

Equilibrium Distributions in Open and Closed Statistical Systems

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Abstract: In this communication, the derivation of the Boltzmann-Gibbs and the Maxwellian distributions is presented from a geometrical point of view under the hypothesis of equiprobability. It is shown that both distributions can be obtained by working out the properties of the volume or the surface of the respective geometries delimited in phase space by an additive constraint. That is, the asymptotic equilibrium distributions in the thermodynamic limit are independent of considering open or closed homogeneous statistical systems.

Key-Words: Equiprobability, asymptotic equilibrium distributions, geometrical derivation

1 Introduction

In this paper, different classical results [1, 2] are obtained from a geometrical interpretation of different multi-agent systems evolving in phase space under the hypothesis of equiprobability.

We start by deriving in section 2 the Boltzmann-Gibbs (exponential) distribution by means of the geometrical properties of the volume of an N -dimensional pyramid. The same result is obtained when the calculation is performed over the surface of a such N -dimensional body. In both cases, the motivation is a multi-agent economic system with an open or closed economy, respectively.

Also, the Maxwellian (Gaussian) distribution is derived in section 3 from geometrical arguments over the volume or the surface of an N -sphere. Here, the motivation is a multi-particle gas system in contact with a heat reservoir (non-isolated or open system) or with a fixed energy (isolated or closed system), respectively.

Last section contains our conclusions.

2 Derivation of the Boltzmann-Gibbs Distribution

2.1 Multi-agent economic open systems

Here we assume N agents, each one with coordinate x_i , $i = 1, \dots, N$, with $x_i \geq 0$ representing the wealth or money of the agent i , and a total available amount of money E :

$$x_1 + x_2 + \dots + x_{N-1} + x_N \leq E. \quad (1)$$

Under random or deterministic evolution rules for the exchanging of money among agents, let us suppose that this system evolves in the interior of the N -dimensional pyramid given by Eq. (1). The role of a heat reservoir, that in this model supplies money instead of energy, could be played by the state or by the bank system in western societies. The formula for the volume $V_N(E)$ of an equilateral N -dimensional pyramid formed by $N + 1$ vertices linked by N perpendicular sides of length E is

$$V_N(E) = \frac{E^N}{N!}. \quad (2)$$

We suppose that each point on the N -dimensional pyramid is equiprobable, then the probability $f(x_i)dx_i$ of finding the agent i with money x_i is proportional to the volume formed by all the points into the $(N - 1)$ -dimensional pyramid having the i th-coordinate equal to x_i . We show now that $f(x_i)$ is the Boltzmann factor (or the Maxwell-Boltzmann distribution), with the normalization condition

$$\int_0^E f(x_i)dx_i = 1. \quad (3)$$

If the i th agent has coordinate x_i , the $N - 1$ remaining agents share, at most, the money $E - x_i$ on the $(N - 1)$ -dimensional pyramid

$$x_1 + x_2 \dots + x_{i-1} + x_{i+1} \dots + x_N \leq E - x_i, \quad (4)$$

whose volume is $V_{N-1}(E - x_i)$. It can be easily proved that

$$V_N(E) = \int_0^E V_{N-1}(E - x_i)dx_i. \quad (5)$$

Hence, the volume of the N -dimensional pyramid for which the i th coordinate is between x_i and $x_i + dx_i$ is $V_{N-1}(E - x_i)dx_i$. We normalize it to satisfy Eq. (3), and obtain

$$f(x_i) = \frac{V_{N-1}(E - x_i)}{V_N(E)}, \quad (6)$$

whose final form, after some calculation is

$$f(x_i) = NE^{-1} \left(1 - \frac{x_i}{E}\right)^{N-1}, \quad (7)$$

If we call ϵ the mean wealth per agent, $E = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{x_i}{E}\right)^{N-1} \simeq e^{-x_i/\epsilon}. \quad (8)$$

The Boltzmann factor $e^{-x_i/\epsilon}$ is found when $N \gg 1$ but, even for small N , it can be a good approximation for agents with low wealth. After substituting Eq. (8) into Eq. (7), we obtain the Maxwell-Boltzmann distribution in the asymptotic regime $N \rightarrow \infty$ (which also implies $E \rightarrow \infty$):

$$f(x)dx = \frac{1}{\epsilon} e^{-x/\epsilon} dx, \quad (9)$$

where the index i has been removed because the distribution is the same for each agent, and thus the wealth distribution can be obtained by averaging over all the agents. This distribution has been found to fit the real distribution of incomes in western societies [3].

This means that the geometrical image of the volume-based statistical ensemble allows us to recover the same result than that obtained from the microcanonical ensemble [4] that we show in the next section.

2.2 Multi-agent economic closed systems

Here, we derive the Boltzmann-Gibbs distribution by considering the system in isolation, that is, a closed economy. Without loss of generality, let us assume N interacting economic agents, each one with coordinate x_i , $i = 1, \dots, N$, with $x_i \geq 0$, and where x_i represents an amount of money. If we suppose now that the total amount of money E is conserved,

$$x_1 + x_2 + \dots + x_{N-1} + x_N = E, \quad (10)$$

then this isolated system evolves on the positive part of an equilateral N -hyperplane. The surface area $S_N(E)$ of an equilateral N -hyperplane of side E is given by

$$S_N(E) = \frac{\sqrt{N}}{(N-1)!} E^{N-1}. \quad (11)$$

Different rules, deterministic or random, for the exchange of money between agents can be given [2]. Depending on these rules, the system can visit the N -hyperplane in an equiprobable manner or not. If the ergodic hypothesis is assumed, each point on the N -hyperplane is equiprobable. Then the probability $f(x_i)dx_i$ of finding agent i with money x_i is proportional to the surface area formed by all the points on the N -hyperplane having the i th-coordinate equal to x_i . We show that $f(x_i)$ is the Boltzmann factor (Boltzmann-Gibbs distribution), with the normalization condition (3).

If the i th agent has coordinate x_i , the $N - 1$ remaining agents share the money $E - x_i$ on the $(N - 1)$ -hyperplane

$$x_1 + x_2 + \dots + x_{i-1} + x_{i+1} + \dots + x_N = E - x_i, \quad (12)$$

whose surface area is $S_{N-1}(E - x_i)$. If we define the coordinate θ_N as satisfying

$$\sin \theta_N = \sqrt{\frac{N-1}{N}}, \quad (13)$$

it can be easily shown that

$$S_N(E) = \int_0^E S_{N-1}(E - x_i) \frac{dx_i}{\sin \theta_N}. \quad (14)$$

Hence, the surface area of the N -hyperplane for which the i th coordinate is between x_i and $x_i + dx_i$ is proportional to $S_{N-1}(E - x_i)dx_i / \sin \theta_N$. If we take into account the normalization condition (3), we obtain

$$f(x_i) = \frac{1}{S_N(E)} \frac{S_{N-1}(E - x_i)}{\sin \theta_N}, \quad (15)$$

whose form after some calculation is

$$f(x_i) = (N-1)E^{-1} \left(1 - \frac{x_i}{E}\right)^{N-2}, \quad (16)$$

If we call ϵ the mean wealth per agent, $E = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{x_i}{E}\right)^{N-2} \simeq e^{-x_i/\epsilon}. \quad (17)$$

As in the former section, the Boltzmann factor $e^{-x_i/\epsilon}$ is found when $N \gg 1$ but, even for small N , it can be a good approximation for agents with low wealth. After substituting Eq. (17) into Eq. (16), we obtain the Boltzmann distribution (9) in the limit $N \rightarrow \infty$ (which also implies $E \rightarrow \infty$). This asymptotic result reproduces the distribution of real economic data [3] and also the results obtained in several models of

economic agents with random exchange interactions [2].

Depending on the physical situation, the mean wealth per agent ϵ takes different expressions and interpretations. For instance, we can calculate the dependence of ϵ on the temperature, which in the micro-canonical ensemble is defined by the derivative of the entropy with respect to the energy. The entropy can be written as $S = -kN \int_0^\infty f(x) \ln f(x) dx$, where $f(x)$ is given by Eq. (9) and k is Boltzmann's constant. If we recall that $\epsilon = E/N$, we obtain

$$S(E) = kN \ln \frac{E}{N} + kN. \quad (18)$$

The calculation of the temperature T gives

$$T^{-1} = \left(\frac{\partial S}{\partial E} \right)_N = \frac{kN}{E} = \frac{k}{\epsilon}. \quad (19)$$

Thus $\epsilon = kT$, and the Boltzmann distribution is obtained in its usual form:

$$f(x)dx = \frac{1}{kT} e^{-x/kT} dx. \quad (20)$$

3 Derivation of the Maxwellian Distribution

3.1 Multi-particle open systems

Let us suppose a one-dimensional ideal gas of N non-identical classical particles with masses m_i , with $i = 1, \dots, N$, and total maximum energy E . If particle i has a momentum $m_i v_i$, we define a kinetic energy:

$$K_i \equiv p_i^2 \equiv \frac{1}{2} m_i v_i^2, \quad (21)$$

where p_i is the square root of the kinetic energy K_i . If the total maximum energy is defined as $E \equiv R^2$, we have

$$p_1^2 + p_2^2 + \dots + p_{N-1}^2 + p_N^2 \leq R^2. \quad (22)$$

We see that the system has accessible states with different energy, which can be supplied by a heat reservoir. These states are all those enclosed into the volume of the N -sphere given by Eq. (22). The formula for the volume $V_N(R)$ of an N -sphere of radius R is

$$V_N(R) = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2} + 1)} R^N, \quad (23)$$

where $\Gamma(\cdot)$ is the gamma function. If we suppose that each point into the N -sphere is equiprobable, then the

probability $f(p_i)dp_i$ of finding the particle i with coordinate p_i (energy p_i^2) is proportional to the volume formed by all the points on the N -sphere having the i th-coordinate equal to p_i . We proceed to show that $f(p_i)$ is the Maxwellian distribution, with the normalization condition

$$\int_{-R}^R f(p_i)dp_i = 1. \quad (24)$$

If the i th particle has coordinate p_i , the $(N - 1)$ remaining particles share an energy less than the maximum energy $R^2 - p_i^2$ on the $(N - 1)$ -sphere

$$p_1^2 + p_2^2 \dots + p_{i-1}^2 + p_{i+1}^2 \dots + p_N^2 \leq R^2 - p_i^2, \quad (25)$$

whose volume is $V_{N-1}(\sqrt{R^2 - p_i^2})$. It can be easily proved that

$$V_N(R) = \int_{-R}^R V_{N-1}(\sqrt{R^2 - p_i^2}) dp_i. \quad (26)$$

Hence, the volume of the N -sphere for which the i th coordinate is between p_i and $p_i + dp_i$ is $V_{N-1}(\sqrt{R^2 - p_i^2}) dp_i$. We normalize it to satisfy Eq. (24), and obtain

$$f(p_i) = \frac{V_{N-1}(\sqrt{R^2 - p_i^2})}{V_N(R)}, \quad (27)$$

whose final form, after some calculation is

$$f(p_i) = C_N R^{-1} \left(1 - \frac{p_i^2}{R^2} \right)^{\frac{N-1}{2}}, \quad (28)$$

with

$$C_N = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{N+2}{2})}{\Gamma(\frac{N+1}{2})}. \quad (29)$$

For $N \gg 1$, Stirling's approximation can be applied to Eq. (29), leading to

$$\lim_{N \gg 1} C_N \simeq \frac{1}{\sqrt{\pi}} \sqrt{\frac{N}{2}}. \quad (30)$$

If we call ϵ the mean energy per particle, $E = R^2 = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{p_i^2}{R^2} \right)^{\frac{N-1}{2}} \simeq e^{-p_i^2/2\epsilon}. \quad (31)$$

The factor $e^{-p_i^2/2\epsilon}$ is found when $N \gg 1$ but, even for small N , it can be a good approximation for particles with low energies. After substituting Eqs. (30)–(31) into Eq. (28), we obtain the Maxwellian distribution

in the asymptotic regime $N \rightarrow \infty$ (which also implies $E \rightarrow \infty$):

$$f(p)dp = \sqrt{\frac{1}{2\pi\epsilon}} e^{-p^2/2\epsilon} dp, \quad (32)$$

where the index i has been removed because the distribution is the same for each particle, and thus the velocity distribution can be obtained by averaging over all the particles.

This newly shows that the geometrical image of the volume-based statistical ensemble allows us to recover the same result than that obtained from the microcanonical ensemble [5] that it is presented in the next section.

3.2 Multi-particle closed systems

We start by assuming a one-dimensional ideal gas of N non-identical classical particles with masses m_i , with $i = 1, \dots, N$, and total energy E . If particle i has a momentum $m_i v_i$, newly we define a kinetic energy K_i given by Eq. (21), where p_i is the square root of K_i . If the total energy is defined as $E \equiv R^2$, we have

$$p_1^2 + p_2^2 + \dots + p_{N-1}^2 + p_N^2 = R^2. \quad (33)$$

We see that the isolated system evolves on the surface of an N -sphere. The formula for the surface area $S_N(R)$ of an N -sphere of radius R is

$$S_N(R) = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} R^{N-1}, \quad (34)$$

where $\Gamma(\cdot)$ is the gamma function. If the ergodic hypothesis is assumed, that is, each point on the N -sphere is equiprobable, then the probability $f(p_i)dp_i$ of finding the particle i with coordinate p_i (energy p_i^2) is proportional to the surface area formed by all the points on the N -sphere having the i th-coordinate equal to p_i . Our objective is to show that $f(p_i)$ is the Maxwellian distribution, with the normalization condition (24).

If the i th particle has coordinate p_i , the $(N-1)$ remaining particles share the energy $R^2 - p_i^2$ on the $(N-1)$ -sphere

$$p_1^2 + p_2^2 \dots + p_{i-1}^2 + p_{i+1}^2 \dots + p_N^2 = R^2 - p_i^2, \quad (35)$$

whose surface area is $S_{N-1}(\sqrt{R^2 - p_i^2})$. If we define the coordinate θ as satisfying

$$R^2 \cos^2 \theta = R^2 - p_i^2, \quad (36)$$

then

$$Rd\theta = \frac{dp_i}{(1 - \frac{p_i^2}{R^2})^{1/2}}. \quad (37)$$

It can be easily proved that

$$S_N(R) = \int_{-\pi/2}^{\pi/2} S_{N-1}(R \cos \theta) R d\theta. \quad (38)$$

Hence, the surface area of the N -sphere for which the i th coordinate is between p_i and $p_i + dp_i$ is $S_{N-1}(R \cos \theta) R d\theta$. We rewrite the surface area as a function of p_i , normalize it to satisfy Eq. (24), and obtain

$$f(p_i) = \frac{1}{S_N(R)} \frac{S_{N-1}(\sqrt{R^2 - p_i^2})}{(1 - \frac{p_i^2}{R^2})^{1/2}}, \quad (39)$$

whose final form, after some calculation is

$$f(p_i) = C_N R^{-1} \left(1 - \frac{p_i^2}{R^2}\right)^{\frac{N-3}{2}}, \quad (40)$$

with

$$C_N = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})}. \quad (41)$$

For $N \gg 1$, Stirling's approximation can be applied to Eq. (41), leading to

$$\lim_{N \gg 1} C_N \simeq \frac{1}{\sqrt{\pi}} \sqrt{\frac{N}{2}}. \quad (42)$$

If we call ϵ the mean energy per particle, $E = R^2 = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{p_i^2}{R^2}\right)^{\frac{N-3}{2}} \simeq e^{-p_i^2/2\epsilon}. \quad (43)$$

As in the former section, the Boltzmann factor $e^{-p_i^2/2\epsilon}$ is found when $N \gg 1$ but, even for small N , it can be a good approximation for particles with low energies. After substituting Eqs. (42)–(43) into Eq. (40), we obtain the Maxwellian distribution (32) in the asymptotic regime $N \rightarrow \infty$ (which also implies $E \rightarrow \infty$).

Depending on the physical situation the mean energy per particle ϵ takes different expressions. For an isolated one-dimensional gas we can calculate the dependence of ϵ on the temperature, which in the microcanonical ensemble is defined by differentiating the entropy with respect to the energy. The entropy can be written as $S = -kN \int_{-\infty}^{\infty} f(p) \ln f(p) dp$, where $f(p)$ is given by Eq. (32) and k is the Boltzmann constant. If we recall that $\epsilon = E/N$, we obtain

$$S(E) = \frac{1}{2} kN \ln \left(\frac{E}{N}\right) + \frac{1}{2} kN (\ln(2\pi) - 1). \quad (44)$$

The calculation of the temperature T gives

$$T^{-1} = \left(\frac{\partial S}{\partial E} \right)_N = \frac{kN}{2E} = \frac{k}{2\epsilon}. \quad (45)$$

Thus $\epsilon = kT/2$, consistent with the equipartition theorem. If p^2 is replaced by $\frac{1}{2}mv^2$, the Maxwellian distribution is a function of particle velocity, as it is usually given in the literature:

$$g(v)dv = \sqrt{\frac{m}{2\pi kT}} e^{-mv^2/2kT} dv. \quad (46)$$

4 Conclusion

We have shown that the Boltzmann factor describes the general statistical behavior of each small part of a multi-component system whose components or parts are given by a set of random variables that satisfy an additive constraint, in the form of a conservation law (closed systems) or in the form of an upper limit (open systems). The derivation of this factor for open systems in a general context has been presented in [6].

Let us remark that these calculations do not need the knowledge of the exact or microscopic randomization mechanisms of the multi-agent system in order to reach the equiprobability. In some cases, it can be reached by random forces [3], in other cases by chaotic [7] or deterministic [8] causes. Evidently, the proof that these mechanisms generate equiprobability is not a trivial task and it remains as a typical challenge in this kind of problems.

In summary, this work has presented a straightforward geometrical argument that recalls us the equivalence between canonical and microcanonical ensembles in the thermodynamic limit in the particular context of physical sciences. For the general context of homogeneous multi-agent systems, we conclude by highlighting the statistical equivalence of the volume-based and surface-based calculations in this type of systems.

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