# Stability Analysis of Continuous-Time Iterative Learning Control Systems with Multiple State Delays

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**Abstract** This paper presents a stability analysis of the iterative learning control (ILC) problem for continuous-time systems with multiple state delays, especially when system parameters are subject to polytopic-type uncertainties. Using the two-dimensional (2-D) analysis approach to ILC, the continuous-discrete Roesser's type linear systems are employed to describe the entire learning dynamics of time-delay systems (TDS) with the development of an expanding operator. Based on such Roesser systems, the 2-D system theory is first used to develop a necessary and sufficient condition for the asymptotic stability of ILC, and then the robust $H_\infty$ control theory is combined to provide a sufficient condition in terms of linear matrix inequalities (LMIs) for the monotonic convergence of ILC. It shows that learning gains can be determined by solving LMIs, which ensure the control input error converging monotonically to zero as a function of iteration. Simulation results show that a robust asymptotically stable ILC scheme can become robust monotonically convergent by adding the P-type learning gains that satisfy a set of LMIs, which can also improve the convergence rate greatly.

**Key words** Iterative learning control, time-delay systems, monotonic convergence, 2-D system theory, robust $H_\infty$ control theory, linear matrix inequality.

*Iterative learning control (ILC)* is an effective technique for such systems that operate repetitively over a finite time interval [1]. The key feature of ILC is to improve the control input law iteratively to achieve a perfect tracking by feeding back the control signals and the associated tracking errors from previous trials. During the iterative learning process, the reasonable transient behavior is desirable to practical applications rather than the asymptotic stability. If an ILC algorithm converges monotonically, good transients can be guaranteed [2]. This motivates the development of a class of ILC algorithms. In [3–7], the authors have considered ILC algorithms that are shown to be monotonically convergent from the time and/or frequency domain points of view. The proposed ILC algorithms in [3–7], however, have been only applied to linear or nonlinear systems without time delays.

Until now, there is only limited literature that considers ILC for time-delay systems (TDS) (e.g., [8–13]). However, TDS exhibit more complicated dynamics and provide more realistic models appropriating to the true situation, as opposed to delay-free systems [14]. Furthermore, to the best of our knowledge, there is no reference in the literature discussing the possibility of designing ILC algorithms with the monotonic convergence for TDS, especially for TDS with explicit parameter uncertainties.

In contrast to [3–7], this paper addresses uncertain TDS and makes a contribution to the ILC literature by presenting a necessary and sufficient condition for the asymptotic stability of ILC, and a sufficient condition in terms of linear matrix inequalities (LMIs) to ensure both boundedness and monotonic convergence of the control errors in the sense of the $\mathcal{L}_2$-norm. It is worth noting that all conditions derived benefit from using 2-D Roesser systems. But, different from the results of ILC based on Roesser systems (e.g., [15–17]), the result of this paper provides new insights into stability analysis of the established 2-D error systems, into which the robust $H_\infty$ control theory is incorporated (see, e.g., [18]). In addition, an expanding operator is introduced based on the delay operator, which replaces time delays in the proofs of ILC stability. Hence, time delays do not play a significant role any longer, while in contrast, the ratio of the learning time length to the unit delay plays a crucial role in establishing the 2-D error systems and implementing LMIs with the Matlab toolbox, as shown by numerical simulation.

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## 1 ILC system description

### 1.1 System description

Consider the following continuous-time system with multiple state delays, which is modeled in the 2-D form of

$$\begin{aligned}\frac{\partial \boldsymbol{x}(t,k)}{\partial t} &= \sum_{i=0}^{l} A_i \boldsymbol{x}(t-i\tau,k) + B\boldsymbol{u}(t,k)\\ \boldsymbol{y}(t,k) &= C\boldsymbol{x}(t,k)\end{aligned} \tag{1}$$

where $t \in [0,T]$ is the continuous-time index, $k \in \mathbf{Z}_+$ is the discrete-iteration index, $\boldsymbol{x}(t,k) \in \mathbf{R}^n$ is the state, $\boldsymbol{u}(t,k) \in \mathbf{R}^r$ is the control input, and $\boldsymbol{y}(t,k) \in \mathbf{R}^q$ is the output. The system matrices $A_i$, $i \in [0,l]$, $B$ and $C$ are of appropriate dimensions, and each delay takes an integral multiple of the fixed delay time $\tau$. In the following discussion, let us denote $\nabla$ as the pure delay operator, where $\nabla : \boldsymbol{\nu}(t,k) \mapsto \boldsymbol{\nu}(t-\tau,k)$ and $\boldsymbol{\nu}$ is a 2-D vector function defined on the time interval $[t-\tau,t]$ for all $k \in \mathbf{Z}_+$, and for matrices $M_i$, $i \in [0,l]$, let us denote $\mathbf{M}(\nabla) \triangleq \sum_{i=0}^{l} M_i \nabla^i$ [14]. Therefore, system (1) can be rewritten in the compact form of

$$\begin{aligned}\frac{\partial \boldsymbol{x}(t,k)}{\partial t} &= \mathbf{A}(\nabla)\boldsymbol{x}(t,k) + B\boldsymbol{u}(t,k)\\ \boldsymbol{y}(t,k) &= C\boldsymbol{x}(t,k).\end{aligned} \tag{2}$$

The following assumptions on system (1) are imposed:

A1) It is assumed that $\boldsymbol{y}_d(t)$ is a realizable desired output trajectory. That is, for any realizable trajectory and an appropriate initial function $\boldsymbol{\varphi}_d(t)$, $t \in [-l\tau, 0]$, there exists a unique control input $\boldsymbol{u}_d(t)$ such that

$$\begin{aligned}\dot{\boldsymbol{x}}_d(t) &= \mathbf{A}(\nabla)\boldsymbol{x}_d(t) + B\boldsymbol{u}_d(t)\\ \boldsymbol{y}_d(t) &= C\boldsymbol{x}_d(t)\end{aligned} \tag{3}$$

with $\boldsymbol{x}_d(t) = \boldsymbol{\varphi}_d(t), t \in [-l\tau, 0]$.

According to this assumption, let us define the sate, control input and tracking error vectors as $\delta\boldsymbol{x}(t,k) = \boldsymbol{x}_d(t) - \boldsymbol{x}(t,k)$, $\delta\boldsymbol{u}(t,k) = \boldsymbol{u}_d(t) - \boldsymbol{u}(t,k)$ and $\boldsymbol{e}(t,k) = \boldsymbol{y}_d(t) - \boldsymbol{y}(t,k)$.

A2) Assume that the standard ILC initial condition is satisfied, i.e., $\boldsymbol{x}(t,k)=\boldsymbol{\varphi}_d(t)$ for $t\in[-l\tau,0]$ and $k\in\mathbf{Z}_+$.

A consequence of A2) is that the zero state error is achieved over the initial time interval $[-l\tau,0]$, i.e., $\delta\boldsymbol{x}(t,k)=0$ for all $t\in[-l\tau,0]$ and $k\in\mathbf{Z}_+$. If this assumption is not satisfied, then ILC may be thought of as a robust problem on initial issues [9, 19]. That is, we assume that $\delta\boldsymbol{x}(t,k)\neq 0$ for all $t\in[-l\tau,0]$, and the objective would be to find the control input $\boldsymbol{u}(t,k)$ that can guarantee the stability of ILC robust against nonzero initial state errors. A possible way to solve the problem is to modify the desired trajectory at the initial stage by making an appropriate interpolation [19].

A3) Parameter Uncertainties Assumptions: It is assumed that the system matrices $A_i$, $i\in[0,l]$ and $B$ in (1) are known to lie within the uncertainty polytope $\Omega$ where

$$\Omega\triangleq\left\{(A_0,\cdots,A_l,B)\middle|(A_0,\cdots,A_l,B)=\sum_{p=1}^{N}\varsigma_p(A_{0p},\cdots,A_{lp},B_p);\varsigma_p\geq 0,\sum_{p=1}^{N}\varsigma_p=1\right\}. \tag{4}$$

### 1.2 ILC scheme description

Our objective is to design an iterative scheme to generate the control input $\boldsymbol{u}(t,k)$ such that the system output $\boldsymbol{y}(t,k)$ converges to $\boldsymbol{y}_d(t)$ and the control input $\boldsymbol{u}(t,k)$ monotonically converges to $\boldsymbol{u}_d(t)$ as $k$ goes to infinity for all $t$ within the time interval $[0,T]$.

To realize the above control objective, this paper considers a PD-type ILC implemented as follows:

- Control input:
$$\boldsymbol{u}(t,k)=\boldsymbol{\alpha}(t,k)+\boldsymbol{\beta}(t,k) \tag{5}$$
- P-type ILC law:
$$\boldsymbol{\alpha}(t,k)=\boldsymbol{\alpha}(t,k-1)+\mathbf{K}(\nabla)\delta\boldsymbol{x}(t,k-1) \tag{6}$$
with $\boldsymbol{\alpha}(t,-1)=0$ and $\delta\boldsymbol{x}(t,-1)=0$;
- D-type ILC law:
$$\boldsymbol{\beta}(t,k)=\boldsymbol{\beta}(t,k-1)+\Gamma\dot{\boldsymbol{e}}(t,k-1) \tag{7}$$
with $\boldsymbol{\beta}(t,-1)=0$ and $\dot{\boldsymbol{e}}(t,-1)=0$.

In the above scheme, $\boldsymbol{\alpha}(t,k)$ and $\boldsymbol{\beta}(t,k)$ are $r\times 1$ vectors, $\Gamma$ is an $r\times q$ matrix, $\dot{\boldsymbol{e}}(t,k)\triangleq\partial\boldsymbol{e}(t,k)/\partial t$, and $\mathbf{K}(\nabla)\triangleq\sum_{i=0}^{l}K_i\nabla^i$, where $K_i$, $i\in[0,l]$ are $r\times n$ matrices.

**Remark 1** After some simple algebraic manipulations, we can rewrite the ILC scheme (5)-(7) in the form of

$$\boldsymbol{u}(t,k)=\boldsymbol{u}(t,k-1)+\mathbf{K}(\nabla)\delta\boldsymbol{x}(t,k-1)+\Gamma\dot{\boldsymbol{e}}(t,k-1). \tag{8}$$

From (8), it is clear that if one takes $\mathbf{K}(\nabla)=0$ (the zero operator), the typical D-type ILC can be derived, which is considered for TDS by Li, Chow and Ho [10]. However, only asymptotic stability of the ILC process is achieved, and the high-overshoot can be generated [5]. In the following, it will be shown that adding the P-type ILC law (6) in the control input (5) helps to achieve monotonic convergence of the ILC process (in the sense of the control input error converging monotonically as a function of iteration) to guarantee good learning transients.

## 2 2-D analysis approach to ILC

The ILC process of (1), (5)-(7) is essentially a 2-D system because of two independent indices: time $t$ and iteration $k$, which is usually formulated in Roesser systems [10, 12]. In this section, we develop a new route to achieve the 2-D ILC dynamics formulation of TDS with Roesser systems. Then, some lemmas on both stability and robust $H_\infty$ performance related to the Roesser systems are provided.

First of all, an expanding operator is developed based on the delay operator, which is defined by

$$\omega(\nabla,j):\boldsymbol{\nu}(t,k)\mapsto\boldsymbol{\nu}^j(t,k)\triangleq\begin{bmatrix}\boldsymbol{\nu}(t,k)\\ \nabla\boldsymbol{\nu}(t,k)\\ \vdots\\ \nabla^j\boldsymbol{\nu}(t,k)\end{bmatrix}.$$

In particular, $\omega(\nabla,0)$ is obviously an identity operator.

### 2.1 2-D system representation

Iterating the ILC scheme (5)-(7) at iterations $k$ and $k+1$, we can obtain

$$\delta\boldsymbol{u}(t,k+1)=\delta\boldsymbol{u}(t,k)-\mathbf{K}(\nabla)\delta\boldsymbol{x}(t,k)-\Gamma\dot{\boldsymbol{e}}(t,k). \tag{9}$$

Subtracting (2) from (3), we have

$$\frac{\partial\delta\boldsymbol{x}(t,k)}{\partial t}=\mathbf{A}(\nabla)\delta\boldsymbol{x}(t,k)+B\delta\boldsymbol{u}(t,k). \tag{10}$$

Since $\boldsymbol{e}(t,k)=C\delta\boldsymbol{x}(t,k)$, $\dot{\boldsymbol{e}}(t,k)$ consequently satisfies

$$\dot{\boldsymbol{e}}(t,k)=C\mathbf{A}(\nabla)\delta\boldsymbol{x}(t,k)+CB\delta\boldsymbol{u}(t,k). \tag{11}$$

Inserting (11) into (9), we thus get

$$\begin{aligned}\delta\boldsymbol{u}(t,k+1)=&-\left[\mathbf{K}(\nabla)+\Gamma C\mathbf{A}(\nabla)\right]\delta\boldsymbol{x}(t,k)\\&+(I-\Gamma CB)\delta\boldsymbol{u}(t,k)\end{aligned} \tag{12}$$

where $I$ is an identity matrix of appropriate orders. To deal with the delay operator $\nabla$ in (10) and (12), we introduce an analysis strategy that considers the time interval $[0,T]$ in an ordered way of a time interval with the length of the unit delay time $\tau$, that is, let $[0,T]\triangleq[0,\tau)\cup[\tau,2\tau)\cup\cdots\cup[l_T\tau,T]$ with $l_T\tau<T$ or $[0,T]\triangleq[0,\tau)\cup[\tau,2\tau)\cup\cdots\cup[(l_T-1)\tau,T]$ with $l_T\tau=T$, where $l_T\triangleq\mathrm{int}\left(\frac{T}{\tau}\right)$ and $\mathrm{int}(\cdot)$ represents the integer part of value. Without any loss of generality, only the latter partition of the time interval $[0,T]$ is considered[1], and for the sake of the following analysis, we formulate the considered interval in a unified form of

$$[0,T]\cup(T,(l_T+1)\tau)\triangleq\bigcup_{j=0}^{l_T}[j\tau,(j+1)\tau). \tag{13}$$

Next, consider partitions $[j\tau,(j+1)\tau)$, $j\in[0,l_T]$ separately. Define two expanding vectors as $\delta\boldsymbol{x}^j(t,k)\triangleq\omega(\nabla,j)\delta\boldsymbol{x}(t,k)$ and $\delta\boldsymbol{u}^j(t,k)\triangleq\omega(\nabla,j)\delta\boldsymbol{u}(t,k)$, and therefore in a compact form, it follows from using (10) and (12) that

$$\Sigma^j:\begin{bmatrix}\dfrac{\partial\delta\boldsymbol{x}^j(t,k)}{\partial t}\\ \delta\boldsymbol{u}^j(t,k+1)\end{bmatrix}=\begin{bmatrix}A^j & B^j\\ -K^j-\Gamma^jC^jA^j & I^j-\Gamma^jC^jB^j\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{x}^j(t,k)\\ \delta\boldsymbol{u}^j(t,k)\end{bmatrix} \tag{14}$$

[1] Whatever case is discussed, we will technically extend the definitions of $\delta\boldsymbol{x}(t,k)$ and $\delta\boldsymbol{u}(t,k)$ to the time interval $(T,(l_T+1)\tau)$, since $(T,(l_T+1)\tau)$ does not work for the ILC stability analysis. In this sense, the considered time interval for both cases will end up with the same form as shown in (13).

where $A^j$, $B^j$, $C^j$, $I^j$, $K^j$ and $\Gamma^j$ are matrices of $(j+1)\times(j+1)$ blocks. If we denote $\varepsilon \triangleq |j-l|$, $A^j$ is defined by

$$A^j=\begin{cases}\begin{bmatrix}A_0 & A_1 & \cdots & A_{j-1} & A_j\\ 0 & A_0 & \ddots & \ddots & A_{j-1}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & A_0 & A_1\\ 0 & 0 & \cdots & 0 & A_0\end{bmatrix}, & j\le l\\ \begin{bmatrix}A_0 & \cdots & A_\varepsilon & \cdots & A_l & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & A_0 & \ddots & A_\varepsilon & \ddots & A_l\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & 0 & \ddots & A_0 & \ddots & A_\varepsilon\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & \cdots & 0 & \cdots & A_0\end{bmatrix}, & j>l\end{cases}$$

$K^j$ is defined in the same way with $A^j$, $B^j$ is defined by

$$B^j=\begin{bmatrix}B & 0 & \cdots & 0 & 0\\ 0 & B & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & B & 0\\ 0 & 0 & \cdots & 0 & B\end{bmatrix}$$

and $C^j$, $I^j$ and $\Gamma^j$ are defined in the same way with $B^j$.

**Remark 2** Using the expanding operator $\omega(\nabla, j)$, the 2-D system $\Sigma^j$ can be developed to disclose the entire learning dynamics of the ILC system (1), (5)-(7). In [10], a similar 2-D analysis approach has been used to address a class of D-type ILC systems with multiple state delays, which however can not clearly provide the accurate 2-D system description of ILC and only derive a vague 2-D structure using Roesser models. Moreover, the 2-D systems in [10] can not, in fact, be determined uniformly over the time interval $[0, T]$, since their dimensions vary with the time index $t$. In contrast to this, the 2-D systems $\Sigma^j$ can be accurately determined for all $j \in [0, l_T]$.

**Remark 3** Let $\boldsymbol{\chi} \in \{\delta\boldsymbol{x}, \delta\boldsymbol{u}\}$. Note that for any $k$, we only care the values of $\boldsymbol{\chi}(t,k)$ over $[0,T]^2$. Hence, we technically assume that $\boldsymbol{\chi}(t,k) = 0$ is satisfied for $t < 0$ and $k \in \mathbf{Z}_+$, since $\boldsymbol{\chi}(0,k) = 0$ holds for $k \in \mathbf{Z}_+$. From the definition of $\omega(\nabla, j)$, it is clear that

$$\boldsymbol{\chi}^j(t,k)=\begin{bmatrix}\boldsymbol{\chi}(t,k)\\ \nabla\boldsymbol{\chi}^{j-1}(t,k)\end{bmatrix}.$$

Obviously, we can derive that $\nabla\boldsymbol{\chi}^{j-1}(t,k)$ for $t \in [j\tau, (j+1)\tau)$ describes $\boldsymbol{\chi}^{j-1}(t,k)$ for $t \in [(j-1)\tau, j\tau)$. Hence, the dynamics of $\boldsymbol{\chi}^j(t,k)$ for $t \in [j\tau, (j+1)\tau)$ contain those of $\boldsymbol{\chi}^{j-1}(t,k)$ for $t \in [(j-1)\tau, j\tau)$, and consequently are equivalent to those of $\boldsymbol{\chi}(t,k)$ for $t \in [0, (j+1)\tau)$. In particular, one has that $\lim_{k\to\infty}\boldsymbol{\chi}^j(t,k) = 0$ for $t \in [j\tau, (j+1)\tau) \Leftrightarrow \lim_{k\to\infty}\boldsymbol{\chi}(t,k) = 0$ for $t \in [0, (j+1)\tau) \Leftrightarrow \lim_{k\to\infty}\boldsymbol{\chi}^j(t,k) = 0$ for $t \in [0, (j+1)\tau)$.

[2] Note that for any $k$, $\delta\boldsymbol{x}(t,k)$ is defined over $t \in [-l\tau, T]$. But, since $\delta\boldsymbol{x}(t,k) = 0$ holds for $t \in [-l\tau, 0]$ and $k \in \mathbf{Z}_+$, we actually only care $\delta\boldsymbol{x}(t,k)$ over $t \in [0,T]$.

**Remark 4** For the 2-D system $\Sigma^j$ in (14), it is developed over the time interval $[j\tau, (j+1)\tau)$. From Remark 3, it is clear that $\Sigma^j$ can still work for all $t \in [0, (j+1)\tau)$. Over the time interval $[j_0\tau, (j_0+1)\tau)$ for all $j_0 < j$ and $j_0 \in \mathbf{Z}_+$, $\Sigma^j$ is degenerated to $\Sigma^{j_0}$ in essence. Hence, the initial time of $\Sigma^j$ indicates $t = 0$ in the following discussion where no confusion arises. In particular, $\Sigma^{l_T}$ can be used to clearly describe the whole learning dynamics of the ILC scheme (5)-(7) over the time interval $[0, T]$.

### 2.2 Some lemmas

To derive the stability of 2-D Roesser systems as the only discrete index goes to infinity, the following lemma is useful and adopted from the literature.

**Lemma 1** (2-D System Theory [10]). Consider a 2-D continuous-discrete linear system described in the Roesser's state-space model of

$$\begin{bmatrix}\dfrac{\partial\boldsymbol{\psi}(t,k)}{\partial t}\\ \boldsymbol{\zeta}(t,k+1)\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}(t,k)\\ \boldsymbol{\zeta}(t,k)\end{bmatrix} \tag{15}$$

where $\boldsymbol{\psi}(t,k) \in \mathbf{R}^n$ and $\boldsymbol{\zeta}(t,k) \in \mathbf{R}^r$ are the horizontal state and the vertical state, respectively. If $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ are constant matrices of appropriate dimensions, and boundary conditions for system (15) satisfy that $\boldsymbol{\psi}(0,k) = 0$ holds for all $k \in \mathbf{Z}_+$ and $\boldsymbol{\zeta}(t,0)$ is finite for all $t \in [0,T]$, then $\lim_{k\to\infty}\left[\boldsymbol{\psi}(t,k)^{\mathrm{T}} \ \ \boldsymbol{\zeta}(t,k)^{\mathrm{T}}\right]^{\mathrm{T}} = 0$ is satisfied for all $t \in [0,T]$ if and only if (iff) the matrix $A_{22}$ is stable, that is, the spectral radius fulfills $\rho(A_{22}) < 1$, where the superscript "T" denotes the transpose of matrix.

For any fixed iteration $k$, the 2-D system (14) can also be taken as a one-dimensional (1-D) system. That is, $\delta\boldsymbol{x}^j(t,k)$ is the system state, $\delta\boldsymbol{u}^j(t,k)$ is the exogenous disturbance signal, and $\delta\boldsymbol{u}^j(t,k+1)$ is the objective function signal to be attenuated. Hence, to derive the (robust) $H_\infty$ performance of such 1-D systems, the follows lemmas are developed or directly adopted from the literature.

**Lemma 2** (Bounded Real Lemma (BRL) [18]). Consider the following system

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \mathcal{A}\boldsymbol{x}(t) + \mathcal{B}\boldsymbol{\varpi}(t)\\ \boldsymbol{z}(t) &= \mathcal{C}\boldsymbol{x}(t) + \mathcal{D}\boldsymbol{\varpi}(t), \quad \boldsymbol{x}(0) = 0\end{aligned} \tag{16}$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is the system state, $\boldsymbol{\varpi}(t) \in \mathcal{L}_2^r[0,\infty)$ is the exogenous disturbance signal, $\boldsymbol{z}(t) \in \mathbf{R}^q$ is the objective function signal to be attenuated, and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are constant matrices of appropriate dimensions. Then, system (16) is asymptotically stable and its $H_\infty$-norm is less than a prescribed scalar $\gamma > 0$ iff there exists a positive definite symmetric matrix $P > 0$ that satisfies

$$\begin{bmatrix}\mathcal{A}^{\mathrm{T}}P + P\mathcal{A} & P\mathcal{B} & \mathcal{C}^{\mathrm{T}}\\ \mathcal{B}^{\mathrm{T}}P & -\gamma I & \mathcal{D}^{\mathrm{T}}\\ \mathcal{C} & \mathcal{D} & -\gamma I\end{bmatrix} < 0. \tag{17}$$

**Lemma 3** (Improved Bounded Real Lemma (IBRL) [18]). System (16) is asymptotically stable and its $H_\infty$-norm is less than a prescribed scalar $\gamma > 0$ iff there exist a positive definite symmetric matrix $Q$, a matrix $Z$ and a sufficiently small positive scalar $\epsilon > 0$ that satisfy

$$\begin{bmatrix} Q-Z-Z^{\mathrm{T}} & Z^{\mathrm{T}}+\epsilon Z^{\mathrm{T}}\mathcal{A}^{\mathrm{T}} & 0 & Z^{\mathrm{T}}\mathcal{C}^{\mathrm{T}} \\ Z+\epsilon\mathcal{A}Z & -Q & \mathcal{B} & 0 \\ 0 & \mathcal{B}^{\mathrm{T}} & -\gamma\epsilon^{-1}I & \epsilon^{-1}\mathcal{D}^{\mathrm{T}} \\ \mathcal{C}Z & 0 & \epsilon^{-1}\mathcal{D} & -\gamma\epsilon^{-1}I \end{bmatrix} < 0. \tag{18}$$

**Lemma 4** If the system matrices of (16) are known to lie within the following uncertainty polytope

$$\widehat{\Omega} \triangleq \left\{ (\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \middle| (\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) = \sum_{p=1}^{N} \varsigma_p(\mathcal{A}_p,\mathcal{B}_p,\mathcal{C}_p,\mathcal{D}_p); \varsigma_p \geq 0, \sum_{p=1}^{N}\varsigma_p = 1 \right\}. \tag{19}$$

then the system (16) over $\widehat{\Omega}$ is robust asymptotically stable and its robust $H_\infty$-norm is less than a prescribed scalar $\gamma > 0$ iff there exist positive definite symmetric matrices $Q_p$, $p \in [1,N]$, a matrix $Z$ and a sufficiently small positive scalar $\epsilon > 0$ that satisfy

$$\begin{bmatrix} Q_p-Z-Z^{\mathrm{T}} & Z^{\mathrm{T}}+\epsilon Z^{\mathrm{T}}\mathcal{A}_p^{\mathrm{T}} & 0 & Z^{\mathrm{T}}\mathcal{C}_p^{\mathrm{T}} \\ Z+\epsilon\mathcal{A}_pZ & -Q_p & \mathcal{B}_p & 0 \\ 0 & \mathcal{B}_p^{\mathrm{T}} & -\gamma\epsilon^{-1}I & \epsilon^{-1}\mathcal{D}_p^{\mathrm{T}} \\ \mathcal{C}_pZ & 0 & \epsilon^{-1}\mathcal{D}_p & -\gamma\epsilon^{-1}I \end{bmatrix} < 0, \quad p \in [1,N]. \tag{20}$$

**Proof.** If the set of LMIs in (20) holds, then we denote $Q = \sum_{p=1}^{N}\varsigma_p Q_p$. It can be easily shown that $Q$ is a positive definite symmetric matrix and can be used as the matrix $Q$ of the LMI (18) which is required by the system matrices over $\widehat{\Omega}$. Using Lemma 3, this proof is immediate. ■

**Lemma 5** (Schur Complement [18]). Given a symmetric matrix $S$ with the form $S = [S_{ij}]$, where $S_{11} \in \mathbf{R}^{r\times r}$, $S_{12} \in \mathbf{R}^{r\times(n-r)}$, $S_{22} \in \mathbf{R}^{(n-r)\times(n-r)}$, then $S < 0$ iff $S_{11} < 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$, or, $S_{22} < 0$ and $S_{11} - S_{12}S_{22}^{-1}S_{21} < 0$.

Besides the above results of systems theory, the following property of operator $\omega(\nabla, j)$, $\forall j \in \mathbf{Z}_+$ is also useful.

**Lemma 6** Given a 2-D function $\boldsymbol{\nu}(t,k)$, if, for any $k \in \mathbf{Z}_+$, it belongs to the space of square integrable vector functions over $[0,\infty)$ and $\boldsymbol{\nu}(t,k) = 0$ holds for $t \leq 0$, then

$$\left\|\boldsymbol{\nu}^j(t,k)\right\|_2 = \sqrt{j+1}\,\|\boldsymbol{\nu}(t,k)\|_2 \tag{21}$$

where $\boldsymbol{\nu}^j(t,k) \triangleq \omega(\nabla,j)\boldsymbol{\nu}(t,k)$, $\forall j \in \mathbf{Z}_+$.

**Proof.** From the definition of $\omega(\nabla,j)$, $\forall j \in \mathbf{Z}_+$, we get

$$\begin{aligned} \left\|\boldsymbol{\nu}^j(t,k)\right\|_2^2 &= \int_0^\infty \left(\boldsymbol{\nu}^j(t,k)\right)^{\mathrm{T}} \left(\boldsymbol{\nu}^j(t,k)\right) \mathrm{d}t \\ &= \int_0^\infty \sum_{m=0}^{j} \left(\nabla^m \boldsymbol{\nu}(t,k)\right)^{\mathrm{T}} \left(\nabla^m \boldsymbol{\nu}(t,k)\right) \mathrm{d}t \\ &= \sum_{m=0}^{j} \int_0^\infty \nabla^m \left(\boldsymbol{\nu}^{\mathrm{T}}(t,k)\boldsymbol{\nu}(t,k)\right) \mathrm{d}t \\ &= \sum_{m=0}^{j} \int_0^\infty \boldsymbol{\nu}^{\mathrm{T}}(t,k)\boldsymbol{\nu}(t,k)\mathrm{d}t \\ &= (j+1)\,\|\boldsymbol{\nu}(t,k)\|_2^2 \end{aligned} \tag{22}$$

where the fact that $\boldsymbol{\nu}(t,k) = 0$ for all $t \leq 0$ and $k \in \mathbf{Z}_+$ is used. From (22), the proof of (21) is immediate. ■

## 3 Stability conditions of ILC

In this section, we consider the asymptotic stability and monotonic convergence of ILC separately.

### 3.1 Asymptotic stability

With Lemma 1 and based on the 2-D system (14), one can state the following result:

**Theorem 1** Let system (1) satisfy Assumptions A1)-A2), and ILC scheme (5)-(7) be applied. The sate error $\delta\boldsymbol{x}(t,k)$, the control input error $\delta\boldsymbol{u}(t,k)$ and the tracking error $\boldsymbol{e}(t,k)$ converge to zero as $k \to \infty$ for all $t$ within the time interval $[0,T]$ iff the learning gain matrix $\Gamma$ can be designed such that the matrix $I - \Gamma CB$ is stable.

**Proof.** Note that the 2-D system $\Sigma^{l_T}$ in (14) can be considered over the time interval $[0,T]$ (see Remark 4), which satisfies: $\delta\boldsymbol{x}^{l_T}(0,k) = 0$ for all $k \in \mathbf{Z}_+$ and finite $\delta\boldsymbol{u}^{l_T}(t,0)$ for all $t \in [0,T]$. Thus, it follows on using Lemma 1 that

$$\lim_{k\to\infty} \left[\delta\boldsymbol{x}^{l_T}(t,k)^{\mathrm{T}} \quad \delta\boldsymbol{u}^{l_T}(t,k)^{\mathrm{T}}\right]^{\mathrm{T}} = 0$$

is satisfied for all $t \in [0,T]$, and therefore (see Remark 3)

$$\lim_{k\to\infty} \left[\delta\boldsymbol{x}(t,k)^{\mathrm{T}} \quad \delta\boldsymbol{u}(t,k)^{\mathrm{T}}\right]^{\mathrm{T}} = 0$$

is satisfied for all $t \in [0,T]$ iff the matrix $I^{l_T} - \Gamma^{l_T}C^{l_T}B^{l_T}$ is stable. Since $\boldsymbol{e}(t,k) = C\delta\boldsymbol{x}(t,k)$ and $I^{l_T} - \Gamma^{l_T}C^{l_T}B^{l_T}$ is stable iff $I - \Gamma CB$ is stable, the proof is immediate. ■

**Corollary 1** Let system (1) satisfy Assumptions A1)-A3), and ILC scheme (5)-(7) be applied. The sate error $\delta\boldsymbol{x}(t,k)$, the control input error $\delta\boldsymbol{u}(t,k)$ and the tracking error $\boldsymbol{e}(t,k)$ converge to zero as $k \to \infty$ for all $t$ within the time interval $[0,T]$ if the learning gain matrix $\Gamma$ can be designed such that $\max_{p\in[1,N]} \|I - \Gamma CB_p\| < 1$[3].

**Proof.** Using the fact that $B = \sum_{p=1}^{N}\varsigma_p B_p$, we have

$$\begin{aligned} \|I - \Gamma CB\| &= \left\|I - \Gamma C\sum_{p=1}^{N}\varsigma_p B_p\right\| \\ &= \left\|\sum_{p=1}^{N}\varsigma_p\left(I - \Gamma CB_p\right)\right\| \\ &\leq \sum_{p=1}^{N}\varsigma_p \|I - \Gamma CB_p\| \\ &< \sum_{p=1}^{N}\varsigma_p \\ &= 1. \end{aligned}$$

Hence, $\rho(I - \Gamma CB) \leq \|I - \Gamma CB\| < 1$, namely, the matrix $I - \Gamma CB$ is stable. According to Theorem 1, this proof is immediately completed. ■

**Remark 5** Theorem 1 implies that the operator $\mathbf{K}(\nabla)$ has no effect on conditions for the asymptotic stability of ILC,

[3] $\|\cdot\|$ can be used by picking any kind of matrix norm such as the $\infty$-norm $\|\cdot\|_\infty$.

and such conditions are independent of the system dynamics appearing in operator $\mathbf{A}(\nabla)$. Note that the stability of $I-\Gamma CB$ implies that the matrix $CB$ has full-column rank. Thus, if gain matrices of the ILC scheme (5)-(7) is selected as $K_i=-\Gamma CA_i$, $i\in[0,l]$ and $\Gamma=\left[(CB)^{\mathrm{T}}CB\right]^{-1}(CB)^{\mathrm{T}}$, then the tracking error can be driven to zero over the whole time interval $[0,T]$ only after one learning iteration. This type of ILC design has been considered by many authors in the literature [10, 13, 16] to improve the system performance. To our knowledge, there is no reference in the literature stating that it is straightforward to derive this type of fast ILC by adding the pure error terms in the typical D-type updating law.

In Corollary 1, a sufficient condition is derived for robust asymptotic stability of ILC when the TDS in (1) are subject to polytopic-type uncertainties. It shows that checking the vertex impulse response matrices of a polytope plant is sufficient to determine the stability properties of the polytope ILC system. To our knowledge, such a robust ILC problem has never been studied by the 2-D analysis approach.

### 3.2 Monotonic convergence

Next, the (robust) monotonic convergence of time-delay ILC systems will be developed by considering Lemmas 2-6.

**Theorem 2** Let system (2) satisfy Assumptions A1)-A2), and ILC scheme (5)-(7) be applied. If there exist a positive definite symmetric matrix $P>0$ and matrices $X$ and $Y$ such that the following LMI holds with $j=l_T-1$

$$\begin{bmatrix} A^{j\mathrm{T}}P+PA^j & PB^j & X^{\mathrm{T}}+A^{j\mathrm{T}}C^{j\mathrm{T}}Y^{\mathrm{T}} \\ B^{j\mathrm{T}}P & -I^j & I^j+B^{j\mathrm{T}}C^{j\mathrm{T}}Y^{\mathrm{T}} \\ X+YC^jA^j & I^j+YC^jB^j & -I^j \end{bmatrix}<0 \tag{23}$$

then $\|\delta\boldsymbol{x}(t,k)\|_{2,[0,T]}$, $\|\delta\boldsymbol{u}(t,k)\|_{2,[0,T]}$ and $\|\boldsymbol{e}(t,k)\|_{2,[0,T]}$ are bounded for all $k\in\mathbf{Z}_+$, and $\lim_{k\to\infty}\|\delta\boldsymbol{x}(t,k)\|_{2,[0,T]}=0$, $\lim_{k\to\infty}\|\boldsymbol{e}(t,k)\|_{2,[0,T]}=0$, and $\lim_{k\to\infty}\|\delta\boldsymbol{u}(t,k)\|_{2,[0,T]}=0$ (monotonic convergence in the sense of the $\mathcal{L}_2$-norm). If the LMI (23) holds, then learning gain matrices $K_i$, $i\in[0,l]$ and $\Gamma$ are computed by

$$\begin{aligned} K_i&=-\left[I\quad 0_{r\times(l_T-1)r}\right]X\left[0_{n\times in}\quad I\quad 0_{n\times(l_T-i-1)n}\right]^{\mathrm{T}}\\ \Gamma&=-\left[I\quad 0_{r\times(l_T-1)r}\right]Y\left[I\quad 0_{q\times(l_T-1)q}\right]^{\mathrm{T}}. \end{aligned} \tag{24}$$

**Proof.** If the LMI (23) holds and learning gain matrices $K_i$, $i\in[0,l]$ and $\Gamma$ are used as in (24), it follows from using Lemma 2 that

$$\left\|G^j(s)\right\|_\infty<1 \tag{25}$$

where $G^j(s)$ is given by

$$\begin{aligned} G^j(s)&=\left[\begin{array}{c|c} A^j & B^j \\ \hline -K^j-\Gamma^jC^jA^j & I^j-\Gamma^jC^jB^j \end{array}\right]\\ &=-\left(K^j+\Gamma^jC^jA^j\right)\left(sI^j-A^j\right)^{-1}B^j\\ &\quad+\left(I^j-\Gamma^jC^jB^j\right). \end{aligned} \tag{26}$$

As a consequence of (14), it follows that

$$\delta\boldsymbol{U}^j(s,k+1)=G^j(s)\delta\boldsymbol{U}^j(s,k) \tag{27}$$

where $\delta\boldsymbol{U}^j(s,k)\triangleq\mathcal{L}\left[\delta\boldsymbol{u}^j(t,k)\right]$. Hence, using the fact that $\left\|\delta\boldsymbol{U}^j(s,k)\right\|_2=\left\|\delta\boldsymbol{u}^j(t,k)\right\|_2$, we get

$$\left\|\delta\boldsymbol{u}^j(t,k+1)\right\|_2\le\left\|G^j(s)\right\|_\infty\left\|\delta\boldsymbol{u}^j(t,k)\right\|_2. \tag{28}$$

Since Lemma 6 results in $\left\|\delta\boldsymbol{u}^j(t,k)\right\|_2=\sqrt{j+1}\,\|\delta\boldsymbol{u}(t,k)\|_2$, (28) leads to

$$\begin{aligned} \|\delta\boldsymbol{u}(t,k)\|_2&\le\left\|G^j(s)\right\|_\infty\|\delta\boldsymbol{u}(t,k-1)\|_2\\ &\le\left\|G^j(s)\right\|_\infty^k\|\delta\boldsymbol{u}(t,0)\|_2. \end{aligned} \tag{29}$$

Using $j=l_T-1$, system (14), and therefore $G^j(s)$, can work over $[0,T)$[4] for the ILC system (1), (5)-(7) (see Remark 4). Note that at the first iteration, i.e., for $k=0$, $\boldsymbol{u}(t,0)=0$ holds, and thus $\|\delta\boldsymbol{u}(t,0)\|_{2,[0,T]}=\|\boldsymbol{u}_d(t)\|_{2,[0,T]}$ is bounded. Consequently (like [6]), over the finite time interval $[0,T]$, one can conclude from (29) that $\|\delta\boldsymbol{u}(t,k)\|_{2,[0,T]}$ is bounded for all $k\in\mathbf{Z}_+$, and $\lim_{k\to\infty}\|\delta\boldsymbol{u}(t,k)\|_{2,[0,T]}=0$ (monotonic convergence in the sense of $\mathcal{L}_2$-norm), where (25) is used.

Also, we can derive from (14) that

$$\begin{aligned} \|\delta\boldsymbol{x}(t,k)\|_2&\le\left\|\left(sI^j-A^j\right)^{-1}B^j\right\|_\infty\|\delta\boldsymbol{u}(t,k)\|_2\\ &\triangleq\left\|G_1^j(s)\right\|_\infty\|\delta\boldsymbol{u}(t,k)\|_2\\ \|\boldsymbol{e}(t,k)\|_2&\le\left\|C^j\left(sI^j-A^j\right)^{-1}B^j\right\|_\infty\|\delta\boldsymbol{u}(t,k)\|_2\\ &\triangleq\left\|G_2^j(s)\right\|_\infty\|\delta\boldsymbol{u}(t,k)\|_2. \end{aligned} \tag{30}$$

Using Lemma 5, we know that if the LMI (23) holds, then

$$A^{j\mathrm{T}}P^j+P^jA^j<0$$

which together with the Lyapunov theory ensures that the matrix $A^j$ is exponentially stable. This implies the boundedness of $\left\|G_1^j(s)\right\|_\infty$ and $\left\|G_2^j(s)\right\|_\infty$. From (30), it thus follows that $\|\delta\boldsymbol{x}(t,k)\|_{2,[0,T]}$ and $\|\boldsymbol{e}(t,k)\|_{2,[0,T]}$ are bounded for all $k\in\mathbf{Z}_+$, and $\lim_{k\to\infty}\|\delta\boldsymbol{x}(t,k)\|_{2,[0,T]}=0$ and $\lim_{k\to\infty}\|\boldsymbol{e}(t,k)\|_{2,[0,T]}=0$. The proof is completed. ■

**Theorem 3** Let system (2) satisfy Assumptions A1)-A2), and ILC scheme (5)-(7) be applied. If the learning gain $\Gamma$ can be designed such that, for a sufficiently small scalar $\epsilon>0$, there exist a positive definite symmetric matrix $Q>0$ and matrices $X$ and $Y$ satisfying the following LMI with $j=l_T-1$

$$\begin{bmatrix} Q-X-X^{\mathrm{T}} & \left(X+\epsilon A^jX\right)^{\mathrm{T}} & 0 & \left(Y-\Gamma^jC^jA^jX\right)^{\mathrm{T}} \\ X+\epsilon A^jX & -Q & B^j & 0 \\ 0 & B^{j\mathrm{T}} & -\epsilon^{-1}I^j & \epsilon^{-1}D^{j\mathrm{T}} \\ Y-\Gamma^jC^jA^jX & 0 & \epsilon^{-1}D^j & -\epsilon^{-1}I^j \end{bmatrix}<0 \tag{31}$$

then $\|\delta\boldsymbol{x}(t,k)\|_{2,[0,T]}$, $\|\delta\boldsymbol{u}(t,k)\|_{2,[0,T]}$ and $\|\boldsymbol{e}(t,k)\|_{2,[0,T]}$ are bounded for all $k\in\mathbf{Z}_+$, and $\lim_{k\to\infty}\|\delta\boldsymbol{x}(t,k)\|_{2,[0,T]}=0$, $\lim_{k\to\infty}\|\boldsymbol{e}(t,k)\|_{2,[0,T]}=0$, and $\lim_{k\to\infty}\|\delta\boldsymbol{u}(t,k)\|_{2,[0,T]}=0$ (monotonic convergence in the sense of the $\mathcal{L}_2$-norm), where $D^j=I^j-\Gamma^jC^jB^j$. If the LMI (31) holds, learning gain matrices $K_i$, $i\in[0,l]$ are computed by

$$\begin{aligned} K_i=&-\left[I\quad 0_{r\times(l_T-1)r}\right]YX^{-1}\\ &\times\left[0_{n\times in}\quad I\quad 0_{n\times(l_T-i-1)n}\right]^{\mathrm{T}}. \end{aligned} \tag{32}$$

[4] Note that the value $T$ has no effect on the $\|\cdot\|_{2,[0,T]}$-norm. Thus, we will take the $\|\cdot\|_{2,[0,T]}$-norm instead of the $\|\cdot\|_{2,[0,T)}$-norm in the proof.

**Proof.** The proof is established by considering Lemma 3 and following the lines of the proof of Theorem 2. ■

Now, using Theorem 3, one can state the following result related to the robust monotonic convergence of ILC against polytopic-type uncertainties:

**Corollary 2** Let system (2) satisfy Assumptions A1)-A3), and ILC scheme (5)-(7) be applied. If the learning gain $\Gamma$ can be designed such that, for a sufficiently small scalar $\epsilon > 0$, there exist positive definite symmetric matrices $Q_p$, $p \in [1, N]$ and matrices $X$ and $Y$ satisfying the following LMIs with $j = l_T - 1$

$$\begin{bmatrix} Q_p - X - X^{\mathrm{T}} & \left(X + \epsilon A_p^j X\right)^{\mathrm{T}} & 0 & \left(Y - \Gamma^j C^j A_p^j X\right)^{\mathrm{T}} \\ X + \epsilon A_p^j X & -Q_p & B_p^j & 0 \\ 0 & B_p^{j\,\mathrm{T}} & -\epsilon^{-1} I^j & \epsilon^{-1} D_p^{j\,\mathrm{T}} \\ Y - \Gamma^j C^j A_p^j X & 0 & \epsilon^{-1} D_p^j & -\epsilon^{-1} I^j \end{bmatrix} < 0, \quad p \in [1, N] \tag{33}$$

then $\|\delta \boldsymbol{x}(t,k)\|_{2,[0,T]}$, $\|\delta \boldsymbol{u}(t,k)\|_{2,[0,T]}$ and $\|\boldsymbol{e}(t,k)\|_{2,[0,T]}$ are bounded for all $k \in \mathbf{Z}_+$, and $\lim_{k\to\infty} \|\delta \boldsymbol{x}(t,k)\|_{2,[0,T]} = 0$, $\lim_{k\to\infty} \|\boldsymbol{e}(t,k)\|_{2,[0,T]} = 0$, and $\lim_{k\to\infty} \|\delta \boldsymbol{u}(t,k)\|_{2,[0,T]} = 0$ (monotonic convergence in the sense of the $\mathcal{L}_2$-norm), where $A_p^j$ (respectively, $B_p^j$) is defined in the same way with $A^j$ (respectively, $B^j$), and $D_p^j = I^j - \Gamma^j C^j B_p^j$. If the LMIs in (33) hold, learning gain matrices $K_i$, $i \in [0, l]$ can also be computed by (32).

**Proof.** The proof is established by considering Lemma 4 and following the lines of the proof of Theorem 2. ■

**Remark 6** From Theorems 2-3, it is clear that conditions for monotonic convergence are very much dependent on the system dynamics appearing in the operator $\mathbf{A}(\nabla)$. This, together with Remark 5, implies that the difference between conditions for monotonic convergence and for asymptotic stability is large [2]. Using Lemma 5 to LMIs (23) and (31), we can obtain the stability of $I - \Gamma CB$, and hence conditions for asymptotic stability of ILC are only necessary for those for monotonic convergence of ILC.

**Remark 7** Corollary 2 implies that a stable ILC design of TDS with polytopic-type uncertainties can become monotonically convergent, and the only requirement is that the set of LMIs in (33) is satisfied. If this condition holds, then the operator $\mathbf{K}(\nabla)$ can also be computed. This results from that the 2-D analysis approach can convert the time-delay ILC systems into the traditional 1-D input-output systems of the control input errors between two sequential iterations. Consequently, the monotonic convergence property of ILC is transformed into the robust $H_\infty$ system performance, based on which the IBRL [18] can be applied to deal with the polytopic-type uncertainties.

**Remark 8** If one takes $\tau = 0$ in system (1), then following the same steps of the proofs of Theorem 1 and Corollary 1, one can prove that conditions for the (robust) asymptotic stability of ILC still work. It seems that the delays do not play a significant role in the stability analysis of ILC. Furthermore, it is an interesting fact that the implementations of the LMIs (23), (31) and (33) depend not on the delay $\tau$ but on $l_T$, i.e., the ratio of the learning time length $T$ to the delay $\tau$. This results from that the expanding operator $\omega(\nabla, j)$ covers the ill effect of the delay operator $\nabla$ in the ILC analysis of the TDS in (1). In particular, Theorems 2-3 and Corollary 2 imply that if $l_T$ keeps the same value, then the same conditions for the monotonic convergence of ILC can be derived regardless of the values of $T$ and $\tau$.

# 4 Simulation results

In this simulation test, let us consider the TDS in (1) with $l = 1$ and matrices given by

$$A_0 = \begin{bmatrix} 0 & 1 \\ -63 & -16 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 60 & -60 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1+g \end{bmatrix}$$

and $C = [0 \;\; 1]$, where $g$ is an uncertain parameter known to reside in the following polytope:

$$\Omega_g = \Big\{ g \big| g = \varsigma_1 g_1 + \varsigma_2 g_2; g_1 = -0.5, g_2 = 0.5, \\ \varsigma_1 \geq 0, \varsigma_2 \geq 0, \varsigma_1 + \varsigma_2 = 1 \Big\}.$$

Let the system state be $\boldsymbol{x}(t) = [x_1(t) \;\; x_2(t)]^{\mathrm{T}}$ and the desired trajectories be described by

$$\boldsymbol{x}_d(t) = \begin{bmatrix} x_{d_1}(t) \\ x_{d_2}(t) \end{bmatrix} = \begin{bmatrix} 4t^3 - 3t^4 \\ 12t^2(1-t) \end{bmatrix}, t \in [0, T]$$

and $\boldsymbol{x}_d(t) = 0$, $t \in [-\tau, 0]$. For different pairs of the delay $\tau$ and learning time $T$, the following two cases are simulated:

**Case I** $\tau = 0.2$, $T = 1$,

**Case II** $\tau = 0.3$, $T = 1.5$

where $l_T = 5$ holds in both cases. Asymptotic stability and monotonic convergence are considered separately.

### 4.1 Asymptotic stability

For the asymptotic stability, we consider the ILC scheme (5)-(7) without the P-type law, i.e., taking $\mathbf{K}(\nabla) = 0$ in (6) and with the D-type law (7) using $\Gamma = 0.5$. From Corollary 1, it follows that $\max\{\|I - \Gamma CB_1\|, \|I - \Gamma CB_2\|\} = 0.75$. Hence, the ILC system is robust asymptotically stable. Figure 1 shows the test results, where without any loss of generality we pick the parameters $\varsigma_1 = 0.6$ and $\varsigma_2 = 0.4$ that are within the polytope uncertainty $\Omega_g$. In the Figure 1, we describe simulation results of Case I in the upper three figures and those of Case II in the lower three figures. For each case, the $\mathcal{L}_2$-norms of state errors $||x_{d_1}(t) - x_1(t,k)||_{2,[0,T]}$ (left), $||x_{d_2}(t) - x_2(t,k)||_{2,[0,T]}$ (middle), and control input error $||u_d(t) - u(t,k)||_{2,[0,T]}$ (right) are described. Obviously, the ILC system is robustly stable. However, the system is only asymptotically stable, and from each subfigure, we observe that even though the ILC system converges as iteration $k$ increases, large transient growth is generated.

Now, if one tests the LMI (23) with $j = l_T - 1$, $X = 0$ and $Y = -\mathrm{diag}\{0.5, 0.5, 0.5, 0.5, 0.5\}$, it is obviously seen that this LMI is infeasible (in this case, the index tmin = 9.7109 is nonnegative). According to Theorem 2, the monotonic convergence of ILC can not be guaranteed to prevent large learning transient growth with such leaning gains.

### 4.2 Monotonic convergence

For the monotonic convergence, we use the same D-type learning gain $\Gamma = 0.5$ as used in the asymptotic stability tests. For the operator $\mathbf{K}(\nabla)$ in the P-type ILC law, it can be determined by solving the set of LMIs in (33) with $j = 4$ and $\epsilon = 0.01$, which is computed by (as given in (32))

$$\mathbf{K}(\nabla) = K_0 + K_1 \nabla, \quad K_0 = \begin{bmatrix} 31.5 & 8 \end{bmatrix}, K_1 = \begin{bmatrix} -30 & 30 \end{bmatrix}.$$

The set of LMIs in (33) is feasible in this case, since it holds that tmin $= -0.4733$. From Corollary 2, it is clear that the ILC system will be robust monotonically convergent. Figure 2 shows the test results of such determined ILC scheme

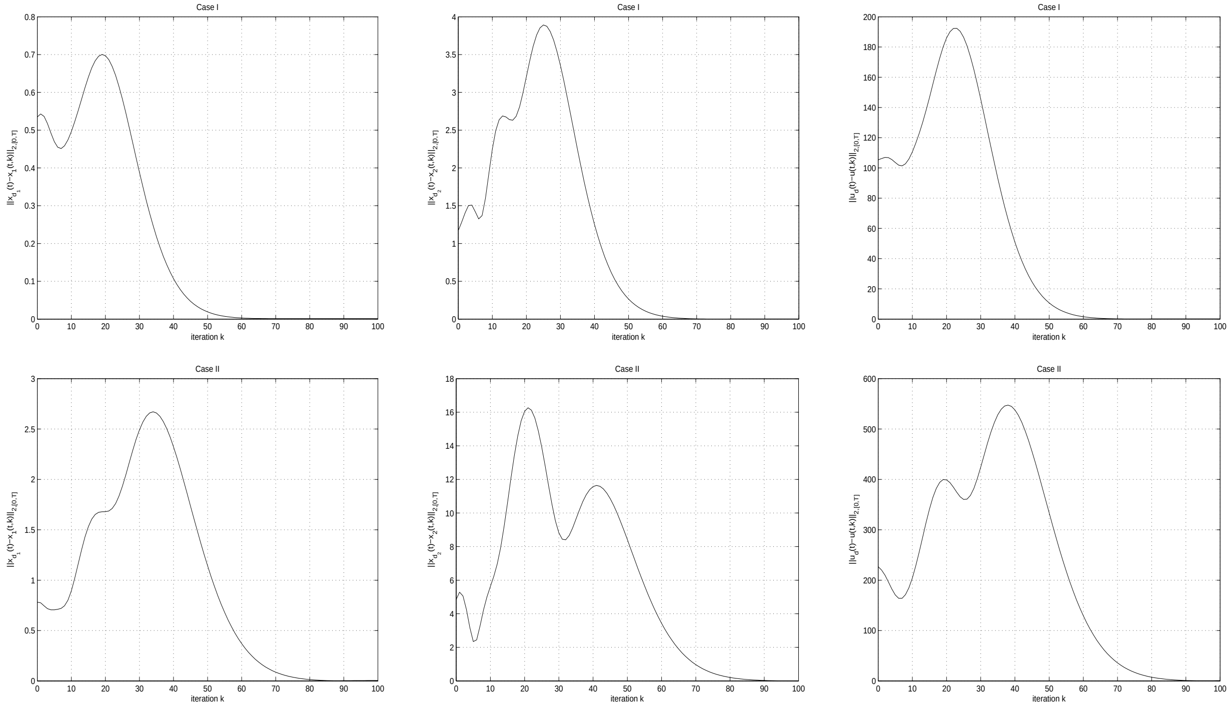


Figure 1 Asymptotic stability. Left: $\left\|x_{d_1}(t)-x_1(t,k)\right\|_{2,[0,T]}$. Middle: $\left\|x_{d_2}(t)-x_2(t,k)\right\|_{2,[0,T]}$. Right: $\|u_d(t)-u(t,k)\|_{2,[0,T]}$.

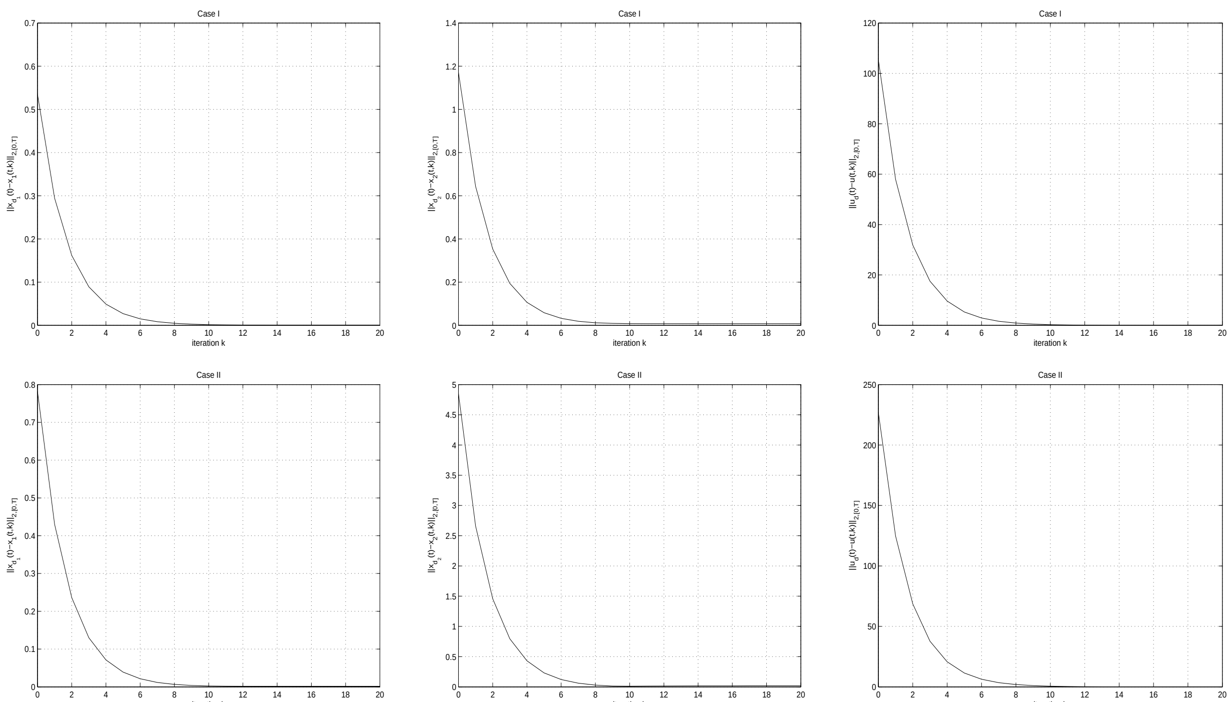


Figure 2 Monotonic convergence. Left: $\left\|x_{d_1}(t)-x_1(t,k)\right\|_{2,[0,T]}$. Middle: $\left\|x_{d_2}(t)-x_2(t,k)\right\|_{2,[0,T]}$. Right: $\|u_d(t)-u(t,k)\|_{2,[0,T]}$.

(5)-(7) in both Cases I and II. As shown in this figure, the $\mathcal{L}_2$-norms of the state errors and control input error decay monotonically to zero as the learning trial increases. Moreover, the convergence rate is greatly improved, which can be clearly seen by comparing both Figures 1 and 2.

The simulation tests verify that the LMI approach can be used to achieve good learning transients of uncertain TDS by converting an asymptotically stable ILC into a monotonically convergent ILC, in which not the delay operator but the expanding operator plays a significant role.

## 5 Conclusions

In this paper, a PD-type ILC has been discussed for TDS subject to polytopic-type uncertainties. After the 2-D ILC error systems are derived, an LMI approach is introduced to deal with such 2-D systems that disclose a connection between the control input errors of two sequential iterations. Hence, this paper considers the asymptotic stability of ILC by the 2-D system theory and the monotonic convergence of ILC by the BRL (IBRL). In particular, sufficient LMI conditions are presented which provide criterions to determine learning gains to achieve the robust monotonic convergence of ILC. These results have been verified through numerical simulation tests.

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