THE STUDY OF THE LEVEL ZERO CROSSING TIME OF A SEMI-MARKOVIAN RANDOM WALK WITH DELAYING SCREEN

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Abstract

In this study, a semi-Markovian random walk with delaying screen at $\beta > 0$ and the first crossing time (γ_1) of the zero level of this process are constructed. Furthermore, the distribution function with its Laplace transform, expected value and variance of random variable (γ_1) are calculated. In addition to these, a formula for the higher order moments of (γ_1) is given.

Key Words: Semi-Markovian random walk, reflecting and delaying screens.

1. Introduction

It is known that most of the problems in stock control theory is often given by using a random walk or a random walk with delaying screens (see [2], [3], [4], [5], etc.). But for the problem considered in this study, one of the screens is reflecting and the other one is delaying, and the process representing the quantity of the stock has been given by using a random walk and a renewal process. Such models were rarely considered in literature. The practical state of the problem mentioned is as the following.

Suppose that some quantity of a stock in a certain storehouse is increasing and decreasing in random discrete portions at random times. Then, it is possible to characterize the level of stock by a process called a semi-Markovian random walk or random walk with delaying screens. Processes of this type have widely been studied in literature (see, for example [1], [6], [7]). But sometimes, for adaquate solution of some problems occuring in stock control theory, we have to consider processes more complex than that of semi-Markovian random walks with a delaying screen. For example, if the borrowed quantity is required to be added to a storehouse immediately when the quantity of demanded stock is more than the total quantity of stock in the storehouse then, it is possible to characterize the level of stock in the storehouse by a stochastic process called a semi-Markovian random walk with reflecting screen. But for the model studied in this paper an additional condition has been considered.

Since the volume of the storehouse is finite in real cases, the supply coming to the storehouse cease until the next demand when the storehouse becomes full. In order to characterize the quantity of stock in the storehouse under these conditions, it is necessary to use a stochastic process called a semi-Markovian random walk with two screens, one of them being reflecting and the other being delaying.

This type of problems may occur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, and etc. We want to note that semi-Markovian random walks with two screens, namely reflecting and delaying, have not been well considered in literature.

It what follows the process mentioned above is mathematically constructed and the probability characteristics of an important boundary functional are obtained.

2. Construction of the Process $X(t)$ and its Boundary Functional γ_1

In order to solve the stock control problem considered above let us construct the following stochastic process. Suppose $\{(\xi_i, \eta_i)\}$, $(i = 1, 2, ...)$ is a sequence of identically and independently distributed pairs of random variables, defined on any probability space $(\Omega, \Im, P)$ such that ξ_i, η_i are independent random variables and ξ_i's are positive valued, i.e., $P\{\xi_i > 0\} = 1$.

Suppose that the distribution functions of ξ_i and η_i are known, i.e.,

$$\Phi(t) = P\{\xi_1 < t\}, F(x) = P\{\eta_1 < x\}; \ t \geq 0, x \in (-\infty, +\infty).$$

Before expressing our process $X(t)$ let us consider the following renewal process $\{T_n : n = 0, 1, 2, ...\}$ and random walk $\{Y_n : n = 0, 1, 2, 3...\}$ defined by

$$Y_0 = T_0 = 0; \ T_n = \sum_{i=1}^{n} \xi_i, \ Y_n = \sum_{i=1}^{n} \eta_i, \ n = 1, 2.... \ .$$

The main probability characteristics of renewal processes and random walks are well known (see, [1], [6], [7]). Let us construct the following random walk with two screens; the one lying on the zero-level as reflecting and the other one lying on β-level as delaying :

$$\chi_1 = \min\{\beta; |\chi_0 + \eta_1|\},$$
$$\chi_n = \min\{\beta; |\chi_{n-1} + \eta_n|\}, \ n \geq 2,$$

where $\chi_0 = z \in [0, \beta]; \beta > 0$.
Now, we can construct mathematically the corresponding random process $X(t)$:

$$X(t) = \chi_n, \quad \text{when} \ t \in [T_n, T_{n+1}) \ n = 0, 1, 2, ... \ .$$

We call the process a semi-Markovian random walk with reflecting and delaying screens. $X(t)$ denotes the level of stock at time t, where the lower screen is reflecting and the upper screen is delaying screen.

Let us denote by γ_1 the first reflection time of the process $X(t)$. We are going to give below a complete characterization of the probability distribution of γ_1. To do this first, we construct $\hat{\chi}_n$-the random walk with one delaying screen on the level β, as follows:

$$\hat{\chi}_n = \min\{\hat{\chi}_{n-1} + \eta_n; \beta\}, \ n \geq 1, \ \hat{\chi}_0 = z \in [0, \beta].$$

Let us, define the random variable ν,

$$\nu = \min\{n \geq 1 : \hat{\chi}_n < 0\}.$$

Then γ_1 is given by

$$\gamma_1 = \sum_{i=1}^{\nu} \xi_i.$$

When we investigate random variable γ_1 and the process $X(t)$ seperately, it is possible to consider γ_1 as the first crossing time to zero level of semi-Markovian random walk with a delaying screen at $\beta > 0$.

The random variable γ_1 is important from the scientific and practical points of view and plays a special role in expressing the probability characteristics of the process $X(t)$.

The main aim of this study is to express the distribution function, Laplace transform of it, and numerical characteristics of γ_1 by the probability characteristics of random walk $\{Y_n\}$ and renewal process $\{T_n\}$.

3. A Study of the Probability Characteristics of γ_1

The following notations will be used throughout this study:

$$
\begin{aligned}
a_n(z) &= P\{z + Y_i \in [0, \beta]; \ i = 1, 2, ..., n-1; \ z + Y_n > \beta\}; \ n \geq 1 \\
b_n(z) &= P\{z + Y_i \in [0, \beta]; i = 1, 2, ..., n\}; \ n \geq 1, \\
b_0(z) &= 1, \forall z \in [0, \beta], \\
A(t; z) &= \sum_{n=1}^{\infty} a_n(z) \cdot \Phi_n(t), \\
B(t; z) &= \sum_{n=0}^{\infty} b_n(z).\Delta\Phi_n(t),
\end{aligned}
$$

$$\Phi_n(t) = P\{T_n < t\}; \ n \geq 1; \ \Phi_0(t) = \left\{ \begin{array}{ll} 1, & t > 0 \\ 0, & t \leq 0 \end{array} \right.,$$

$$\Delta \Phi_n(t) = \Phi_n(t) - \Phi_{n+1}(t); \ n \geq 0,$$

$\widetilde{M}(p;z) = \int_0^\infty e^{-pt}.M(t;z)dt$ - the Laplace transform of the function $M(t;z)$ with respect to t.

$M^*(p;Z) = [\int_0^\infty e^{-pt}.d_tM(t;z)$-the Laplace-Stieltjes transform of the function $M(t;z)$ with respect to t.

We will give the convolution of $M_1(t;z)$ and $M_2(t;v)$ functions as follows:

$$M_1(t;z) * M_2(t;v) = \int_0^t M_1(ds;z)M_2(t-s;v)$$

where $M_1(ds;z) = d_sM_1(s;z)$.

Now, we can formulate the first result of this study as the following.

Theorem 1. *If* $\{(\xi_i; \eta_i)\}, (i = 1, 2, ...)$ *is the initial sequence of random pairs mentioned above then, in terms of the probability characteristics of renewal process* $\{T_n\}$ *and random walk* $\{Y_n\}$, *the distribution function of* γ_1 *is given by*

$$P_z\{\gamma_1 < t\} = 1 - B(t;z) - A(t;z) * B(t;\beta) - \sum_{m=2}^\infty A(t;z) * (A(t;\beta))_*^{m-1} * B(t;\beta)$$

for all $z \in [0, \beta]$.

Proof. Let $N(t;z)$ denote the following conditional probability

$$N(t;z) = P_z\{\gamma_1 \geq t\} = P\{\gamma_1 \geq t/X(0) = z\}$$

and let $\nu_\beta(t)$ be the number of times $X(t)$ reaches the delaying screen during the interval $[0,t]$. According to the total probability formula we have

$$N(t;z) = \sum_{m=0}^\infty P_z\{\gamma_1 \geq t; \nu_\beta(t) = m\}$$

and to make it simple, we put

$$n_m(t;z) = P_z\{\gamma_1 \geq t; \nu_\beta(t) = m\}, \ m = 0, 1, ...$$

Now express each $n_m(t > z)$ by the probability characteristics of both $\{T_n\}$ and $\{Y_n\}$ separately. First, let us start with . We have

$$n_0(t; z) = P_z\{\gamma_1 \geq t; \nu_\beta(t) = 0\}$$

$$= \sum_{n=0}^{\infty} P\{z + Y_j \in [0, \beta]; j = 1, 2, ..., n\} . \Delta \Phi_n(t)$$

$$= \sum_{n=0}^{\infty} b_n(z) . \Delta \Phi_n(t) = B(t; z)$$

Calculation of $n_1(t; z)$:

$$n_1(t; z) = P_z\{\gamma_1 \geq t; \nu_\beta(t) = 1\}$$

$$= \sum_{n=1}^{\infty} P\{T_n \leq t < T_{n+1}; \nu_\beta(t) = 1; \gamma_1 \geq t\}$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \int_0^t P\{z + Y_j \in [0, \beta]; j = 1, 2, ..., k - 1; z + Y_k > \beta; T_k > \beta; T_k \in ds\}$$

$$.P\{\beta + Y_j \in [0, \beta]; j = 1, 2, ..., n - k; T_{n-k} \geq t - s < T_{n-k+1}\}$$

$$= \int_p^t \left[\sum_{\ell=0}^{\infty} b_\ell(\beta) \cdot \Delta \Phi_\ell(t - s) \right] \left[\sum_{k=1}^{\infty} a_k(z) d\Phi_k(s) \right].$$

Using the expression for the $A(t; z)$ and $B(t; \beta)$, we get

$$n_1(t; z) = \int_0^t A(ds; z) \cdot B(t - s; \beta) = A(t; z) * B(t; \beta).$$

To discover the general formula for $n_m(t; z)$ in terms of probability characteristics of the renewal process $\{T_n\}$ and random walk $\{Y_n\}$ we will also calculate $n_2(t; z)$:

$$n_2(t; z) = \sum_{n=2}^{\infty} P_z\{T_n \leq t < T_{n+1}; \nu_\beta(t) = 2, \gamma_1 \geq t\}$$

$$= \sum_{n=2}^{\infty} \sum_{2 \leq k+\ell \leq n} \int_{2 \leq k+u \leq t} \int P\{z + Y_i \in [0, \beta]; j = 1, 2, ..., k - 1; z + Y_k > \beta\}$$

$$.P\{T_k \in ds\} . P\{\beta + Y_j \in [0, \beta]; j = 1, 2y..., \ell - 1; \beta + Y_\ell > \beta\}$$

$$.P\{T_\ell \in du\} . P\{\beta + Y_j \in [0, \beta]; j = 1, 2, ..., n - \ell - k\}$$

$$.P\{T_{n-k-\ell} \leq t - s - u < T_{n-k-\ell+1}\}$$

$$= \int_{0 \leq s+u \leq t} \int A(ds; z) A(du; \beta) B(t - s - u; \beta),$$

where $A(ds; z) = d_s A(s; z)$. As such,

$$n_2(t; z) = A(t; z) * (t; \beta) * B(t; \beta)$$

is obtained.

Analogously, it is possible to prove that

$$n_m(t; z) = A(t; z) * (A(t > \beta))_*^{m-1} * B(t; \beta), m \geq 2.$$

Substituting all of these expressions in the formula for $N(t; z)$ given above, we obtain

$$N(t > z) = B(t, z) + A(t; z) * B(t; \beta) + \sum_{m=2}^{\infty} A(t; z) * (A(t; \beta))_*^{m-1} * B(t; \beta).$$

Therefore, the distribution function of γ_1 can be expressed as follows:

$$P_z\{\gamma_1 < t\} = 1 - N(t; z)$$

$$= 1 - B(t; z) - A(t; z) * B(t; \beta) - \sum_{m=2}^{\infty} A(t; z) * (A(t; \beta))_*^{m-1} * (t; \beta).$$

We have thus expressed distribution of γ_1 in terms of the probability characteristics of $\{T_n\}$ and $\{Y_n\}$. Hence the theorem is proved.

Notice that in particular, if the random variables $\xi_i (i = 1, 2, ...)$ have distribution functions of exponential type with the parameter $\lambda > 0$, then the functions $A(t; z)$ and $B(t; z)$ can be calculated more easily. $\qquad \square$

Corollary 1. *If the conditions of Theorem 1 are satisfied and ξ_1 has the exponential distribution function with parameter $\lambda > 0$, then $A(ds; z)$ and $B(t; z)$ have the following forms:*

$$A(ds; z) = \sum_{k=1}^{\infty} a_k(z) \frac{\lambda(\lambda s)^{k-1}}{(k-1)!} e^{-\lambda s} \cdot ds,$$

$$B(t; z) = \sum_{k=0}^{\infty} b_k(z) \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Proof. In this case, $\Phi_k(s) = P\{T_k < s\} = P\left\{ \sum_{i=1}^{k} \xi_i < s \right\}$ is the Erlang distribution function of $k - th$ order. Its density function is

$$\Phi'_k(s) = \lambda \frac{(\lambda s)^{k-1}}{(k-1)!} e^{-\lambda s}, k = 1, 2, \dots.$$

Now, we have

$$\Delta \Phi_k(t) = P\{T_k \le t < T^{k+1}\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

Bringing this formula into the expressions for the $A(ds; z)$ and $B(t; z)$ we complete the proof of Corollary 1.

We remark that in the solution of some practical problems it is required to calculate numerical characteristics of γ_1, i.g., expected value $(E\gamma_1)$, variance $(Var\gamma_1)$ and etc. In order to do this we need to calculate the Laplace transform of $N(t; z)$. Let us denote by $\widetilde{N}(t; z)$ this Laplace transform. Now, we can formulate Theorem 2. $\qquad\square$

Theorem 2. *If the conditions of Theorem 1 are satisfied then the following expression is true, for the Laplace transform of* $N(t; z)$:

$$\widetilde{N}(p; z) = \frac{\widetilde{B}(p; z) + A^*(p; z)\widetilde{B}(p; \beta) - A^*(p; \beta)\widetilde{B}(p; z)}{1 - A^*(p; \beta)}$$

for all $z \in [0, \beta]$. *In particular, when* $z = \beta$, *then*

$$\widetilde{N}(p; \beta) = \widetilde{B}(p; \beta)(1 - A^*(p; \beta))^{-1},$$

where

$$\widetilde{B}(p; \beta) = \frac{1 - \Phi^*(p)}{p} \sum_{n=0}^{\infty} b_n(z)(\Phi^*(p))^n,$$

$$A^*(p; z) = \sum_{k=1}^{\infty} a_k(z)(\Phi^*(p))^k,$$

$$\Phi^*(p) = \int_0^{\infty} e^{-pt} d\Phi(t) = E(e^{-p\xi}).$$

Proof. Applying the Laplace transform to the results of the Theorem 1, we get

$$\begin{aligned}
\widetilde{N}(p; z) &= \widetilde{B}(p; z) + A^*(p; z)\widetilde{B}(p; \beta) + \sum_{m=2}^{\infty} A^*(p; z)(A^*(p; \beta))^{m-1}\widetilde{B}(p; \beta) \\
&= \widetilde{B}(p; z) + A^*(p; z)\widetilde{B}(p; \beta)(1 - A^*(p; \beta))^{-1} \\
&= \left[\widetilde{B}(p; z) + A^*(p; z)\widetilde{B}(p; \beta) - A^*(p; \beta)B\widetilde{(p; z)}\right](1 - A^*(p; \beta))^{-1}.
\end{aligned}$$

In particular, substituting z by β, we have

$$\widetilde{N}(p;\beta) = \widetilde{B}(p;\beta)(1 - A^*(p\beta))^{-1}$$

Let us calculate $\widetilde{B}(p;z)$, for all $z \in [0,\beta]$.

$$\widetilde{B}(p;z) = \int_0^\infty e^{-pt}B(t;z)dt = \sum_{n\geq 0} b_n(z) \int_0^\infty e^{-pt}\Delta\Phi_n(t)dt$$

$$= \sum_{n\geq 0} b_n(z) \cdot \frac{\Phi_n^*(p) - \Phi_{n-1}^*(p)}{p} = \frac{1 - \Phi(p)}{p} \sum_{n\geq 0} b_n(z)(\Phi(p))^n.$$

Now, let us calculate $A^*(p;z)$, for all $z \in [0,\beta]$.

$$A^*(p;z) = \int_0^\infty e^{-pt}A(dt;z) = \int_0^\infty e^{-pt} \sum_{k=1}^\infty a_k(z)d\Phi_k(t)$$

$$= \sum_{k=1}^\infty a_k(z)\Phi_k^*(p) = \sum_{k=1}^\infty a_k(z)(\Phi^*(p))^k.$$

Hence, Theorem 2 is proved.

In order to express the expected value and the variance of γ_1 by mean of the probability characteristics of renewal process $\{T_n\}$ and random walk $\{Y_n\}$, let us introduce the following notions:

$$A(z) = \sum_{n=1}^\infty, \quad A_1(z) = \sum_{n=1}^\infty n.a_n(z);$$

$$B(z) = \sum_{n=0}^\infty b_n(z), \quad B_1(z) = \sum_{n=0}^\infty n.b_n(z)$$

for all $z \in [0,\beta]$. Now, we can formulate the following theorem: □

Theorem 3. *If conditions of Theorem 1 hold, and $E(\xi_1^2) < \infty, A(\beta) < 1$, then the expected value and variance of γ_1 are expressed by the probability characteristics of renewal process $\{T_n\}$ and random walk $\{Y_n\}$ as follows:*

1) $E_z\gamma_1 = E\xi_1 \cdot [B(z) + B(\beta)A(z) - B(z)A(\beta)] \cdot (1 - A(\beta))^{-1}$.

In particular, when , we have

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$$E_\beta \gamma_1 = E\xi_1 \cdot B(\beta) \cdot (1 - A(\beta))^{-1}.$$

$$2) \ E_z(\gamma_1^2) = E(\xi_1^2) \cdot \left[B(z) + \frac{B(\beta)A(z)}{1 - A(\beta)} \right]$$

$$= 2(E\xi_1)^2 \cdot \left[B_1(z) + \frac{B_1(\beta)A(z) + A_1(z)B(\beta)}{1 - A(\beta)} + \frac{A_1(\beta)B(\beta)A(z)}{(1 - A(\beta))^2} \right],$$

and particularly, when $z = \beta$, we have

$$E_\beta(\gamma_1^2) = E(\xi_1^2) \cdot \frac{B(\beta)}{1 - A(\beta)} + 2(E\xi_1)^2 \cdot \frac{B_1(\beta) + A_1(\beta)B(\beta) - B_1(\beta)A(\beta)}{(1 - A(\beta))^2}$$

and

$$3) \ Var(\gamma_1) = E_z(\gamma_1^2) - (E_z\gamma_1)^2$$

where $E_z(\cdot), Var_z(\cdot)$ are symbols of conditional expected value and variance with the condition that $X(0) = z, z \in [0, \beta]$.

Proof. It is known that

$$E_z\gamma_1 = \int_0^\infty N(t;z)dt = \widetilde{N}(p;z)\Big|_{p=0}.$$

Calculating $\widetilde{B}(p;z)$ and $A^2(p;z)$ at the point $p = 0$, we have

$$\widetilde{B}(0;z) = E\xi_1 \sum_{n=0}^\infty b_n(z) = E\xi_1.B(z)$$

$$A^*(0;z) = \sum_{k=1}^\infty a_k(z) = A(z).$$

Substituting these expressions in the formula for , we get

$$E_z\gamma_1 = \widetilde{N}(p;z)|_{p=0}$$
$$= E\xi_1.B(z) + E\xi_1.B(\beta)A(z) + E\xi_1.A(z)B(\beta)A(\beta) + \cdots + E\xi_1.A(z)(A(\beta))^k B(\beta) + \cdots$$
$$= E\xi_1.B(z) + \frac{E\xi_1.A(z)B(\beta)}{1 - A(\beta)}$$
$$= E\xi_1.[B(z) - B(z)A(\beta) + A(z)B(\beta)](1 - A(\beta))^{-1}.$$

In particular, when $z = b$,, we have then a simpler formula:

$$E_\beta \gamma_1 = E\xi_1 . B(\beta) . (1 - A(\beta))^{-1}.$$

In order to calculate the $E_z(\gamma_1^2)$, let us use the following formula:

$$E_z(\gamma_1^2) = -2 \left. \frac{\partial}{\partial p} [\widetilde{N}(p; z)] \right|_{p=0} .$$

It is shown that

$$A^*(0; z) = A(z) \text{ ve } \widetilde{B}(0, z) = E\xi_1 . B(z)$$

Taking the derivative of $\widetilde{B}(p; z)$ and $A^*(p; z)$ with respect to p, and then the limit for $p \to 0$ we get

$$\cdot \quad \lim_{p \to 0} \left[\frac{\partial A^*(p; z)}{\partial p} \right] = -E\xi_1 . A_1(z),$$

$$\lim_{p \to 0} \left[\frac{\partial \widetilde{B}(p; z)}{\partial p} \right] = -\frac{1}{2} E(\xi_1^2) B(z) - (E\xi_1)^2 B(z).$$

If the above obtained expression is replaced in the formula of $E_z(\gamma_1^2)$, then we get

$$\begin{aligned} E_z \gamma_1^2 &= E(\xi_1^2) \cdot \left[B(z) + \frac{B(\beta)A(z)}{1 - A(\beta)} \right] \\ &+ 2(\xi_1)^2 \cdot \left[B_1(z) + \frac{B_1(\beta)A(z) + A_1(z)B(\beta)}{1 - A(\beta)} + \frac{A_1(\beta)B(\beta)A(z)}{(1 - A(\beta))^2} \right] . \end{aligned}$$

In particular, if we substitute β for z, then we get a simpler formula for $E_\beta(\gamma_1^2)$ as

$$E_\beta \gamma_1^2 = E(\xi_1^2) \cdot \frac{B(\beta)}{1 - A(\beta)} + 2(E\xi_1)^2 \cdot \frac{B_1(\beta) + A_1(\beta)B(\beta) - B_1(\beta)A(\beta)}{(1 - A(\beta))^2} .$$

By using this formula the variance of γ_1 is obtained, as

$$Var_z(\gamma_1) = E_z(\gamma_1^2) - (E_z \gamma_1)^2, \forall z \in [0, \beta].$$

This completes the proof. $\qquad \square$

Remark Note that if conditions of Theorem 3 hold and if the first $n(n \geq 3)$ moments of the random variable ξ_1 exist and $A(\beta) < 1$, in this case we can calculate the higher order moments of by γ_1 using the following formul:

$$E_z(\gamma_1^k) = k \cdot (-1)^{k-1} \cdot \frac{\partial^{k-1}}{\partial p^{k-1}} [\widetilde{N}(p;z)]\Big|_{k=0} , \quad k = 3, 4, ..., n.$$

Acknowledgement

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Tutan Ekrana Sahip Yapı-Markov Rastgele Yürüyüş Sürecinin Sıfır Seviyesini Kesip Geçmesi Anının İncelenmesi

Özet

Bu çalışmada stok kontrol teorisinde önemli rol oynayan $\beta > 0$ seviyesinde tutan ekrana sahip yarı-Markov rastgele yürüyüş süreci ve bu sürecin ilk kez sıfır seviyesini kesip geçmesi anı (γ_1) inşa edilmiştir. Ayrıca γ_1 rastgele değişkeninin dağılım fonksiyonu, onun Laplace dönüşümü, beklenen değer ve varyansı hesaplanmıştır. Ek olarak yüksek mertebeden momentler için bir formül verilmiştir.

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