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STRUCTURE OF m-DIMENSIONAL IMPLICITLY DEFINED SURFACES IN n-DIMENSIONAL EUCLIDEAN SPACE E_n

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Abstract

We consider the structure of the surface in the given point, if we vary all its normals in this point.

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1. Surfaces $F_m \subset E_n$

We consider a C^2-regular m-dimensional surface F_m in n-dimensional Euclidean space E_n. Let its equations be

$$\bar{r}(x_1, \dots, x_m) = \{f_1(x_1, \dots, x_m); \dots; f_n(x_1, \dots, x_m)\}, \tag{1}$$

where $x_i, i = 1, \dots, m$, are Decartes coordinates in E_n.

We take unit ortogonal basis of normals for F_m as

$$\bar{n}_k = \{d^1_{|k}; \dots; d^n_{|k}\} \quad k = 1, \dots, n-m.$$

Let the arbitrary normal vector in this basis be

$$\bar{N} = \{\alpha^k d^1_{|k}; \dots; \alpha^k d^n_{|k}\}. \tag{2}$$

The coefficients of the second quadratic form have relatively arbitrary unit normals:

$$b_{ij}(\bar{n}) = \frac{1}{|\bar{N}|} \sum_{l=1}^{m} \alpha^k d^l_{|k} f^l_{ij}, \quad \bar{n} = \frac{\bar{N}}{|\bar{N}|}, \tag{3}$$

where f_{ij}^l denotes the second partial derivative of the function f_l with respect to x_i and x_j.

Later, we use external curvatures of F_m relatively unit normals

$$K_e(\bar{n}) = \frac{det(b_{ij}(\bar{n}))}{det(g_{ij})}.$$

2. Structural theorem

We shall prove the following theorem.

Structural Theorem. Let F_m be a C^2-regular, m-dimensional, implicitly defined nongenerate [1] surface in $E_n, n \geq 4$. Then at every point of F_m we have the following characteristics:

a) There exist at most m hiperplanes $\prod_0^i$ in the normal space $\prod$ of F_m such that external curvatures $K_e(\bar{n}) = 0$ if $\bar{n} \in \prod_0^i$.

b) F_m is k-**saddle** relatively normal $\bar{n}$, if $\bar{n}$ lies in any quadrant $\prod_k$ between k positive half-spaces $^+\prod_0^i$ and (m-k) negative half-spaces $^-\prod_0^i \subset \prod$. It denotes $\bar{n} \in \prod_k = (^+\prod_0^1; \ldots ;^- \prod_0^{k+1}; \ldots ;^- \prod_0^m)$. All this means that $\forall \bar{n} \in \prod_k$ the second quadratic form $II(\bar{n})$ relatively $\bar{n}$ with coefficients (3) has canonical form

$$II(\bar{n}) = c_{11}(\bar{n})dv_1^2 + \ldots + c_{kk}(\bar{n})dv_k^2 - c_{k+1,k+1}(\bar{n})dv_k^2 - \ldots - c_{mm}(\bar{n})dv_m^2. \quad (4)$$

External curvates $K_e(\bar{n}) \neq 0$ if $\bar{n} \in \prod_k$.

c) F_m is **convex** relatively normal $\bar{n}$, if $\bar{n}$ lies in quadrant $\prod_m$ or symmetric quadrant $\prod_{-m}$, i. e. if $\bar{n} \in \prod_{\pm m}$ that all coefficients $c_{ii}(\bar{n})$ in canonical form for $II(\bar{n})$ have the same signs.

d) If normal $\bar{n} \in \prod_0^1 \cap \prod_0^2 \cap \ldots \cap \prod_0^m$, then form $II(\bar{n}) \equiv 0$, and external curvature $K_e(\bar{n}) \equiv 0$. Surface F_m is planar relatively this $\bar{n}$.

e) If $\bar{n} \in \prod_0^1 \cap \prod_0^2 \cap \ldots \cap \prod_0^s$, then canonical for $II(\bar{n})$ has m-s terms. External curvature $K_e(\bar{n}) = 0$. Surface F_m is cilinder with s-dimensional generators.

Proof. It is in general similar to the situation with nonparametrizied surface F_m [2]. The second quadratic form $II(\bar{n})$ relatively $\bar{n}$ for C^2-regular surface $F_m \subset E_n$ in general case is □

$$II(\bar{n}) = b_{ij}(\bar{n})du^i du^j.$$

We observe that external curvature $K_e(\bar{n})$ is dicsriminant of the form $II(\bar{n})$ but with coefficients $\frac{1}{g}$ (g is discriminant of metric quadratic form for F_m). In fact, this discriminant for $II(\bar{n})$ is

$$\Delta(\bar{n}) = \begin{vmatrix} b_{11}(\bar{n}) & \dots & b_{1m}(\bar{n}) \\ \multicolumn{3}{c}{\dotfill} \\ b_{m1}(\bar{n}) & \dots & b_{mm}(\bar{n}) \end{vmatrix} = gK_e(\bar{n}).$$

We substitute (3) and get

$$K_e(\bar{n}) = \frac{1}{g \mid \bar{N} \mid^m} \begin{vmatrix} \sum_{l=1}^n \alpha^k d^l_{|k} f^l_{11} & \dots & \sum_{l=1}^n \alpha^k d^l_{|k} f^l_{lm} \\ \multicolumn{3}{c}{\dotfill} \\ \sum_{l=1}^n \alpha^k d^l_{|k} f^l_{ml} & \dots & \sum_{l=1}^n \alpha^k d^l_{|k} f^l_{mm} \end{vmatrix}. \quad (5)$$

Now we fix an arbitrary point M_0 on the surface F_m. Then $f^l_{ij}(M_0)$ are the constants. If the basis $\bar{n}_k$ is fixed in M_0, then the coordinates $d^l_{|k}(M_0)$ are the constants too. These multipliers are the coefficients for α^k in (5).

We decompose determinant via sum of the determinants, in which every column is one product $\alpha^p d^l_{|p} f^l_{ij}$. We denote these determinants $D_q, q = 1, \dots, (n-m)^n n^m$. We can carry out the factors α^p in every column of D_q over the D_q.

Now we perform Gauss method to (5) and bring it to diagonal form. It is known that such transformations not change the value of any determinant. Hence the value of external curvature $K_e(\bar{n})$ does not change as well.

As a result we obtain that determinant (5) has the form (after that all coefficients α^k we insert again under D_q, and we shall add all D_q)

$$K_e(\bar{n}) = \frac{1}{g \mid \bar{N} \mid^m} \begin{vmatrix} \sum_{k=1}^n \alpha^k \varphi^k_{11} & \dots & 0 \\ \multicolumn{3}{c}{\dotfill} \\ 0 & \dots & \sum_{k=1}^n \alpha^k \varphi^k_{mm} \end{vmatrix}. \quad (6)$$

Here, φ^k_{ii} are the contants in the fixed point M_0. The coefficients α^k in Egn. (5) remain unchanged.

At the same the diagonal form (6) corresponds to the reduction of the second quadratic form $II(\bar{n})$ to canonical form.

Now we shall vary the normal $\bar{N}$ in the whole normal space $\prod$ of the point M_0 on F_m, e. we shall arbitrary vary the coefficients α^k.

Then every sum in (6)

$$\widetilde{c}_{ii}\ (\bar{N}) = \sum_{k=1}^{n} \alpha^k \varphi_{ii}^k \tag{7}$$

can becomes zero.

But every equation

$$\alpha^1 \varphi_{11}^1 + \ldots + \alpha^n \varphi_{11}^n = 0 \tag{8}$$

is the equation of a hyperplane $\prod_0^i$ in normal space $\prod$. If normal $\bar{n} \in \prod_0^i$, then external curvature $K_e(\bar{n}) = 0$.

Every hyperplane $\prod_0^i$ divides the normal space $\prod$ on the positive half-space $^+\prod_0^i$ and negative half-space $^-\prod_0^i$. In the quadrants between all hyperplanes $\prod_0^i$ of point M_0, on which they $(\prod_0^i)$ divide $\prod$, we have the following properties:

1) Consider $\bar{n}$ in the quadrants between all positive half-spaces $^+\prod_0^i$. Denote this by $\bar{n} \in (^+\prod_0^i; \ldots ;^+\prod_0^m)$. Then the second quadratic form relatively $\bar{n}$ has canonical form $II(\bar{n}) = c_{11}(\bar{n})dv_1^2 + \ldots + c_{mm}(\bar{n})dv_m^2$, where all $c_{ii}(\bar{n}) > 0$. The surface F_m is **convex** in this quadrant. Here external curvatures $K_e(\bar{n}) > 0$ for every $\bar{n}$.

2) The symmetric quadrant, which contains all negative half-spaces $^-\prod_0^i$, we denote by $-\bar{n} \in (^-\prod_0^1; \ldots ;^-\prod_0^m)$. Canonical second quadratic form is $II(-\bar{n}) = -c_{11}(\bar{n})dv_1^2 - \ldots - c_{mm}(\bar{n})dv_m^2$. All its coefficients have the same signs. The surface F_m is also convex in this quadrant. Moreover, the external curvature, $K(-\bar{n})$, remains positive for even m and it change sign to negative for odd m.

3) Consider quadrants $\prod_k$ where we take k half-spaces $^+\prod_0^i$. We write this as $\bar{n} \in (^+\prod_0^1; \ldots ;^+\prod_0^k ;^-\prod_0^{k+1}; \ldots ;^-\prod_0^m)$. Here, the canonical second quadratic form is $II(-\bar{n}) = c_{11}(\bar{n})dv_1^2 + \ldots + c_{kk}(\bar{n})dv_k^2 - c_{k+1,k+1}(\bar{n})dv_{k+1}^2 - \ldots - c_{mm}(\bar{n})dv_m^2$. The surface F_m with that canonical second quadratic form we shall call k-**saddle**. External curvatures $K_e(\bar{n}) \neq 0$ in $\prod_k$.

4) Symmetric quadrant to $\prod_k$ we denote $\prod_{-k}$. The type of F_m in $\prod_{-k}$ changes from k-saddle to (m-k)-saddle.

5) If the normal $\bar{n} \in \prod_0^1 \cap \prod_1^2 \cap \ldots \cap \prod_0^m$ then the second quadratic form $II(\bar{n}) \equiv 0$. External curvature $K_e(\bar{n}) \equiv 0$. The surface F_m is planar relatively that normals $\bar{n}$.

6) For the normals $\bar{n} \in \prod_0^1 \cap \prod_0^2 \cap \ldots \cap \prod_0^s$ canonical second quadratic form $II(\bar{n})$ has $(m-s)$ terms. External curvature $K_e(\bar{n}) = 0$, the surface F_m is a cylinder with s-dimensional generators.

The theorem is proved.

3. Corollaries of structural theorem

We shall reduce separately two special cases of our structural theorem.

Corollary 1. *Let us given C^2-regular, implicitly defined, nongenerate surface $F_2 \subset E_n, n > 4$. Then in every point of F_2 we have:*

a) *Exist at most two hyperplanes $\prod_0^i$ in the normal space $\prod$ of F_2 such that external curvatures $K_e(\bar{n}) = 0$ if $\bar{n} \in \prod_0^i$.*

b) *F_2 is* **convex** *in the positive space between hyperplanes $\prod_0^i$ in the normal space $\prod$, i. e. if $\bar{n} \in ({}^+\prod_0^1, {}^+\prod_0^2)$. External curvatures $K_e(\bar{n}) > 0$ relatively that $\bar{n}$. In the symmetric space in $\prod$ too.*

c) *F_2* **is saddle** *in the complementary domain between hyperplanes $\prod_0^i$ in $\prod$, i. e. if $\bar{n} \in ({}^+\prod_0^1; -\prod_0^2)$. External curvatures $K_e(\bar{n}) < 0$ relatively this $\bar{n}$. In the symmetric space in $\prod$ too.*

d) *If $\bar{n} \in \prod_0^1 \cap \prod_0^2$ then F_2 is planar, and $K_e(\bar{n}) \equiv 0$.*

Corollary 2. *Let us given C^2-regular, implicitly defined, nongenerate surface $F_2 \subset E_4$. Then in every point of F_2 we have:*

a) *Exist at most two normals $\bar{n}^1$ and $\bar{n}^2$ in the normal plane $\prod$ of F_2 such that external curvatures $K_e(\bar{n}^1) = 0, K_e(\bar{n}^2) = 0$;*

b) *Exist two symmetric domains in $\prod$ between the directions of the vectors $\bar{n}^1$ and $\bar{n}^2$, in which the surface F_2 is* **convex**. *External curvatures $K_e(\bar{n} > 0$ there.*

c) *In the other two symmetric domains in $\prod$ the surface F_2 is* **saddle**. *External curvatures $K_e(\bar{n}) < 0$ there.*

Thus structure of the surfaces $F_2 \subset E_n, \geq 4$, in the different domains in the normal space $\prod$ is the same as the local structure of the regular surfaces in E_3. They are characterized the same signs of external curvatures like in E_3.

Corollary 3. *Regular surfaces in E_3 locally are the cuts of 2-dimensional regular surface of E_4 in any point along various directions of normals, there are convex, saddle or cylinder surfaces.*

Remark. *Structural theorem is generalization to the surfaces $F_m \subset E_n$ of the property of C^2-regular curve $\bar{r} = \bar{r}(s)$ from E_3: its curvatures $k(\bar{n}) = (\bar{r}''(\bar{n}), \bar{n})$ relatively normals $\bar{n}$ vary in the interval with ends $\pm k = \pm \mid \bar{r}''(s) \mid$, they are analogs of coefficients $b_{ij}(\bar{n})$. Therefore curvatures $k(\bar{n})$ have one "null" direction $\bar{n}^0$ in every point of the curve.*

Structural theorem holds for the surfaces F_m from the space R_n of constant curvature. It is obtained with help of the Cayley-Klein map for R_n which is a geodesic map.

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