Robust Stabilization for Nonlinear Differential Inclusion Systems Subject to Disturbances

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Abstract The stabilization problem of nonlinear differential inclusion systems with disturbances is dealt with in this paper. First, based on the convex hull Lyapunov function approach, a continuous state feedback law is designed, which can globally asymptotically stabilize this kind of system without disturbances. Secondly, by the state feedback, the reachable set under two classes of bounded disturbances are achieved simultaneously. Finally, a numerical example is used to illustrate the effectiveness of the proposed design technique.

Key words nonlinear differential inclusion, stabilization, convex hull Lyapunov functions, disturbance rejection

The control of the differential inclusion (DI) systems is extensively studied recently because of its practical and theoretical significance [1-6]. The DI systems are considered a generalization of the system of differential equations. Many practical systems can be described by DI systems [7-13]. In [7], the asymptotic stability of linear differential inclusion (LDI) systems is investigated, especially, some specific families of LDI systems, such as polytopic LDI systems, norm-bound LDI systems, Diagonal norm-bound LDI systems, were studied extensively. For LDI systems, the convex hull Lyapunov function (CHLF) was proposed [8] since it was difficult to construct a common Lyapunov function for a set-valued function. In [9], a nonlinear control design method for LDI systems was presented by using the CHLF. Stability and performance for saturated systems were analyzed via the CHLF in [10]. A saturated control design method for LDI systems was obtained by the CHLF in [11]. In [12], a frequency-domain approach is proposed to analyze the globally asymptotic stability of DI systems with discrete and distributed time-delays. Sliding mode control for polytopic DI systems was studied in [13]. The CHLF is also a powerful tool to deal with nonlinear differential inclusion (NDI) systems. In this paper, we study a class of NDI system based on the CHLF. Moreover the NDI system of our interest can include numerous nonlinear systems, for example, [14-16]. To the best of the authors' knowledge, this paper is the first to provide sufficient conditions and a state feedback law to stabilize such kind of NDI systems.

The rest of the paper is organized as follows: Section 2 gives the preliminaries of the paper which includes the description of the problem and a necessary lemma. In Section 3, a continuous state feedback law is designed based on the CHLF, which can globally asymptotically stabilize this kind of system without disturbances. By the state feedback, the reachable set under two classes of bounded disturbances are achieved simultaneously. Section 4 uses an example to illustrate the effectiveness of the proposed method. Section 5 concludes the paper.

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1 System description and preliminaries

Consider the following NDI system with disturbances

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \boldsymbol{y}(t) \end{bmatrix} \in co \left\{ \begin{bmatrix} A_i\boldsymbol{x}(t) + B_i\boldsymbol{u}(t) + T_i\boldsymbol{\omega}(t) + \\ G_i\boldsymbol{\gamma}_i(\boldsymbol{x}(t)) + \boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t)), \\ h_i(\boldsymbol{x}(t)) \end{bmatrix} \right\}_{i=1,2,\cdots N} \tag{1}$$

where co denotes the convex hull of a set and $\boldsymbol{x}(t) \in \mathbf{R}^n$, $\boldsymbol{u}(t) \in \mathbf{R}^m$ and $\boldsymbol{\omega}(t) \in \mathbf{R}^r$ are the state, the input and the unmeasurable disturbance respectively. $\boldsymbol{y}(t) \in \mathbf{R}^q$ is the output. $A_i \in \mathbf{R}^{n\times n}, B_i \in \mathbf{R}^{n\times m}, T_i \in \mathbf{R}^{n\times r}$ and $G_i \in \mathbf{R}^{n\times p}$ are given real matrices. $\boldsymbol{\gamma}_i : \mathbf{R}^n \to \mathbf{R}^p$ with $\boldsymbol{\gamma}_i(\mathbf{0}) = \mathbf{0}$, $\boldsymbol{\varphi}_i : \mathbf{R}^q \times \mathbf{R}^m \to \mathbf{R}^n$ and $\boldsymbol{h}_i : \mathbf{R}^n \to \mathbf{R}^q$ are continuous.

A positive-definite (semidefinite) matrix P is denoted as $P > 0$ ($P \geq 0$). When we say positive-definite (semidefinite), it is implied that the matrix is symmetric. Let $P \in \mathbf{R}^{n\times n}$, $P > 0$, and a $\rho \in (0, \infty)$. Then denote a subset of $\mathbf{R}^n$ as follows

$$\varepsilon(P, \rho) = \{\boldsymbol{x} \in \mathbf{R}^n : \boldsymbol{x}^{\mathrm{T}} P\boldsymbol{x} \leq \rho\}.$$

If $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x}$, a level set of $V(.)$, denoted as $L_V(\rho)$, is defined as

$$L_V(\rho) = \{\boldsymbol{x} \in \mathbf{R}^n : V(\boldsymbol{x}) \leq \rho\} = \varepsilon(P, \rho).$$

The CHLF is constructed from a family of positive definite matrices. Let $Q_j \in \mathbf{R}^{n\times n}, Q_j = Q_j^{\mathrm{T}} > 0, j = 1, 2, \cdots J$, and

$$S^J = \{s : | \, s = (s_1, s_2, \cdots, s_J) : s_1+s_2+\cdots+s_J = 1, s_j \geq 0\}.$$

Then the CHLF is defined as

$$V_c(\boldsymbol{x}) = \min_{s\in S^J} \boldsymbol{x}^{\mathrm{T}} (\sum_{j=1}^{J} s_jQ_j)^{-1}\boldsymbol{x}. \tag{2}$$

It is obvious that $V_c(\boldsymbol{x})$ is a positive definite function. From the definition of $V_c(\boldsymbol{x})$, we have

$$V_c(\boldsymbol{x}) = \min\{\alpha : \alpha \geq \boldsymbol{x}^{\mathrm{T}}(\sum_{j=1}^{J} s_jQ_j)^{-1}\boldsymbol{x}, \\ (s_1, s_2, \cdots, s_J) \in S^J\}.$$

By the Schur complement, $V_c(\boldsymbol{x})$ and the optimal value of s can be computed by solving a linear matrix inequality constraint

$$\begin{aligned} V_c(\boldsymbol{x}) = &\min_{s_1,\cdots s_j} \alpha \\ s.t. &\begin{bmatrix} \alpha & \boldsymbol{x}^{\mathrm{T}} \\ \boldsymbol{x} & \sum\limits_{j=1}^{N} s_jQ_j \end{bmatrix} \geq 0, \\ &\sum_{j=1}^{N} s_j = 1, \quad s_j \geq 0 \end{aligned}$$

which is an optimization problem and can be easily solved with the techniques presented in [7].

Define a function $s^*(\boldsymbol{x})$ as follows

$$s^*(\boldsymbol{x}) = \arg\min_{s\in S^J} \boldsymbol{x}^{\mathrm{T}}(\sum_{j=1}^{J} s_jQ_j)^{-1}\boldsymbol{x}. \tag{3}$$

We see that for a given $\boldsymbol{x}$ the optimal value of s is $s^*(\boldsymbol{x})$ such that $V_c(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}(\sum\limits_{j=1}^{J} s_j^*Q_j)^{-1}\boldsymbol{x}$. Generally, s^* is uniquely determined by x and is a continuous function of $\boldsymbol{x}$ except for some degenerated cases.

The following Lemma 1 was given in [9].

Lemma 1. Let $\boldsymbol{x} \in \mathrm{R}^n$. For simplicity and without loss of generality, assume that $s_k^*(\boldsymbol{x}) > 0$ for $k = 1, 2, \cdots J_0$ and $s_k^*(\boldsymbol{x}) = 0$ for $k = J_0 + 1, \cdots J$. Denote $E_k = \{\boldsymbol{x} : V_c(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} Q_k^{-1} \boldsymbol{x} = 1\}$ and

$$Q(s^*) = \sum_{k=1}^{J_0} s_k^* Q_k, \quad \boldsymbol{x}_k = Q_k Q(s^*)^{-1} \boldsymbol{x}, k = 1, 2, \cdots J_0.$$

Then $V_c(\boldsymbol{x}_k) = V_c(\boldsymbol{x}) = \boldsymbol{x}_k^{\mathrm{T}} Q_k^{-1} \boldsymbol{x}_k$ and $\boldsymbol{x}_k \in (V_c(\boldsymbol{x}))^{1/2} E_k$, for $k = 1, 2, \cdots J_0$. Moreover, $\boldsymbol{x} = \sum\limits_{k=1}^{J_0} s_k^* \boldsymbol{x}_k$, and for $k = 1, 2, \cdots J_0$,

$$\nabla V_c(\boldsymbol{x})^{\mathrm{T}} = \nabla V_c(\boldsymbol{x}_k)^{\mathrm{T}} = 2 Q_k^{-1} \boldsymbol{x}_k = 2 Q(s^*)^{-1} \boldsymbol{x}, \qquad (4)$$

where $\nabla V_c(\boldsymbol{x})$ denotes the gradient of V_c at $\boldsymbol{x}$.

2 Main results

This section begins with the problem of stabilization. For this objective, we only consider NDI systems without disturbances, *i.e.,*

$$\left[\begin{array}{c} \dot{\boldsymbol{x}}(t) \\ \boldsymbol{y}(t) \end{array}\right] \in co \left\{ \left[\begin{array}{c} A_i \boldsymbol{x}(t) + B_i \boldsymbol{u}(t) + G_i \boldsymbol{\gamma}_i(\boldsymbol{x}(t)) \\ +\boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t)), \\ \boldsymbol{h}_i(\boldsymbol{x}(t)) \end{array}\right] \right\} \qquad (5)$$
$$i = 1, 2, \cdots N$$

Moreover, we assume that there exist $\alpha_i > 0$, $\beta_i > 0$, $\lambda_i > 0$, $i = 1, 2, \cdots, N$ such that

$$\begin{array}{ll} \|\boldsymbol{\gamma}_i(\boldsymbol{x}(t))\| \le \alpha_i \|\boldsymbol{x}(t)\|, & \boldsymbol{x}(t) \in \mathrm{R}^n, \\ \|\varphi_i(\boldsymbol{y}(t), \boldsymbol{u}(t))\| \le \beta_i \|\boldsymbol{y}(t)\|, & \boldsymbol{y}(t) \in \mathrm{R}^q, \boldsymbol{u}(t) \in \mathrm{R}^m, \\ \|\boldsymbol{h}_i(\boldsymbol{x}(t))\| \le \lambda_i \|\boldsymbol{x}(t)\|, & \boldsymbol{x}(t) \in \mathrm{R}^n. \end{array} \qquad (6)$$

Theorem 1. Suppose the system (5) satisfies condition (6). Let $Q_k \in \mathbf{R}^{n \times n}, k = 1, 2, \cdots, J$ be positive definite matrices, $V_c(\boldsymbol{x})$ be the function defined in (2). If there exist matrices $Y_k \in \mathbf{R}^{m \times n}$, and $\mu_k > 0, k = 1, 2, \cdots, J$ such that the following inequalities hold

$$\begin{array}{l} A_i Q_k + Q_k A_i^{\mathrm{T}} - B_i Y_k - Y_k^{\mathrm{T}} B_i^{\mathrm{T}} + 2I + J(\|G_i\|^2 \alpha_i^2 \\ +\beta_i^2 \max\limits_i \lambda_i^2) Q_k^2 + \mu_k Q_k \le 0, \\ i = 1, 2, \cdots, N \end{array} \qquad (7)$$

where I is the $n \times n$ identity matrix.
Denote

$$\begin{array}{l} Y(s^*) = \sum\limits_{k=1}^{J} s_k^* Y_k, \ Q(s^*) = \sum\limits_{k=1}^{J} s_k^* Q_k, \\ F(s^*) = Y(s^*) Q(s^*)^{-1} \end{array} \qquad (8)$$

where $s^*(\boldsymbol{x})$ is the function defined in (3). Then the system (5) under the control law

$$\boldsymbol{u}(t) = -F(s^*) \boldsymbol{x}(t) \qquad (9)$$

is globally asymptotically stable. Moreover, if $s^*(\boldsymbol{x}(t))$ is continuous then the function defined in (8) is continuous too.

Proof. We consider an arbitrary $\boldsymbol{x}(t) \in \mathbf{R}^n$. By Lemma 1, $\boldsymbol{x}(t)$ is a convex combination of a set of $\boldsymbol{x}_k^{,}s$, each of which belongs to a certain $\boldsymbol{x}_k \in (V_c(\boldsymbol{x}))^{1/2} E_k$. For simplicity and without loss of generality, assume that $s_k^*(\boldsymbol{x}) > 0$ for $k = 1, 2, \cdots J_0$ and $s_k^*(\boldsymbol{x}) = 0$ for $k = J_0 + 1, \cdots J$. Then $\boldsymbol{x} = \sum\limits_{k=1}^{J_0} s_k^* \boldsymbol{x}_k$, from Lemma 1 $\nabla V_c(\boldsymbol{x}) = 2Q(s_k^*)^{-1} \boldsymbol{x}$ and $Q(s^*)^{-1} \boldsymbol{x} = Q_k^{-1} \boldsymbol{x}_k$, $k = 1, 2, \cdots J_0$, we have $F(s^*) \boldsymbol{x} = \sum\limits_{k=1}^{J_0} s_k^* Y_k Q_k^{-1} \boldsymbol{x}_k$.

Multiplying (7) from left and right by Q_k^{-1}, we have

$$\begin{array}{l} Q_k^{-1} A_i + A_i^T Q_k^{-1} - Q_k^{-1} B_i Y_k Q_k^{-1} - Q_k^{-1} Y_k^{\mathrm{T}} B_i^{\mathrm{T}} Q_k^{-1} \\ +2(Q_k^{-1})^2 + (\|G_i\|^2 \alpha_i^2 + \beta_i^2 \max\limits_i \lambda_i^2) J I + \mu_k Q_k^{-1} \le 0, \\ i = 1, 2, \cdots, N. \end{array} \qquad (10)$$

It follows that for any $\boldsymbol{x}_k \in (V_c(\boldsymbol{x}))^{1/2} E_k$,

$$\begin{array}{l} \sum\limits_{k=1}^{J_0} s_k^* \boldsymbol{x}_k^{\mathrm{T}}(t) [Q_k^{-1}(A_i - B_i Y_k Q_k^{-1}) + (A_i - B_i Y_k Q_k^{-1})^{\mathrm{T}} Q_k^{-1} \\ +2(Q_k^{-1})^2 + (\|G_i\|^2 \alpha_i^2 + \beta_i^2 \max\limits_i \lambda_i^2) J I] \boldsymbol{x}(t) \\ \le -\sum\limits_{k=1}^{J_0} \mu_k s_k^* \boldsymbol{x}_k^{\mathrm{T}}(t) Q_k^{-1} \boldsymbol{x}_k(t) \\ \le -\mu \sum\limits_{k=1}^{J_0} s_k^* \boldsymbol{x}_k^{\mathrm{T}}(t) Q_k^{-1} \boldsymbol{x}_k(t), i = 1, 2, \cdots, N, \end{array} \qquad (11)$$

where $\mu = \min\limits_{k=1,2,\cdots,J} \mu_k$.

For every $\boldsymbol{x}(t) \in \mathbf{R}^n$, and in view of (6), it can be deduced

$$\begin{array}{l} \nabla V_c(\boldsymbol{x}(t))^{\mathrm{T}}((A_i - B_i F(s^*) \boldsymbol{x}(t) + G_i \boldsymbol{\gamma}_i(\boldsymbol{x}(t)) \\ \quad +\boldsymbol{\varphi}_i(\boldsymbol{y}(t), u(t))) \\ = 2\boldsymbol{x}^{\mathrm{T}}(t) Q(s^*)^{-1} \sum\limits_{k=1}^{J_0} s_k^*(A_i \boldsymbol{x}_k(t) - B_i Y_k Q_k^{-1} \boldsymbol{x}_k(t)) \\ \quad +2\boldsymbol{x}^{\mathrm{T}}(t) Q(s^*)^{-1}(G_i \boldsymbol{\gamma}_i(\boldsymbol{x}(t)) + \boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t))) \\ \le \sum\limits_{k=1}^{J_0} 2 s_k^* \boldsymbol{x}_k^{\mathrm{T}}(t) Q_k^{-1}(A_i - B_i Y_k Q_k^{-1}) \boldsymbol{x}_k(t) \\ \quad + \sum\limits_{k=1}^{J_0} 2 s_k^* \boldsymbol{x}_k^T(t) Q_k^{-2} \boldsymbol{x}_k(t) \\ \quad +(\alpha_i^2 \|G_i\|^2 + \beta_i^2 \max\limits_i \lambda_i^2) \sum\limits_{k=1}^{J_0} J_0 (s_k^*)^2 \boldsymbol{x}_k^{\mathrm{T}}(t) \boldsymbol{x}_k(t). \end{array}$$

Since $0 < s_k^* < 1, k = 1, 2, \cdots J_0$, we have

$$\begin{array}{l} \nabla V_c(\boldsymbol{x}(t))^{\mathrm{T}}(A_i \boldsymbol{x}(t) - B_i F(s^*) \boldsymbol{x}(t) + G_i \gamma_i(\boldsymbol{x}(t)) \\ \quad +\boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t))) \\ \le \sum\limits_{k=1}^{J_0} s_k^* \boldsymbol{x}_k^{\mathrm{T}}(t) [Q_k^{-1}(A_i - B_i Y_k Q_k^{-1}) \\ \quad +(A_i - B_i Y_k Q_k^{-1})^{\mathrm{T}} Q_k^{-1} \\ \quad +2Q_k^{-2} + (\alpha_i^2 \|G_i\|^2 + \beta_i^2 \max\limits_i \lambda_i^2) J_0 I] \boldsymbol{x}_k(t) \end{array} \qquad (12)$$

Using (11) and (12), then it can be deduced that

$$\begin{array}{l} \nabla V_c(\boldsymbol{x}(t))^T (A_i \boldsymbol{x}(t) - B_i F(s^*) \boldsymbol{x}(t) + G_i \boldsymbol{\gamma}_i(\boldsymbol{x}(t)) \\ \quad +\boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t))) \\ \le -\mu \sum\limits_{k=1}^{J_0} s_k^* \boldsymbol{x}_k^{\mathrm{T}}(t) Q_k^{-1} \boldsymbol{x}_k(t) \\ = -\mu V_c(\boldsymbol{x}(t)) \end{array} \qquad (13)$$

Thus the system (5) under the control law (9) is globally asymptotically stable. Since $Y(s^*)$ and $Q(s^*)$ are continuous in s^*, and $Q(s^*) > 0$, the continuity of $\boldsymbol{u} = -F(s^*) \boldsymbol{x}(t)$ follows from that of $s^*(\boldsymbol{x})$. This completes the proof.□

Let us consider the system (1) is suffering from the unit energy disturbances, *i.e.,*

$$\|\boldsymbol{\omega}(t)\|_2 = \left(\int_0^\infty \boldsymbol{\omega}^{\mathrm{T}}(t) \boldsymbol{\omega}(t) dt \right)^{1/2} \le 1.$$

The control design objective is disturbance rejection, *i.e.,* to keep the state close to the origin. The disturbance rejection performance can be characterized by the reachable set. The level set $L_{V_c}(1)$ can be considered as an estimate for the reachable set.

Theorem 2. Suppose the system (1) satisfies condition (6). Let $Q_k \in \mathbf{R}^{n\times n}, k=1,2,\cdots,J$ be positive definite matrices, $V_c(\boldsymbol{x})$ be the function defined in (2). If there exist matrices $Y_k \in \mathbf{R}^{m\times n}$ such that

$$\left[\begin{array}{cc} M_{ik} & T_i \\ T_i^{\mathrm{T}} & -I \end{array}\right] \le 0, i=1,2,\cdots,N; k=1,2,\cdots,J \tag{14}$$

where

$$\begin{aligned} M_{ik} = & A_iQ_k + Q_kA_i^{\mathrm{T}} - B_iY_k - Y_k^{\mathrm{T}}B_i^{\mathrm{T}} + 2I \\ & + J(\|G_i\|^2\alpha_i^2 + \beta_i^2 \max_i \lambda_i^2)Q_k^2 \end{aligned} \tag{15}$$

then for all $\boldsymbol{\omega}(t)$ bounded by $\|\boldsymbol{\omega}(t)\|_2 \le 1$ and initial condition $\boldsymbol{x}_0 = \mathbf{0}$, the solutions of the closed-loop system (1) and (9) satisfy $\boldsymbol{x}(t) \in L_{V_c}(1) = \{x \in R^n : V(x) \le 1\}$ for all $t > 0$.

Proof. Multiplying (14) from left and right by $diag\{Q_k^{-1}, I\}$, we have

$$\left[\begin{array}{cc} Q_k^{-1}M_{ik}Q_k^{-1} & Q_k^{-1}T_i \\ T_i^{\mathrm{T}}Q_k^{-1} & -I \end{array}\right] \le 0 \tag{16}$$

By (15) and (16), for any $\boldsymbol{x}_k(t) \in (V_c(\boldsymbol{x}))^{1/2}E_k, \boldsymbol{\omega}(t) \in \mathbf{R}^r$, the following inequality holds

$$\begin{aligned} & 2\boldsymbol{x}_k^{\mathrm{T}}(t)Q_k^{-1}((A_i - B_iY_kQ_k^{-1})\boldsymbol{x}_k(t) + T_i\omega(t)) \\ & + \boldsymbol{x}_k^{\mathrm{T}}(t)(2Q_k^{-2} + JI(\|G_i\|^2\alpha_i^2 \\ & + \beta_i^2 \max_i \lambda_i^2))\boldsymbol{x}_k(t) - \boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \le 0 \end{aligned} \tag{17}$$

Similar to the proof of Theorem 1, for any $\boldsymbol{x}(t) \in \mathbf{R}^n, \boldsymbol{\omega}(t) \in \mathbf{R}^r$, we have

$$\begin{aligned} & \nabla V_c(\boldsymbol{x}(t))^{\mathrm{T}}(A_i\boldsymbol{x}(t) - B_iF(s^*)\boldsymbol{x}(t) \\ & \quad + T_i(\boldsymbol{\omega}(t)) + G_i\boldsymbol{\gamma}_i(\boldsymbol{x}(t)) + \boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t))) \\ & \le \sum_{k=1}^{J_0} 2s_k^*\boldsymbol{x}_k^{\mathrm{T}}(t)Q_k^{-1}((A_i - B_iY_kQ_k^{-1})\boldsymbol{x}_k(t) + T_i\boldsymbol{\omega}(t)) \\ & \quad + \sum_{k=1}^{J_0} s_k^*\boldsymbol{x}_k^{\mathrm{T}}(t)(2Q_k^{-2} + J_0I(\alpha_i^2\|G_i\|^2 + \beta_i^2 \max_i \lambda_i^2))\boldsymbol{x}_k(t) \end{aligned} \tag{18}$$

where $\boldsymbol{x}_k(t) \in (V_c(\boldsymbol{x}))^{1/2}E_k$.

By (17),(18), the derivative of $V_c(\boldsymbol{x}(t))$ along with the trajectories of the closed-loop system (1) and (9) holds

$$\begin{aligned} & \dot{V}_c(\boldsymbol{x}(t), \boldsymbol{\omega}(t)) \in co\{\nabla V_c(\boldsymbol{x}(t))^{\mathrm{T}}(A_i\boldsymbol{x}(t) - B_iF(s^*)\boldsymbol{x}(t) \\ & \quad + T_i(\boldsymbol{\omega}(t)) + G_i\boldsymbol{\gamma}_i(\boldsymbol{x}(t)) + \boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t)))\} \\ & \le \boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \end{aligned} \tag{19}$$

Now, suppose $\boldsymbol{x}_0 = \mathbf{0}$ and $\|\boldsymbol{\omega}(t)\|_2 \le 1$. Then by integrating both sides, we have $V_c(\boldsymbol{x}(t)) \le \int_0^\infty \boldsymbol{\omega}(t)^{\mathrm{T}}\boldsymbol{\omega}(t)dt \le 1$ and hence $\boldsymbol{x}(t) \in L_{V_c}(1) = \{x \in R^n : V(x) \le 1\}$ for all $t > 0$. This completes the proof.□

Similar to the unit energy disturbance case, the following considers the bounded disturbances, *i.e.,* there exists $D > 0$, such that

$$\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \le D, \forall t \ge 0 \tag{20}$$

Theorem 3. Suppose the system (1) satisfying condition (6). Let $Q_k \in \mathbf{R}^{n\times n}, k=1,2,\cdots,J$ be positive definite matrices, $V_c(\boldsymbol{x})$ be the function defined in (2). If there exist matrices $Y_k \in \mathbf{R}^{m\times n}$, and positive numbers $\beta > 0$ such that

$$\begin{aligned} & \left[\begin{array}{cc} M_{ik} + \beta Q_k & T_i \\ T_i^{\mathrm{T}} & -\beta I \end{array}\right] \le 0, \\ & i=1,2,\cdots,N; k=1,2,\cdots,J \end{aligned} \tag{21}$$

where M_{ik} is given by (15), then $L_{V_c}(\sqrt{D})$ is an invariant set, which means that all trajectories starting from $L_{V_c}(\sqrt{D})$ will stay inside for any possible disturbance satisfying $\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \le D, \forall t \ge 0$. Moreover, for all $\boldsymbol{x}_0 \in \mathbf{R}^n$ and all possible disturbances, $\boldsymbol{x}(t)$ will converge to $L_{V_c}(\sqrt{D})$.

Proof. Multiplying (21) from left and right by $diag\{Q_k^{-1}, I\}$, we have

$$\left[\begin{array}{cc} Q_k^{-1}M_{ik}Q_k^{-1} + \beta Q_k^{-1} & Q_k^{-1}T_i \\ T_i^{\mathrm{T}}Q_k^{-1} & -\beta I \end{array}\right] \le 0 \tag{22}$$

Similar to the proof of Theorem 2, the derivative of $V_c(\boldsymbol{x}(t))$ along with the trajectories of the closed-loop system (1) and (9) holds

$$\begin{aligned} & \dot{V}_c(\boldsymbol{x}(t), \boldsymbol{\omega}(t)) \in co\{\nabla V_c(\boldsymbol{x}(t))^{\mathrm{T}}(A_i\boldsymbol{x}(t) - B_iF(s^*)\boldsymbol{x}(t) \\ & \quad + T_i(\boldsymbol{\omega}(t)) + G_i\boldsymbol{\gamma}_i(x(t)) + \boldsymbol{\varphi}_i(\boldsymbol{y}(t), \boldsymbol{u}(t)))\} \\ & \le -\beta V_c(\boldsymbol{x}(t)) + \beta\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \end{aligned} \tag{23}$$

for all $\boldsymbol{x}(t) \in \mathbf{R}^n, \boldsymbol{\omega}(t) \in \mathbf{R}^r$. Since $\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \le D$, for $V_c(\boldsymbol{x}) = D$, we have $\dot{V}_c \le 0$ and V_c is non-increasing. Hence $L_{V_c}(\sqrt{D})$ is an invariant set. If $V_c(\boldsymbol{x}) > D$, then $\dot{V}_c$ is strictly decreasing and every trajectory starting from outside of $L_{V_c}(\sqrt{D})$ converges to $L_{V_c}(\sqrt{D})$. This completes the proof.□

3 Example

Consider a second-order NDI system

$$\begin{aligned} & \left[\begin{array}{c} \dot{\boldsymbol{x}}(t) \\ y(t) \end{array}\right] \in co\left\{\left[\begin{array}{l} A_i\boldsymbol{x}(t) + B_iu(t) \\ +G_i\gamma_i(\boldsymbol{x}(t)) + \boldsymbol{\varphi}_i(y(t), u(t)), \\ h_i(\boldsymbol{x}(t)) \end{array}\right]\right\} \\ & i = 1, 2 \end{aligned} \tag{24}$$

where

$$A_1 = \left[\begin{array}{cc} 0 & 20 \\ -1 & 0 \end{array}\right], \; B_1 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \; G_1 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right],$$

$$A_2 = \left[\begin{array}{cc} 1 & 20 \\ -1 & 0 \end{array}\right], B_2 = \left[\begin{array}{c} 0 \\ 1.2 \end{array}\right], \; G_2 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right],$$

$$\begin{aligned} & \gamma_1(\boldsymbol{x}(t)) = sin\boldsymbol{x}_1(t), \boldsymbol{\varphi}_1(y(t), u(t)) = [\tfrac{y(t)}{2}, 0]^{\mathrm{T}}, \\ & h_1(\boldsymbol{x}(t)) = \tfrac{\boldsymbol{x}_2(t)}{2}, \gamma_2(\boldsymbol{x}(t)) = \boldsymbol{x}_1(t)sin\boldsymbol{x}_2(t), \\ & \boldsymbol{\varphi}_2(y(t), u(t)) = [0, \tfrac{y(t)siny(t)}{2}]^{\mathrm{T}}, \\ & h_2(\boldsymbol{x}(t)) = \tfrac{\boldsymbol{x}_1(t)+\boldsymbol{x}_2(t)}{2}. \end{aligned}$$

It is easy to know

$$\begin{aligned} & \|G_1\| = \|G_2\| = 1, \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = \tfrac{1}{2}, \\ & \lambda_1 = \tfrac{1}{2}, \lambda_2 = \tfrac{\sqrt{2}}{2}. \end{aligned}$$

V_c is composed of

$$\begin{aligned} Q_1 &= \left[\begin{array}{cc} 0.4181 & -0.1025 \\ -0.1025 & 0.0834 \end{array}\right], \\ Q_2 &= \left[\begin{array}{cc} 0.4100 & -0.1216 \\ -0.1216 & 0.0818 \end{array}\right]. \end{aligned}$$

There exist $Y_1 = [0.9742\ 1.5890]$, $Y_2 = [0.7170\ 1.6755]$, $\mu_1 = 0.5$, and $\mu_2 = 0.8$, such that the condition (7) in Theorem 1 holds. For each $\boldsymbol{x}(t) \in \mathbf{R}^2$, let

$$Y(s_1^*) = s_1^* Y_1 + (1 - s_1^*)Y_2, Q(s_1^*) = s_1^* Q_1 + (1 - s_1^*)Q_2,$$
$$F(s_1^*) = Y(s_1^*)Q(s_1^*)^{-1}$$

where s_1^* is defined as

$$s_1^*(\boldsymbol{x}) = \arg\min_{s \geq 0} \boldsymbol{x}^{\mathrm{T}}(sQ_1 + (1-s)Q_2)^{-1}\boldsymbol{x}.$$

Then the closed-loop system under

$$u = -F(s_1^*)\boldsymbol{x}(t) \tag{25}$$

is globally asymptotically stable when $\boldsymbol{x}(t) \in \mathbf{R}^2$.

Figures 1 and 2 show the time response of the state trajectories $\boldsymbol{x}(t)$ under state feedback control law (25), the output trajectories $y(t)$ and the control law $u(t)$ respectively. For any initial state $\boldsymbol{x}_0 \in \mathbf{R}^2$, it has a similar simulation result.

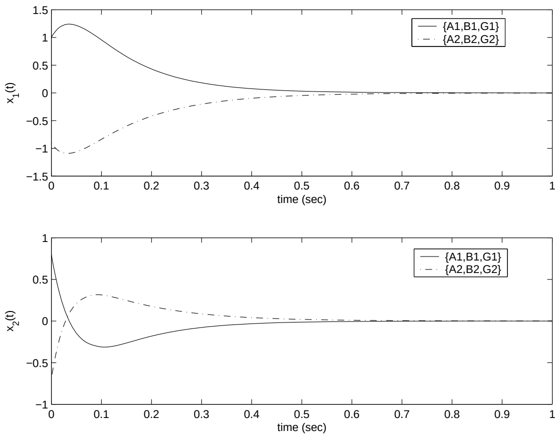

Fig.1 State trajectories $\boldsymbol{x}(t)$ of the two subsystems in (24) under state feedback control law (25)
(the initial state are $[1, 0.8]^{\mathrm{T}}, [-0.9, -0.7]^{\mathrm{T}}$, respectively)

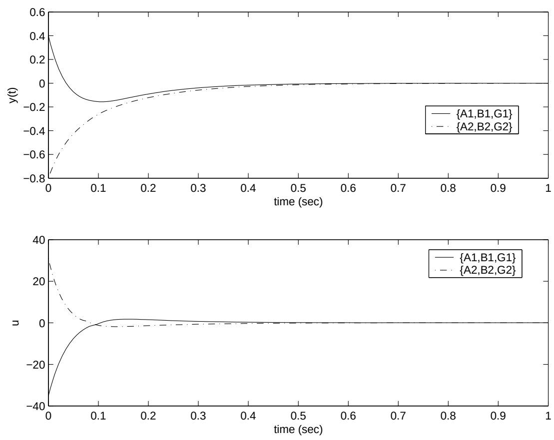

Fig.2 Output trajectories $y(t)$ and control law $u(t)$ of the two subsystems in (24), respectively

4 Conclusions

In this paper, we deal with the stabilization problem of a class of NDI system with disturbances. Based on the CHLF, a continuous state feedback law is designed at first to globally asymptotically stabilize this kind of system without disturbances. And then by the state feedback, the reachable set under two classes of bounded disturbances are estimated simultaneously. Finally, the effectiveness of the proposed method is illustrated by a simulation example.

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