta)} \quad = \quad & R(u+\eta/2) + R(u-\eta/2) + \\ & + \frac{A(u+\eta)}{Q(u+\eta)} - \frac{B(u)}{Q(u)} + \frac{A(u)}{Q(u)} - \frac{B(u-\eta)}{Q(u-\eta)}. \end{aligned} \tag{26}$$

The term $\frac{A(u)-B(u)}{Q(u)}$ in the r.h.s. must vanish because otherwise it would have extra poles that are absent in the l.h.s. of (26) (the degrees of the polynomials in the numerator being less than the degree of the polynomial in the denominator). Therefore, $A(u) = B(u)$ and hence

$$\frac{\phi(u)}{Q(u+\eta/2)Q(u-\eta/2)} = R(u) + \frac{A(u+\eta/2)}{Q(u+\eta/2)} - \frac{A(u-\eta/2)}{Q(u-\eta/2)}. \tag{27}$$

If $F(u)$ is a function such that

$$R(u) = F(u+\eta/2) - F(u-\eta/2), \tag{28}$$

we can rewrite (24) as

$$\begin{aligned} & \frac{P(u+\eta/2)}{Q(u+\eta/2)} - \frac{P(u-\eta/2)}{Q(u-\eta/2)} = \\ & \quad = F(u+\eta/2) + \frac{A(u+\eta/2)}{Q(u+\eta/2)} - F(u-\eta/2) - \frac{A(u-\eta/2)}{Q(u-\eta/2)}. \end{aligned} \tag{29}$$

This means that we can choose the second solution of Baxter equation (20) in the form [16]

$$P(u) = F(u)Q(u) + A(u). \tag{30}$$

The newly constructed solution of the Baxter equation is not a periodic function in the general case. It is a periodic function if and only if $F(u)$ is a periodic solution of (28).

We defer the question of the existence of such a solution and first consider the consequences if it exists. We define the function

$$\begin{aligned} t_k(u) = {} & Q\left(u+(k+1)\eta/2\right) P\left(u-(k+1)\eta/2\right) \\ & - P\left(u+(k+1)\eta/2\right) Q\left(u-(k+1)\eta/2\right), \end{aligned} \tag{31}$$

where k is any nonnegative integer. Comparing this with (22) and (23), we can see that $t_0(u) = \phi(u)$ and $t_1(u) = t(u)$. From (31), it is easy to obtain [16] the following functional equations for $t_k(u)$:

$$t_k\left(u+\eta/2\right)t_k\left(u-\eta/2\right)-t_{k+1}(u)t_{k-1}(u)=\phi\left(u+(k+1)\eta/2\right)\phi\left(u-(k+1)\eta/2\right),$$

$$\begin{aligned} t_k(u)t_1\left(u-(k+1)\eta/2\right) &= t_{k+1}\left(u-\eta/2\right)\phi\left(u-k\eta/2\right) \\ &\quad + t_{k-1}\left(u+\eta/2\right)\phi\left(u-(k+2)\eta/2\right). \end{aligned} \tag{32}$$

These functional relations [17] coincide with the relations for eigenvalues of the fused transfer matrices obtained by Zhou [9] for the XXZ model and by Behrend, Pearce, and O'Brien [19] for the ABF models with fixed boundary conditions.

We now return to the question of the existence of the periodic solution of (28). Recalling $\phi(-u) = -\phi(u)$ and definition (25) of $R(u)$, we conclude that $R(u)$ is also an odd trygonometric polynomial with $\deg R(u) = 2N + 2 - 4M \geq 2$ if the number of the reversed spins M is not more than $N/2$. Such a polynomial can therefore be written as

$$R(u) = \sum_{m=1}^{2N+2-4M} R_m \sin mu, \tag{33}$$

where R_m are coefficients defined by (25).

If η/π is irrational, a periodic solution of (28) exists and has the form

$$F(u) = -\frac{1}{2}\sum_{m=1}^{2N+2-4M} \frac{R_m}{\sin\frac{m\eta}{2}}\cos mu. \tag{34}$$

But if η/π is a rational number, the situation is more sophisticated. Specifically, we concentrate on the case $\eta = \pi/(p+1)$, which corresponds to the $LM(p, p+1)$ model; from (34), we obtain

$$F(u) = -\frac{1}{2}\sum_{m=1}^{2N+2-4M} \frac{R_m}{\sin\frac{m\pi}{2(p+1)}}\cos mu. \tag{35}$$

Therefore, if we try to find the solution in a sector where $2N + 2 - 4M \geq 2(p+1)$ or, equivalently, $S_z \geq p/2$, we can see from (35) that a periodic solution does not exist. The reason is just a resonance.

We therefore conclude that a periodic $F(u)$ and the second periodic solution of the Baxter equation exist if we apply it to subspaces of the configuration space with $S_z < p/2$. This inequality coincides with the condition for the quantum group reduction from part II. We thus arrive at the main statement of this section:

In the case where $\eta = \pi/(p+1)$, the second periodic solution of the Baxter equation exists if and only if the configuration space of the model has undergone a quantum group reduction.

For this case, we have additional relations for the eigenvalues of the transfer matrices $t_k(u)$. Equation (31) now becomes

$$t_k(u) = Q\left(u - \frac{(k+1)\pi}{2(p+1)}\right) P\left(u + \frac{(k+1)\pi}{2(p+1)}\right) - Q\left(u + \frac{(k+1)\pi}{2(p+1)}\right) P\left(u - \frac{(k+1)\pi}{2(p+1)}\right). \tag{36}$$

Therefore, we can see that

$$t_p(u) = 0 \tag{37}$$

and

$$t_{p-1-k}(u) = -t_k(u + \pi/2). \tag{38}$$

We thus obtain the second main statement of this section:

The quantum group reduction of the model is equivalent to the truncation of functional relations (32).[1]

5. Solution of the truncated functional equations in the case $\eta = \pi/4$.

In the case $\eta = \pi/4$ (or $q^4 = -1$), the truncated fuctional equations become

$$t(u + \pi/8)t(u - \pi/8) = \phi(u + \pi/4)\phi(u - \pi/4) - \phi(u)\phi(u + \pi/2). \tag{39}$$

Substituting $\phi(u) = \sin 2u \sin^{2N} u$ in (39), we obtain

$$t(u + \pi/8)t(u - \pi/8) = 2^{2N}[\sin^{2N+2} 2u - \cos^{2N+2} 2u]. \tag{40}$$

These equations were considered and solved in [20] as the equations for the transfer matrix of the Ising model on the cylinder. For $T(u) = \frac{t(u)}{\sin 2u}$, we can rewrite the last equation as

$$2^{2N-1}T(u + \pi/8)T(u - \pi/8) = \prod_{k=1}^{N/2} |\varphi_k(u + \pi/8)|^2 |\varphi_k(u - \pi/8)|^2, \tag{41}$$

where we suggest that N be even,

$$\varphi_k(u) = \sin(2u + \pi/4) - \omega^k \sin(2u - \pi/4), \tag{42}$$

and $\omega = \exp \frac{i\pi}{N+1}$.

[1] This statement was also verified numerically by J. Suzuki [18].

Using $\varphi_k(u+\pi/8) = -\omega^k \varphi_k(u-\pi/8)$, we find $2^{N/2}$ real solutions of (41):

$$T(u) = T_{n_1,\dots,n_{N/2}} \equiv 2^{\frac{1-2N}{2}} \prod_{k=1}^{N/2} \left| \varphi_k \left(u + \frac{\pi n_k}{4} \right) \right|^2, \tag{43}$$

where $n_1, \dots, n_{N/2} = 0, 1$. In another form, the last expression is

$$T_{n_1,\dots,n_{N/2}} = 2^{\frac{1-2N}{2}} \prod_{k=1}^{N/2} \left[1 + (-1)^{n_k} \cos \frac{\pi k}{N+1} \cos 4u \right]. \tag{44}$$

In the integers $\{n_k\}$, we recognize the occupation numbers of the fermions in the one-dimensional Ising model. This is not suprising, because $LM(3,4)$ exactly coincides with the Ising model, as was mentioned above [5, 7, 8].

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