On Thermal Radiation of Black Holes

A.A.Kyasov and G.V.Dedkov[1],

Nanoscale Physics Group, Kabardino –Balkarian State University, Nalchik

Abstract . We calculate intensity of thermal radiation (via Hawking effect) and evaporation time of a stationary nonrotating black hole using Kirchhoff 's law and the electrodynamic membrane paradigm. It is shown that both quantities significantly depend on the relative thickness of membrane and real part of its static dielectric permittivity.

1.Introduction

Following [1,2], stationary black holes without rotation in the vicinity of their event horizon generate photons having thermal spectrum with temperature

$$T = \frac{\hbar c^3}{8\pi GM k_B}, \qquad (1)$$

where $\hbar$, k_B and G are Planck's, Boltzann's and gravitational constants, c the speed of light in free space and M the mass of a black hole. It is usual to believe that intensity of this radiation is governed by Stefan's law, i.e. $dE/dt = 4\pi\sigma_S r_g{}^2 T^4$, where $r_g = 2GM/c^2$ is Schwarzshild's radius and σ_S is Stefan's constant. Solving equation $c^2 dM/dt = dE/dt$ with respect to M yields the characteristic evaporation time for an isolated black hole, $\tau_B = 5120\pi(G^2 M^3/\hbar c^4)$.

However, generally speaking, the above estimations are valid only in the limit of geometrical optics, at $r_g/\lambda_T >> 1$, $\lambda_T = \dfrac{2\pi c\hbar}{k_B T}$. In this case, it is assumed that the black hole emits radiation just like an absolutely absorbing black sphere of radius r_g. Nevertheless, it is not so: from the above expressions for r_g, T and λ_T it follows just an opposite relation: $r_g/\lambda_T = \dfrac{1}{8\pi^2} << 1$. This results in great difference in the rate of evaporation of a black hole in comparison with the common accepted estimation, thus stimulating our further consideration.

[1] Corresponding author e-mail: gv_dedkov@mail.ru

Recently, it has been realized [3] that an event horizon of a black hole can be considered as a conducting sphere with surface resistance of $R_H = \frac{4\pi}{c} = 377\,Ohm$. This result can be used to get the absorption cross –section of the black hole with respect to the low –frequency photons ($\lambda >> r_g$) and further, applying Kirchoff's law, to calculate the intensity of thermal radiation. So, this paper aims at an estimation of the radiation intensity of black hole with account of the known electrodynamic and thermodynamic properties.

2. Theoretical consideration

According to the membrane paradigm [3], in the presence of the external electromagnetic field the event horizon of a black hole behaves like a conducting surface. Introducing an effective thickness Δr of the membrane, and assuming the same resistivity within Δr, one can relate total resistance R_H of the black hole with the conductivity of the membrane using a model of two concentric spheres with radii $a_1 = r_g$, $a_2 = r_g + \Delta r$, where the spherical shell is filled by homogenious conducting matter [4]

$$R_H = \frac{1}{4\pi\sigma}\left(\frac{1}{a_1} - \frac{1}{a_2}\right) \tag{2}$$

with σ being the conductivity. Assuming $\Delta r / r_g << 1$, we get from (2)

$$\sigma = \frac{\Delta r}{4\pi R_H r_g{}^2} = \frac{c}{(4\pi)^2}\frac{\Delta r}{r_g{}^2} \tag{3}$$

On the other hand, following [5], the dielectric permittivity of the corresponding conducting medium can be written as

$$\varepsilon(\omega) = \varepsilon_0 + \mathrm{i}\frac{4\pi\sigma}{\omega} \tag{4}$$

Eqs.(3), (4) enable to determine polarizability of the membrane by analogy with the polarizability of spherical shell of thickness $\Delta a = a_2 - a_1$, occupied by substance having the dielectric permittivity $\varepsilon(\omega)$ (see Appendix with substitution $\varepsilon_1 = 1,\ \varepsilon_2 = \varepsilon_2(\omega)$ in (A2))

$$\alpha(\omega) = a_2{}^3 \frac{(1 - a_2{}^3 / a_1{}^3)(2\varepsilon(\omega)+1)(\varepsilon(\omega)-1)}{2(\varepsilon(\omega)-1)^2 - (a_2{}^3 / a_1{}^3)(2\varepsilon(\omega)+1)(\varepsilon(\omega)+2)} \tag{5}$$

Substituting $a_1 = r_g$, $a_2 = r_g + \Delta r$ into (5) and expanding it up to a linear order in $x = \Delta r / r_g$, one obtains

$$\alpha(\omega) = x\frac{r_g{}^3}{3}\frac{(\varepsilon(\omega)-1)(2\varepsilon(\omega)+1)}{\varepsilon(\omega)} \tag{6}$$

Because $\lambda_T >> r_g$, the related cross-section for absorption of electromagnetic radiation is given by

$$\sigma_a(\omega) = \frac{4\pi\,\omega}{c}\alpha''(\omega) \tag{7}$$

where $\alpha''(\omega)$ is the corresponding imaginary part. Using (3), (6) yields

$$\alpha''(\omega) = \frac{r_g{}^3 x^2}{3t}\left(2 + \frac{t^2}{\varepsilon_0{}^2 t^2 + x^2}\right),\ t \equiv \omega/\omega_T,\ \omega_T = k_B T/\hbar \tag{8}$$

Applying Kirchoff's law, the intensity of thermal radiation of the black hole is given by

$$I = c\int_0^\infty \sigma_a(\omega)\frac{\hbar\,\omega^3}{\pi^2 c^3}\frac{1}{\exp(\hbar\,\omega/k_B T)-1}d\omega \tag{9}$$

Introducing (7), (8) into (9) with account of (1) and $r_g = 2GM/c^2$ yields

$$I(M,x,\varepsilon_0) = \frac{1}{3072\pi^6}\left(\frac{\hbar\,c^6}{G^2 M^2}\right)f(x,\varepsilon_0) \tag{10}$$

$$f(x,\varepsilon_0) = x^2\cdot\int_0^\infty \frac{t^3}{\exp(t)-1}\left(2+\frac{t^2}{\varepsilon_0{}^2 t^2 + x^2}\right)dt \tag{11}$$

The evaporation time $\tau(M,x,\varepsilon_0)$ is calculated from $c^2 dM/dt = I(M,x,\varepsilon_0)$ and reads

$$\tau(M,x,\varepsilon_0) = \frac{1024\pi^6}{f(x,\varepsilon_0)}\left(\frac{G^2 M^3}{\hbar\,c^4}\right) \tag{12}$$

Contrary to that, usage of the generally accepted expression for the total absorption cross – section of a black hole, $\sigma_a = \pi\,r_g{}^2$, which is grounded on an approximation of geometrical optics, being incorrect in this case, coupled with Kirchoff's law results in conventional expressions (see, for instance, [6]):

$$I_B = \pi\,r_g{}^2 c\int_0^\infty \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\hbar\,\omega/k_B T)-1}d\omega = \frac{1}{15360\pi}\left(\frac{\hbar\,c^6}{G^2 M^2}\right) \tag{13}$$

$$\tau_B = 5120\pi\left(\frac{G^2 M^3}{\hbar\,c^4}\right) \tag{14}$$

Comparing (10), (13) and (12), (14) shows that, in general, the radiation intensity and evaporation time of black holes essentially depend not only on the mass M, but on parameters x,ε_0, too. It seems to be quite natural to put forward the assumption $x << 1$, but this still leaves the value of the constant ε_0 to be arbitrary. Correspondingly, both intensity of radiation and evaporation time may change in a rather broad interval of values. In the above calculation we have not taken into account the presence of other type particles (besides photons) having thermal spectrum.

3. Numerical estimations

It is interesting to numerically compare the intensities (10), (13) and evaporation times (12), (14). Figs. 1, 2 show the calculated fractions I / I_B and τ / τ_B in dependence of x and ε_0. We see from the figures that the obtained results significantly depend on x , ε_0 and their interrelation. Particularly, at $\varepsilon_0 << x << 1$ we get $I / I_B \approx 2$. On the other hand, at $\varepsilon_0 \geq 1$ we get $I / I_B \approx 0.2x^2 << 1$. Correspondingly, the relative evaporation time τ / τ_B changes from nearly 30.6 at $\varepsilon_0 << x << 1$ to $\sim 5 / x^2 >> 1$ at $\varepsilon_0 \geq 1$. As seen from Figs.1,2, the evaporation time of black holes can be many orders of value larger as compared with generally accepted estimate. This makes it possible for some primary black holes with relatively small mass to survive and to be observed even at present time.

4.Conclusions

As it follows yet from Kirchoff's law, the intensity of thermal radiation of a black hole of any mass differs from that one corresponding to absolutely absorbing black sphere with the same radius, as the condition $\lambda_T / r_g = 8\pi^2 >> 1$ directly opposes to the validity condition of geometrical optics, $\lambda_T / r_g << 1$. Consequently, the currently used formula $dE / dt = 4\pi r_g{}^2 \sigma_S T^4$ (where σ_S is Stefan's constant) proves to be incorrect.

Numerical calculations of the radiation intensity of black holes based on the membrane paradigm manifest strong dependence on the relation between a relative thickness of the membrane and a real part of the static dielectric permittivity. More precise judgement needs the corresponding parameters to be specified. In particular, it may happen that evaporation time of black holes with a small mass will exceed the generally accepted estimate by many orders of value at a definite relation between $\Delta r / r_g$ and ε_0. From this it follows that, if such black holes might formerly born during the universe evolution, they might survive up to the present time. The lack of observations of black holes having small mass ($M < M_\oplus$) may provoke both a more precise determination of the parameters ε_0, x and give rise to definite problems relating to mutual concord between the theory and experiment.

Appendix

The static polarizability of a two –layer dielectric sphere (Fig.3) can be found solving Laplace equation for the electric potential. If the dielectric sphere is placed in homogeneous external electric field **E** being directed along z -axis (Fig.3), the Laplace equation $\Delta\phi = 0$ should be solved subjected to boundary conditions of continuity ϕ and normal projection of the electric displacement at $r = a_1$ and $r = a_2$. In outer space $r > a_2$ we finally obtain

$$\phi(r,\theta) = -Er\cos\theta + \frac{\alpha E}{r^2}\cos\theta \qquad \text{(A1)}$$

$$\alpha = a_2^{\,3}\,\frac{(1+2\varepsilon_2)(\varepsilon_1-\varepsilon_2)a_1^{\,3} + (\varepsilon_1+2\varepsilon_2)(\varepsilon_2-1)a_2^{\,3}}{2(\varepsilon_2-1)(\varepsilon_1-\varepsilon_2)a_1^{\,2} + (\varepsilon_1+2\varepsilon_2)(\varepsilon_2+2)a_2^{\,3}} \qquad \text{(A2)}$$

where α is the static polarizability. Making use the transformations $\varepsilon_i \to \varepsilon_i(\omega)$, $i = 1,2$ in (A2), we immediately get the dynamic polarizability $\alpha(\omega)$. The corresponding long –wave approximation holds at $\omega a_2 / c << 1$. Eq.(5) follows from (A2) at $\varepsilon_1 = 1, \varepsilon_2 \to \varepsilon(\omega)$

To get the polarizability of a thin dielectric spherical shell with thickness Δa, one must put $a_1 = a$, $a_2 = a + \Delta a$, $\varepsilon_1 = 1$ in Eq.(A2). Then in a linear expansion (A2) in $\Delta a / a$ we retrieve Eq.(6) in the particular case of $\Delta a = \Delta r$, $a = r_g$.

References

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FIGURE 1

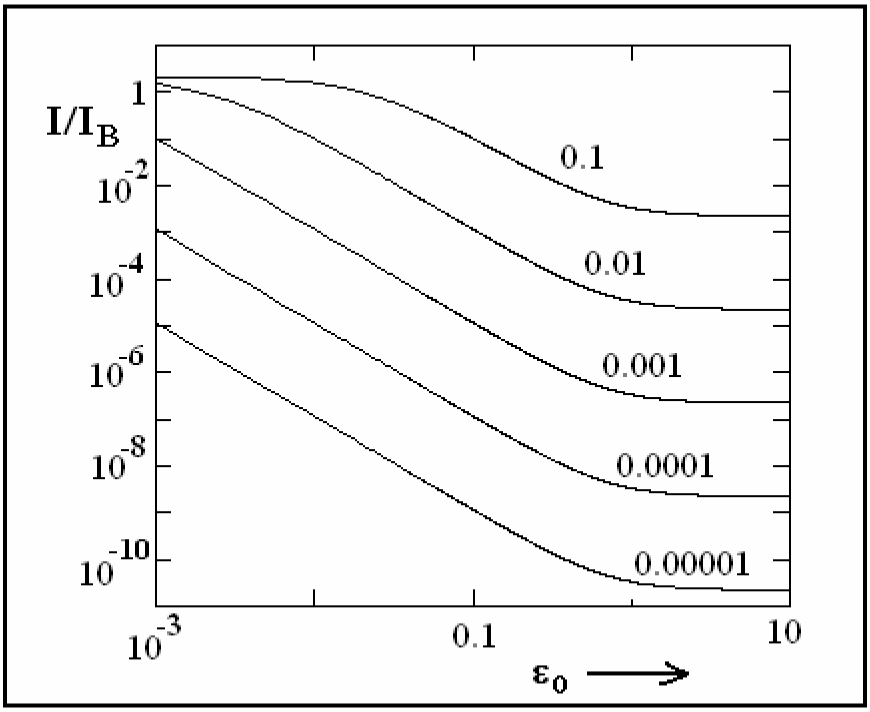

FIGURE 2

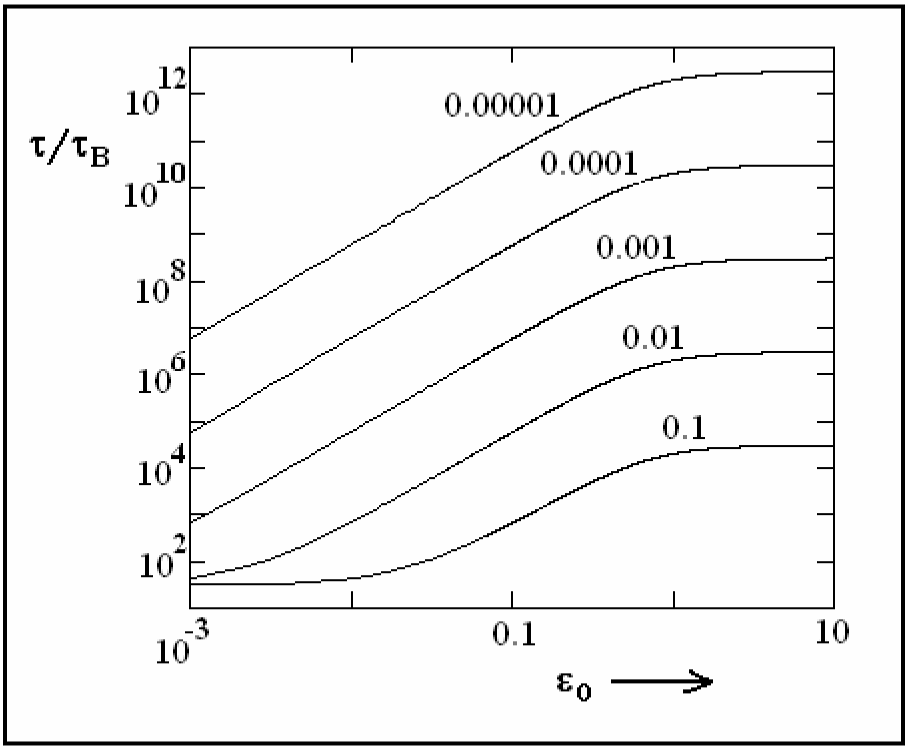
τ/τ_B
10^12
10^10
10^8
10^6
10^4
10^2
10^-3
0.1
10
ε_0
0.00001
0.0001
0.001
0.01
0.1

FIGURE 3

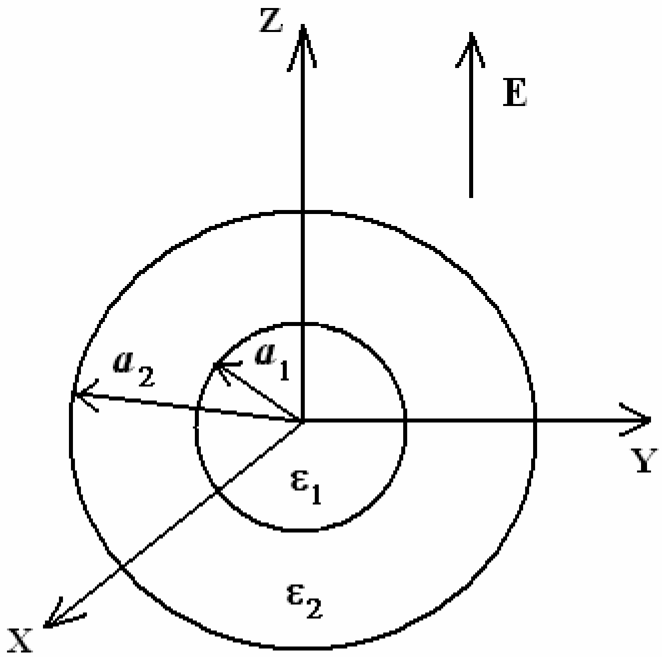
Z
E
Y
X
a_2
a_1
ε_1
ε_2

FIGURE CAPTIONS

Fig.1. Relative intensity of the thermal radiation of black holes , I / I_B vs. ε_0 . Different lines correspond to the relative thickness of the event horizon membrane, $x = \Delta r / r_g$.

Fig.2. Relative evaporation time of black holes, τ / τ_B . Different lines correspond to the same values of x as on Fig.1.

Fig.3. Schematic view of the two –shell dielectric sphere placed in the external electric field $\mathbf{E}$, and the coordinate system used.