Some Diffusion Processes Associated With Two Parameter Poisson-Dirichlet Distribution and Dirichlet Process

Shui Feng and Wei Sun

McMaster University and Concordia University

The two parameter Poisson-Dirichlet distribution $PD(\alpha, \theta)$ is the distribution of an infinite dimensional random discrete probability. It is a generalization of Kingman's Poisson-Dirichlet distribution. The two parameter Dirichlet process $\Pi_{\alpha,\theta,\nu_0}$ is the law of a pure atomic random measure with masses following the two parameter Poisson-Dirichlet distribution. In this article we focus on the construction and the properties of the infinite dimensional symmetric diffusion processes with respective symmetric measures $PD(\alpha, \theta)$ and $\Pi_{\alpha,\theta,\nu_0}$. The methods used come from the theory of Dirichlet forms.

1 Introduction

The Poisson-Dirichlet distribution $PD(\theta)$ was introduced by Kingman in [11] to describe the distribution of gene frequencies in a large neutral population at a particular locus. The component $P_k(\theta)$ represents the proportion of the k-th most frequent allele. The Dirichlet process Π_{θ,ν_0} first appeared in [6] in the context of Bayesian statistics. It is a pure atomic random measure with masses distributed according to $PD(\theta)$.

In the context of population genetics, both the Poisson-Dirichlet distribution and the Dirichlet process appear as approximations to the equilibrium behavior of certain large populations evolving under the influence of mutation and random genetic drift. To be precise, let $C_b(S)$ be the set of bounded, continuous functions on a locally compact, separable metric space S, $\mathcal{M}_1(S)$ denote the space of all probability measures on S equipped with the usual weak topology, and $\nu_0 \in \mathcal{M}_1(S)$. We consider the operator B of the form

$$Bf(x) = \frac{\theta}{2} \int (f(y) - f(x))\nu_0(dy), f \in C_b(S).$$

Define

$$\mathcal{D} = \{u : u(\mu) = f(\langle \phi, \mu \rangle),\ f \in C_b^\infty(\mathbf{R}), \phi \in C_b(S), \mu \in \mathcal{M}_1(S)\},$$

where $C_b^\infty(\mathbf{R})$ denotes the set of all bounded, infinitely differentiable functions on $\mathbf{R}$. Then the Fleming-Viot process with neutral parent independent mutation or the labeled infinitely-many-neutral-alleles model is a pure atomic measure-valued Markov process with generator

$$Lu(\mu) = \langle B\nabla u(\mu)(\cdot), \mu\rangle + \frac{f''(\langle \phi, \mu\rangle)}{2}\langle \phi, \phi\rangle_\mu, u \in \mathcal{D},$$

where

$$\nabla u(\mu)(x) = \delta u(\mu)/\delta\mu(x) = \lim_{\varepsilon\to 0+} \varepsilon^{-1}\{u((1-\varepsilon)\mu + \varepsilon\delta_x) - u(\mu)\},$$
$$\langle \phi, \psi\rangle_\mu = \langle \phi\psi, \mu\rangle - \langle \phi, \mu\rangle\langle \psi, \mu\rangle,$$

and δ_x stands for the Dirac measure at $x \in S$. For compact space S and diffusive probability ν_0, i.e., $\nu_0(x) = 0$ for every x in S, it is known (cf. [2]) that the labeled infinitely-many-neutral-alleles model is reversible with reversible measure Π_{θ,ν_0}.

Introduce a map Φ from $\mathcal{M}_1(S)$ to the infinite dimensional ordered simplex

$$\nabla_\infty = \{(x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^\infty x_i = 1\}$$

so that $\Phi(\mu)$ is the ordered masses of μ. Then the labeled infinitely-many-neutral-alleles model is mapped through Φ to another symmetric diffusion process, called the unlabeled infinitely-many-neutral-alleles model, on ∇_∞ with generator

$$A^0 = \frac{1}{2}\left\{\sum_{i,j=1}^\infty x_i(\delta_{ij} - x_j)\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^\infty \theta x_i \frac{\partial}{\partial x_i}\right\}, \tag{1.1}$$

defined on an appropriate domain. The symmetric measure of this process is $PD(\theta)$.

For any $0 \leq \alpha < 1$ and $\theta > -\alpha$, let $U_k, k = 1, 2, ...,$ be a sequence of independent random variables such that U_k has $Beta(1-\alpha, \theta + k\alpha)$ distribution. Set

$$V_1^{\alpha,\theta} = U_1,\ V_n^{\alpha,\theta} = (1-U_1)\cdots(1-U_{n-1})U_n,\ n \geq 2,$$

and let $\mathbf{P}(\alpha, \theta) = (\rho_1, \rho_2, ...)$ denote $(V_1^{\alpha,\theta}, V_2^{\alpha,\theta}, ...)$ in descending order. The distribution of $(V_1^{\alpha,\theta}, V_2^{\alpha,\theta}, \ldots)$ is called the two parameter GEM distribution. The law of $\mathbf{P}(\alpha, \theta)$ is called the two parameter Poisson-Dirichlet distribution, denoted by $PD(\alpha, \theta)$. For a locally compact, separable metric space S, and a sequence of i.i.d. S-valued random variables $\xi_k, k = 1, 2, ...$ with common diffusive distribution ν_0 on S, let

$$\Xi_{\alpha,\theta,\nu_0} = \sum_{k=1}^\infty \rho_k \delta_{\xi_k}. \tag{1.2}$$

The distribution of $\Xi_{\alpha,\theta,\nu_0}$, denoted by $Dirichlet(\theta, \alpha, \nu_0)$ or $\Pi_{\alpha,\theta,\nu_0}$, is called the two-parameter Dirichlet process. Clearly $PD(\theta)$ and Π_{θ,ν_0} correspond to $\alpha = 0$ in $PD(\alpha, \theta)$ and $\Pi_{\alpha,\theta,\nu_0}$, respectively.

As was indicated in [18] and the references therein, the two parameter Poisson-Dirichlet distribution and Dirichlet process are natural generalizations of their one parameter counterparts and possess many similar structures including the urn construction, GEM representation, sampling formula, etc. The cases of $\theta = 0, \alpha$ are associated with distributions of the lengths of excursions of Bessel processes and Bessel bridge, respectively. It is thus natural to investigate the two parameter generalizations of the labeled and unlabeled infinitely-many-neutral-alleles models. One would hope that these dynamical models will enhance our understanding of the two parameter distributions.

Several papers have appeared recently discussing the stochastic dynamics associated with the two parameter distributions. A symmetric diffusion process appears in [5], where the symmetric measure is the GEM distribution. An infinite dimensional diffusion process is constructed in [16] generalizing the unlabeled infinitely-many-neutral-alleles model to the two-parameter setting. In [1], $PD(\alpha, \theta)$ is shown to be the unique reversible measure of a continuous time Markov chain constructed through an exchangeable fragmentation coalescence process. But it is still an open problem to construct the two parameter measure-valued process generalizing the Fleming-Viot process with parent independent mutation.

In this article, we will consider two diffusion processes that are analogous to the unlabeled and labeled infinitely-many-neutral-alleles models. In Section 2, an unlabeled two parameter infinitely-many-neutral-alleles diffusion model is constructed via the classical gradient Dirichlet forms. This process is shown to coincide with the process constructed in [16]. Besides establishing the existence and uniqueness of the process, we also obtain results on the sample path properties, the large deviations for occupation time process, and the model with interactive selection. The construction of the labeled infinitely-many-neutral-alleles diffusion model turns out to be much harder. Here the evolution of the system involves both the masses and the locations. Note that the one parameter model with finite many types is the Wright-Fisher diffusion, and the partition property of the infinite type model makes it possible for the finite dimensional approximation. However, in the two parameter setting, the finite type model itself is already a challenging problem not to mention the loss of the partition property. In Section 3, we construct a general bilinear form that, if closable, will generate the needed diffusion process. If the type space contains only two elements or the type space is general but $\alpha = -\kappa$ and $\theta = m\kappa$ for some $\kappa > 0$ and integer $m \geq 2$, then the above bilinear form is closable and a symmetric diffusion can be constructed accordingly. The closability problem in the general case boils down to the establishment of boundedness of a linear functional. An auxiliary result is enclosed at the end of the article to demonstrate the difficulty involved in establishing the boundedness. If the bilinear form is indeed not closable, then its relaxation may be considered.

2 Unlabeled Model

Let

$$\overline{\nabla}_\infty := \{x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0,\ \sum_{i=1}^{\infty} x_i \leq 1\}$$

be the closure of ∇_∞ in the product space $[0,1]^\infty$. For $0 \leq \alpha < 1$ and $\theta > -\alpha$, we extend the two parameter Poisson-Dirichlet distribution $PD(\alpha,\theta)$ from ∇_∞ to $\overline{\nabla}_\infty$. To simplify notation, we still use $PD(\alpha,\theta)$ to denote this extended distribution. Let $a(x)$ be the infinite matrix whose (i,j)-th entry is $x_i(\delta_{ij} - x_j)$. Denote by $\mathcal{P}$ the algebra generated by 1, φ_2, $\varphi_3, \ldots, \varphi_m, \ldots$, where $\varphi_m(x) = \sum_{i=1}^{\infty} x_i^m$. We consider the bilinear form $\mathcal{A}$ of the form

$$\mathcal{A}(u,v) = \frac{1}{2}\int_{\overline{\nabla}_\infty} \langle \nabla u, a(x)\nabla v\rangle dPD(\alpha,\theta), \quad u, v \in \mathcal{P}.$$

Theorem 2.1 *The symmetric bilinear form $(\mathcal{A},\mathcal{P})$ is closable on $L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))$ and its closure $(\mathcal{A}, D(\mathcal{A}))$ is a regular Dirichlet form.*

Proof Define

$$A = \frac{1}{2}\left\{\sum_{i=1}^{\infty} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^{\infty} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{\infty}(\theta x_i + \alpha)\frac{\partial}{\partial x_i}\right\}. \tag{2.1}$$

The case of $\alpha = 0$ corresponds to A^0 defined in (1.1). One finds that for any $u, v \in \mathcal{P}$,

$$A^0(uv) = A^0 u \cdot v + A^0 v \cdot u + \langle \nabla u, a(x)\nabla v\rangle.$$

Hence

$$A(uv) = Au \cdot v + Av \cdot u + \langle \nabla u, a(x)\nabla v\rangle. \tag{2.2}$$

We claim that

$$\int_{\overline{\nabla}_\infty} Au dPD(\alpha,\theta) = 0, \quad \forall u \in \mathcal{P}. \tag{2.3}$$

In fact, let $m_1,\ldots,m_k\in\{2,3,\ldots\}$ and $k\geq 1$. Then we obtain by (2.1), (1.1) and [4, (2.13)] that

$$\begin{aligned}
A(\varphi_{m_1}\cdots\varphi_{m_k}) &= \sum_{i=1}^{k}\left[\binom{m_i}{2}-\frac{m_i\alpha}{2}\right]\varphi_{m_i-1}\prod_{j\neq i}\varphi_{m_j}+\sum_{i<j}m_im_j\varphi_{m_i+m_j-1}\prod_{l\neq i,j}\varphi_{m_l}\\
&\quad -\left\{\sum_{i=1}^{k}\left[\binom{m_i}{2}+\frac{m_i\theta}{2}\right]+\sum_{i<j}m_im_j\right\}\prod_{i=1}^{k}\varphi_{m_i} \qquad (2.4)\\
&= \sum_{i=1}^{k}\left[\binom{m_i}{2}-\frac{m_i\alpha}{2}\right]\varphi_{m_i-1}\prod_{j\neq i}\varphi_{m_j}+\sum_{i<j}m_im_j\varphi_{m_i+m_j-1}\prod_{l\neq i,j}\varphi_{m_l}\\
&\quad -\frac{1}{2}m(m-1+\theta)\prod_{i=1}^{k}\varphi_{m_i}.
\end{aligned}$$

Denote by $\{n_1,n_2,\ldots,n_l\}$ an arbitrary partition of $\{m_1,m_2,\ldots,m_k\}$. That is, each $n_i=m_{i_1}+\cdots+m_{i_{j_i}}$ for some distinct indexes $i_1,\ldots,i_{j_i}$, and $\{1,2,\ldots,k\}=\cup_{i=1}^{l}\{i_1,\ldots,i_{j_i}\}$. By Ewens-Pitman's sampling formula, we get

$$\begin{aligned}
&\int_{\overline{\nabla}_\infty}A(\varphi_{m_1}\ldots\varphi_{m_k})dPD(\alpha,\theta)\\
=&\sum_{n_1,n_2,\ldots,n_l}\left\{\sum_{i=1}^{l}\frac{n_i}{2}(n_i-1-\alpha)\frac{(-\frac{\theta}{\alpha})(-\frac{\theta}{\alpha}-1)\cdots(-\frac{\theta}{\alpha}-l+1)}{\theta(\theta+1)\cdots(\theta+m-2)}\right.\\
&\quad\cdot\left(\prod_{j\neq i}(-\alpha)\cdots(-\alpha+n_j-1)\right)(-\alpha)\cdots(-\alpha+n_i-2)\\
&\quad-\frac{1}{2}m(m-1+\theta)\frac{(-\frac{\theta}{\alpha})(-\frac{\theta}{\alpha}-1)\cdots(-\frac{\theta}{\alpha}-l+1)}{\theta(\theta+1)\cdots(\theta+m-1)}\\
&\quad\left.\cdot\prod_{j}(-\alpha)\cdots(-\alpha+n_j-1)\right\}\\
=&\ 0,
\end{aligned}$$

where the value of the right hand side is obtained by continuity when $\alpha=0$ or $\theta=0$. Similarly, by Ewens-Pitman's sampling formula, we can further check that

$$\int_{\overline{\nabla}_\infty}(Au)vdPD(\alpha,\theta)=\int_{\overline{\nabla}_\infty}(Av)udPD(\alpha,\theta),\quad\forall u,v\in\mathcal{P}. \qquad (2.5)$$

By (2.2), (2.3) and (2.5), we get

$$\mathcal{A}(u,v)=-\int_{\overline{\nabla}_\infty}(Au)vdPD(\alpha,\theta),\quad\forall u,v\in\mathcal{P}.$$

Hence the symmetric bilinear form $(\mathcal{A}, \mathcal{P})$ is closable on $L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))$ by ([13, Proposition 3.3]). The closure $(\mathcal{A}, D(\mathcal{A}))$ of $(\mathcal{A}, \mathcal{P})$ is a symmetric closed form. To prove that $(\mathcal{A}, D(\mathcal{A}))$ is a regular Dirichlet form, it is enough to show that $(\mathcal{A}, D(\mathcal{A}))$ is a Markovian form. To this end, we will show that $(\mathcal{A}, D(\mathcal{A}))$ is the same as the closure $(\mathcal{A}, \overline{\mathcal{B}})$ of $(\mathcal{A}, \mathcal{B})$ with

$$\mathcal{B} := \{u \in L^2(\overline{\nabla}_\infty; PD(\alpha,\theta)) : u = f \circ \pi_k \text{ for some } k, f \in C_0^\infty(\mathbf{R}^\mathbf{k})\},$$

where $\pi_k : \overline{\nabla}_\infty \to \mathbf{R}^\mathbf{k}, (x_1, \ldots, x_k, \ldots) \to (x_1, \ldots, x_k)$. Note that $(\mathcal{A}, \mathcal{B})$ is clearly Markovian (cf. [7, Page 4]) and this property is preserved by its closure (cf. [7, Theorem 3.1.1]).

Let $m \geq 2$. Then one can show that $\varphi_m \in \overline{\mathcal{B}}$ by considering the approximation sequence $\{\varphi_m^N(x) := \sum_{i=1}^N x_i^m\}_{N \in \mathbf{N}}$. Thus $\mathcal{P} \subset \overline{\mathcal{B}}$. To show that $\mathcal{B} \subset D(\mathcal{A})$, we need to show that any finite-dimensional smooth function of the coordinates $x_1, x_2, \ldots$ belongs to $D(\mathcal{A})$. This can be done by polynomial approximation and noting the fact that

$$x_1 = \lim_{m\to\infty} (\varphi_m)^{1/m}, \quad x_2 = \lim_{m\to\infty} (\varphi_m - x_1^m)^{1/m}, \ldots,$$

where the convergence takes place pointwise on $\overline{\nabla}_\infty$.

□

It is worth noting that $PD(\alpha,\theta)$ is the unique probability measure on $\overline{\nabla}_\infty$ such that (2.3) is satisfied. In fact, suppose that $\mu \in \mathcal{M}_1(\overline{\nabla}_\infty)$ satisfying

$$\int_{\overline{\nabla}_\infty} Au d\mu = 0, \quad \forall u \in \mathcal{P}.$$

Note that for any $m \geq 2$

$$A\varphi_m = A^0\varphi_m - \frac{m\alpha}{2}\varphi_{m-1} = \left[\binom{m}{2} - \frac{m\alpha}{2}\right]\varphi_{m-1} - \left[\binom{m}{2} + \frac{m\theta}{2}\right]\varphi_m.$$

The fact of $\int_{\overline{\nabla}_\infty} A\varphi_m d\mu = 0$ implies that

$$\begin{aligned} \int_{\overline{\nabla}_\infty} \varphi_m d\mu &= \frac{\Gamma(m-\alpha)\Gamma(\theta+1)}{\Gamma(1-\alpha)\Gamma(\theta+m)} \\ &= \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)\Gamma(1-\alpha)} \int_0^1 u^m \frac{(1-u)^{\theta+\alpha+1}}{u^{\alpha+1}} du. \end{aligned}$$

Then, we obtain by [18, (6)] that

$$\int_{\overline{\nabla}_\infty} \varphi_m d\mu = \int_{\overline{\nabla}_\infty} \varphi_m dPD(\alpha,\theta), \quad \forall m \in \mathbf{N}. \tag{2.6}$$

Furthermore, we obtain by (2.4), (2.6) and induction that

$$\int_{\overline{\nabla}_\infty} u d\mu = \int_{\overline{\nabla}_\infty} u dPD(\alpha,\theta), \quad \forall u \in \mathcal{P}.$$

Since $\mathcal{P}$ is measure-determining, $\mu = PD(\alpha,\theta)$.

By the theory of Dirichlet forms, there exists an essentially unique Hunt process $(X, (P_x)_{x\in\overline{\nabla}_\infty})$ on $\overline{\nabla}_\infty$ with the stationary distribution $PD(\alpha,\theta)$ such that X is associated with the Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ (cf. [7, Chapter 7]). Note that $A1 = 0$. By [20, Proposition 2.3], one finds that X is a conservative diffusion process. Denote $P_{PD(\alpha,\theta)}(\cdot) = \int_{\overline{\nabla}_\infty} P_x(\cdot) PD(\alpha,\theta)(dx)$. Then we have the following proposition.

Proposition 2.2 *The process X with initial distribution $PD(\alpha,\theta)$ never leaves ∇_∞, i.e.,*

$$P_{PD(\alpha,\theta)}(X_t \in \nabla_\infty, \forall t \geq 0) = 1. \tag{2.7}$$

In addition, the process X is ergodic, i.e.,

$$\lim_{t\to\infty} \left\| T_t f - \int_{\overline{\nabla}_\infty} f dPD(\alpha,\theta) \right\|_{L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))} = 0, \quad \forall f \in L^2(\overline{\nabla}_\infty; PD(\alpha,\theta)), \tag{2.8}$$

where $(T_t)_{t\geq 0}$ denotes the semigroup associated with $(\mathcal{A}, D(\mathcal{A}))$ on $L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))$.

Proof We first prove (2.7) by approximation. For $N \in \mathbf{N}$, denote $\varphi_1^N(x) := \sum_{i=1}^N x_i$, $x \in \overline{\nabla}_\infty$. Then $\lim_{N\to\infty} \|\varphi_1^N - \varphi_1\|_{L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))} = 0$. For $N > M$, we have that

$$\begin{aligned} \mathcal{A}(\varphi_1^N - \varphi_1^M, \varphi_1^N - \varphi_1^M) &\leq \frac{1}{2} \sum_{i=M+1}^N \int_{\overline{\nabla}_\infty} x_i PD(\alpha,\theta)(dx) \\ &\to 0 \text{ as } N, M \to \infty. \end{aligned}$$

Thus $\{\varphi_1^N\}_{N\in\mathbf{N}}$ is an $\mathcal{A}$-Cauchy sequence such that φ_1^N converges to φ_1 in $L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))$ as $N \to \infty$. By [7, Lemma 5.1.2], one finds that for any $T > 0$,

$$P_{PD(\alpha,\theta)}\left(\sum_{i=1}^N X_i(t) \text{ converges uniformly on } [0,T] \text{ as } N \to \infty\right) = 1.$$

Then $P_{PD(\alpha,\theta)}(t \to \sum_{i=1}^\infty X_i(t) \text{ is continuous}) = 1$. Since for any fixed t, $P_{PD(\alpha,\theta)}(\sum_{i=1}^\infty X_i(t) = 1) = PD(\alpha,\theta)\{\sum_{i=1}^\infty x_i = 1\} = 1$, (2.7) holds.

Next we turn to the proof of the ergodicity. In fact, it is enough to verify (2.8) by considering the following family of functions

$$\mathcal{T} := \{\varphi_{m_1} \cdots \varphi_{m_k} : m_1, \ldots, m_k \in \{2, 3, \ldots\}, k \geq 1\}.$$

Let $f \in \mathcal{T}$. By (2.4), there exists a constant $\lambda > 0$ and $g \in \mathcal{T}$ with degree(g) < degree(f) such that $Af = -\lambda f + g$. Then

$$T_t f = e^{-\lambda t} f + e^{-\lambda t} \int_0^t e^{\lambda s} T_s g ds, \quad \forall t \geq 0. \tag{2.9}$$

Taking integration on both sides of (2.9), we obtain by the symmetry of $(T_t)_{t \geq 0}$ that

$$\int_{\overline{\nabla}_\infty} f dPD(\alpha, \theta) = e^{-\lambda t} \int_{\overline{\nabla}_\infty} f dPD(\alpha, \theta) + e^{-\lambda t} \int_0^t e^{\lambda s} \left(\int_{\overline{\nabla}_\infty} g dPD(\alpha, \theta) \right) ds. \tag{2.10}$$

Subtracting (2.10) from (2.9), we get

$$\begin{aligned}
& \left\| T_t f - \int_{\overline{\nabla}_\infty} f dPD(\alpha, \theta) \right\|_{L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))} \\
\leq \; & e^{-\lambda t} \left\| f - \int_{\overline{\nabla}_\infty} f dPD(\alpha, \theta) \right\|_{L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))} \\
& + e^{-\lambda t} \int_0^t e^{\lambda s} \left\| T_s g - \int_{\overline{\nabla}_\infty} g dPD(\alpha, \theta) \right\|_{L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))} ds.
\end{aligned}$$

Then we can establish (2.8) by using induction on the degree of f.

□

Remark 2.3 *The unlabeled two parameter infinitely-many-neutral-alleles diffusion model considered in this section is directly motivated by [16]. In [16], Petrov used up/down Markov chains and an approximation method to construct the model. In this section, we use the theory of Dirichlet forms to give a completely different construction. Our construction might be more direct and simpler. More importantly, our observation that the model is given by the classical gradient Dirichlet form enables us to use this powerful analytic tool to generalize various basic properties of the infinitely-many-neutral-alleles diffusion model from the one parameter setting to the two parameter setting. The Dirichlet form constructed here differs from the Dirichlet form associated with the GEM process in [5] even on symmetric functions.*

There are many problems about the unlabeled two parameter infinitely-many-neutral-alleles diffusion model which deserve further investigation. As applications of Theorem 2.1, we present below several properties of the model via Dirichlet forms, including a sample path property, a result on large deviations, and the construction of models with selection.

Theorem 2.4 *Let X be the unlabeled two parameter infinitely-many-neutral-alleles diffusion model and let $k \geq 1$. Denote $A_k := \nabla_\infty \cap \{\sum_{i=1}^k x_i = 1\}$ and $D_k := \nabla_\infty \cap \{\sum_{i=1}^k x_i = 1\} \cap \{x_k > 0\}$.*

(i) If $\theta + \alpha k < 1$, then any subset of the $(k-1)$-dimensional simplex A_k with non-zero $(k-1)$-dimensional Lebesgue measure is hit by X with positive probability.
(ii) If $\theta + \alpha k \geq 1$, then D_k is not hit by X.

Proof We will establish (i) and (ii) by generalizing [19, Propositions 2 and 3] to the two parameter setting. The results are based on Fukushima's classical result (cf. [7, Theorem 4.2.1]), which says that a Borel set B is hit by X if and only if B has non-zero capacity. We use $\mathrm{Cap}(B)$ to denote the capacity of a Borel set B (cf. [7, Chapter 2]). Recall that

$$\mathrm{Cap}(B) = \inf_{\substack{B\subset A \\ A \text{ is open}}} \mathrm{Cap}(A)$$

and

$$\mathrm{Cap}(A) = \inf\left\{\mathcal{A}(u,u) + \int_{\overline{\nabla}_\infty} u^2 dPD(\alpha,\theta) : u \in D(\mathcal{A}), u \geq 1 \text{ on } A,\ PD(\alpha,\theta) - a.e.\right\}$$

if A is an open set.

For $k = 1$, let ν_1 denote the Dirac measure at $(1, 0, 0, \ldots)$. For $k \geq 2$, let $S_{k-1} := \{x \in \mathbf{R^{k-1}} : x_1 \geq \cdots \geq x_{k-1} \geq 0, \sum_{i=1}^{k-1} x_i \leq 1\}$ be equipped with $(k-1)$-dimensional Lebesgue measure and let ν_k denote the measure induced by the map $\xi : S_{k-1} \to A_k$, $\xi(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, 1 - \sum_{i=1}^{k-1} x_i, 0, 0, \ldots)$. In order to show that A_k has non-zero capacity if $\theta + \alpha k < 1$, it is enough to show that there is a dimension-independent constant $c > 0$ such that

$$\left(\int_{\overline{\nabla}_\infty} u d\nu_k\right)^2 \leq c\left(\mathcal{A}(u,u) + \int_{\overline{\nabla}_\infty} u^2 dPD(\alpha,\theta)\right), \quad \forall u \in D(\mathcal{A}) \cap C(\overline{\nabla}_\infty). \tag{2.11}$$

For $n \geq k$, denote $B_k = S_n \cap \{\sum_{i=1}^k x_i = 1\}$ and use μ_n, ν_{kn} to denote respectively the image measures of $PD(\alpha,\theta)$, ν_k under the projection of ∇_∞ onto the first n coordinates. Then (2.11) is equivalent to

$$\left(\int_{B_k} f d\nu_{kn}\right)^2 \leq c\int_{S_n}\left(\frac{1}{2}\langle \nabla f, a\nabla f\rangle + f^2\right) d\mu_n, \quad \forall f \in C_0^\infty(\mathbf{R^n}). \tag{2.12}$$

To prove (2.12), we will make use of a new coordinate system. Denote

$$r = 1 - \sum_{i=1}^k (x_i - x_{k+1})$$

and

$$S'_{n-k} = \{x \in \mathbf{R^{n-k}} : x_1 \geq \cdots \geq x_{n-k} \geq 0, (k+1)x_1 + x_2 + \cdots + x_{n-k} \leq 1\}.$$

Consider the map $\phi : S_n \cap (0 < r < 1) \to S_{k-1} \times (0,1) \times S'_{n-k}$,

$$\begin{aligned} \phi(x_1, \ldots, x_n) &= (u_1, \ldots, u_{k-1}, u_k, u_{k+1}, \ldots, u_n) \\ &= \left(\frac{x_1 - x_{k+1}}{r}, \ldots, \frac{x_{k-1} - x_{k+1}}{r}, r, \frac{x_{k+1}}{r}, \ldots, \frac{x_n}{r} \right). \end{aligned}$$

ϕ is a one-to-one onto map with the inverse

$$\begin{aligned} x_1 &= (1 - u_k)u_1 + u_k u_{k+1}, \\ &\vdots \\ x_{k-1} &= (1 - u_k)u_{k-1} + u_k u_{k+1}, \\ x_k &= (1 - u_k)(1 - (u_1 + \cdots + u_{k-1})) + u_k u_{k+1}, \\ x_{k+1} &= u_{k+1} u_k, \\ &\vdots \\ x_n &= u_n u_k. \end{aligned} \tag{2.13}$$

One can check that the Jocobian of ϕ^{-1} is $(1 - u_k)^{k-1} u_k^{n-k}$.

Denote by h the density function of μ_n with respect to n-dimensional Lebesgue measure. By [9, Theorem 5.5], we have that

$$h(x_1, \ldots, x_n) = c_{n,\alpha,\theta} \prod_{j=1}^{n} x_j^{-(\alpha+1)} \left(1 - \sum_{i=1}^{n} x_i\right)^{\theta+\alpha n-1} \rho_{\alpha,\theta+\alpha n}\left(\frac{1 - \sum_{i=1}^{n} x_i}{x_n}\right), \tag{2.14}$$

where

$$c_{n,\alpha,\theta} = \prod_{i=1}^{n} \frac{\Gamma(\theta + 1 + (i-1)\alpha)}{\Gamma(1-\alpha)\Gamma(\theta + i\alpha)} = \begin{cases} \theta^n, & \alpha = 0, \\ \frac{\Gamma(\theta+1)\Gamma(\theta/\alpha+n)\alpha^{n-1}}{\Gamma(\theta+\alpha n)\Gamma(\theta/\alpha+1)\Gamma(1-\alpha)^n}, & 0 < \alpha < 1 \end{cases}$$

and $\rho_{\alpha,\theta+\alpha n}$ is a two parameter version of Dickman's function, i.e.,

$$\rho_{\alpha,\theta+\alpha n}(s) = P(sV_1^{\alpha,\theta+\alpha n} < 1), \quad s \geq 0.$$

Note that $(1 - \sum_{i=1}^{n} x_i)/x_n$ is only a function of $u_{k+1}, \ldots, u_n$ by (2.13). Hence we obtain by (2.14) that the joint density of $(u_1, \ldots, u_n)$ under $\mu_n \circ \phi^{-1}$ is given by

$$h(u_1, \ldots, u_n) = \psi(u_{k+1}, \ldots, u_n)(x_1 \cdots x_k)^{-(\alpha+1)}(1 - u_k)^{k-1} u_k^{\theta+\alpha k-1}$$

for some function ψ. Note that the product $x_1 \cdots x_k$ is only a function of $u_1, \ldots, u_{k+1}$. Hence the conditional density satisfies

$$h(u_1, \ldots, u_k | u_{k+1}, \ldots, u_n) = \frac{(1 - u_k)^{k-1} u_k^{\theta+\alpha k-1}(x_1 \cdots x_k)^{-(\alpha+1)}}{\int_0^1 \cdots \int_0^1 (1 - u_k)^{k-1} u_k^{\theta+\alpha k-1}(x_1 \cdots x_k)^{-(\alpha+1)} du_1 \cdots du_k}.$$

Therefore, there exists a constant $c_1 > 0$, which depends on α, θ, k and $\varepsilon > 0$ but not n, such that on $\{u_{k+1} \geq \varepsilon\}$ we have

$$h(u_1, \ldots, u_k | u_{k+1}, \ldots, u_n) \geq c_1 (1 - u_k)^{k-1} u_k^{\theta + \alpha k - 1}.$$

We now prove (2.12). Without loss of generality, we assume that f vanishes for $r(= u_k) \geq 1/2$. In fact, if this condition is not satisfied, we may obtain (2.12) by multiplying f by a finite-dimensional smooth function $\gamma \in \mathcal{B}$, which is equal to 1 when $r = 0$ and vanishes for $r \geq 1/2$. Denote by $\sigma(du_{k+1}, \ldots, du_n)$ the distribution of $u_{k+1}, \ldots, u_n$ under $\mu_n \circ \phi^{-1}$. We choose $\varepsilon > 0$ such that $p := \sigma(u_{k+1} > \varepsilon) > 0$. Define $A = S_{k-1} \times (0,1) \times S'_{n-k}$ and $A_\varepsilon = S_{k-1} \times (0,1) \times [S'_{n-k} \cap (u_{k+1} > \varepsilon)]$. To simplify notation, we denote by I the integral on the left hand side of (2.12). Then

$$\begin{aligned}
|I| &= \left| \int_{S_{k-1}} f\left(u_1, u_2, \ldots, u_{k-1}, 1 - \sum_{i=1}^{k-1} u_i, 0, \ldots, 0\right) du_1 \cdots du_{k-1} \right| \\
&= \left| -\int_0^1 \int_{S_{k-1}} \partial_k f((1-u_k)u_1 + u_k u_{k+1}, \ldots, u_n u_k) du_1 \cdots du_{k-1} du_k \right| \\
&= \left| \frac{1}{p} \int_{A_\varepsilon} \partial_k (f \circ \phi^{-1}(u)) du_1 \ldots du_k \sigma(du_{k+1} \cdots du_n) \right| \\
&\leq \frac{1}{p} \left(\int_{A_\varepsilon} u_k [\partial_k (f \circ \phi^{-1}(u))]^2 u_k^{\theta + \alpha k - 1} du_1 \ldots du_k \sigma(du_{k+1} \cdots du_n) \right)^{1/2} \\
&\quad \times \left(\int_{A_\varepsilon} u_k^{-(\theta + \alpha k)} du_1 \ldots du_k \sigma(du_{k+1} \cdots du_n) \right)^{1/2} \\
&\leq C \left(\int_A u_k [\partial_k (f \circ \phi^{-1}(u))]^2 h(u_1, \ldots, u_k | u_{k+1}, \ldots, u_n) du_1 \ldots du_k \sigma(du_{k+1} \cdots du_n) \right)^{1/2} \\
&\leq C \left(\int_A \langle \nabla f(\phi^{-1}(u)), a(\phi^{-1}(u)) \nabla f(\phi^{-1}(u)) \rangle \mu_n \circ \phi^{-1}(du) \right)^{1/2} \\
&= C \int_{S_n} \langle \nabla f, a(x) \nabla f \rangle \mu_n(dx) \\
&= C \mathcal{A}(f, f),
\end{aligned}$$

which proves (2.12). Here C denotes a generic constant whose value may change from line to line but independent of n. For the last inequality we have used the following estimate

$$u_k [\partial_k (f \circ \phi^{-1}(u))]^2 \leq C \langle \nabla f(\phi^{-1}(u)), a(\phi^{-1}(u)) \nabla f(\phi^{-1}(u)) \rangle \text{ for } u_k \leq \frac{1}{2},$$

which is given by [19, Lemma 3].

We now establish (ii). For $k = 1$, by (2.14), we have that

$$h(x_1) \le c x_1^{-(\alpha+1)}(1-x_1)^{\theta+\alpha-1}$$

for some constant $c > 0$. For $n \ge 1$, we choose $g_n \in C^\infty(\mathbf{R})$ satisfying $g_n(x) = 0$ if $x \le n$, $g_n(x) = 1$ if $x \ge 2n$, and $0 \le g_n(x) \le 1$ for all $x \in \mathbf{R}$. Also, we require that $g_n'(x) \le 2/n$ for all $x \in \mathbf{R}$. Set $u_n = g_n \circ \ln((1-x_1)^{-1})$. Then, if $\theta + \alpha \ge 1$, we have that

$$\begin{aligned}
\mathrm{Cap}(A_1) &\le \mathcal{A}(u_n, u_n) + \int_{\overline{\nabla}_\infty} u_n^2 dPD(\alpha,\theta) \\
&\le \frac{c}{2}\int_{1-\exp(-n)}^{1-\exp(-2n)} [g_n'(\ln(1-x_1)^{-1}]^2(1-x_1)^{-2}x_1(1-x_1) \\
&\quad \cdot x_1^{-(\alpha+1)}(1-x_1)^{\theta+\alpha-1}dx_1 + PD(\alpha,\theta)\{x_1 \ge (1-\exp(-n))\} \\
&\le \frac{2^{1+\alpha}c}{n^2}\int_{1/2}^{1-\exp(-2n)} (1-x_1)^{\theta+\alpha-2}dx_1 + PD(\alpha,\theta)\{x_1 \ge (1-\exp(-n))\} \\
&\to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

For $k \ge 2$, we fix an $\varepsilon > 0$. Choose $w \in C^\infty(\mathbf{R})$ satisfying $w(x) = 0$ if $x \le \varepsilon$ and $w(x) = 1$ if $x > 2\varepsilon$. Let $s = \sum_{i=1}^k x_i$ and define $u_n = g_n \circ \ln((1-s)^{-1})$. Set $v_n(x) = u_n(x)w(x_k)$. Note that $v_n = 1$ on an open subset containing $(s = 1) \cap (x_k \ge 2\varepsilon)$ and v_n vanishes if $x_k \le \varepsilon$. For a large n, the support of v_n is contained in the set $(1-s)x_k^{-1} \le 1$. Moreover, we obtain by (2.14) that there exists a constant $C(\varepsilon,\alpha,\theta) > 0$ such that

$$h(x_1,\ldots,x_k) \le C(\varepsilon,\alpha,\theta)(1-s)^{\theta+\alpha k-1} \text{ on the support of } v_n. \tag{2.15}$$

Since

$$\nabla(u_n w) = w\nabla u_n + u_n \nabla w,$$

we get

$$\mathcal{A}(v_n, v_n) \le \int_{\overline{\nabla}_\infty} w^2\langle \nabla u_n, a\nabla u_n\rangle dPD(\alpha,\theta) + \int_{\overline{\nabla}_\infty} u_n^2 \langle w, a\nabla w\rangle dPD(\alpha,\theta). \tag{2.16}$$

Similar to the $k = 1$ case, we can use (2.15) to show that the first term of the right hand side of (2.16) tends to 0 as $n \to \infty$ if $\theta + \alpha k \ge 1$. Since $u_n^2\langle w, a\nabla w\rangle \to 0$ as $n \to \infty$, $PD(\alpha,\theta)$-a.e., we conclude that $\mathrm{Cap}((s = 1) \cap (x_k \ge 2\varepsilon)) = 0$. Since $\varepsilon > 0$ is arbitrary, $\mathrm{Cap}(D_k) = 0$. The proof is complete. $\square$

Remark 2.5 *In [19], Schmuland showed that, in the one parameter model, A_k is hit by X if and only if $\theta < 1$. The phase transition is between infinite and any finite alleles and occurs* *at $\theta = 1$. In the two parameter model, our Theorem 2.4 shows that the phase transition is between infinite and certain finite alleles (number of alleles is no more than $k_c = [\frac{1-\theta}{\alpha}]$). The maximum number of finite alleles can be hit is $[\frac{1}{\alpha}]$ corresponding to $\theta = 0$. So the number of alleles is either infinity or less than or equal to $[\frac{1}{\alpha}]$. This creates a barrier between finite alleles and infinite alleles. The results indicate an essential difference between the one parameter model and the two parameter model, which deserves a better explanation in terms of coalescent.*

We denote by $(L, D(L))$ the generator of the Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ (cf. Theorem 2.1) on $L^2(\overline{\nabla}_\infty; PD(\alpha, \theta))$. Note that $Lu = Au$ for all $u \in \mathcal{P}$, where A is defined in (2.1). For $m \geq 2$, define $\lambda_m = m(m-1+\theta)/2$ and denote by $\pi(m)$ the number of partitions of the integer m.

Proposition 2.6 *The spectrum of $(L, D(L))$ consists of the eigenvalues $\{0, -\lambda_2, -\lambda_3, \ldots\}$. 0 is a simple eigenvalue and for each $m \geq 2$, the multiplicity of $-\lambda_m$ is $\pi(m) - \pi(m-1)$.*

Proof The spectrum characterization has been obtained in [16] using the up/down Markov chains and approximation. However, a bit more transparent derivation can be given using our (2.4). Note that (2.4) is a consequence of Pitman's sampling formula and already indicates the structure of the spectrum of $(L, D(L))$. With [3, (1.4)] replaced with our (2.4), Proposition 2.6 then follows from an argument similar to that used in the proof of [3, Theorem 2.3].

□

We now present a result on the large deviations for occupation time process. It shows that the Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ appears naturally as the function governing the large deviations. Define

$$L_t(C) := \frac{1}{t}\int_0^t 1_C(X_s)ds, \quad \forall C \in \mathcal{B}(\overline{\nabla}_\infty),$$

where $\mathcal{B}(\overline{\nabla}_\infty)$ denotes the Borel σ-algebra of $\overline{\nabla}_\infty$. We equip $\mathcal{M}_1(\overline{\nabla}_\infty)$ with the τ-topology, which is generated by open sets of the form

$$U(\nu; \varepsilon, F) := \left\{\mu \in \mathcal{M}_1(\overline{\nabla}_\infty) \middle| \left|\int F d\mu - \int F d\nu\right| < \varepsilon\right\},$$

where $\varepsilon > 0$, $\nu \in \mathcal{M}(\overline{\nabla}_\infty)$ and $F \in B_b(\overline{\nabla}_\infty)$, the set of bounded Borel measurable functions on $\overline{\nabla}_\infty$. The next result follows from [15, Theorems 1 and 2].

Proposition 2.7 *Let U be a τ-open subset and K be a τ-compact subset of $\mathcal{M}_1(\overline{\nabla}_\infty)$. Then for $\mathcal{A}$-q.e. $x \in \overline{\nabla}_\infty$ we have that*

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x[L_t \in U] \geq -\inf\{\mathcal{A}(u,u) | u \in D(\mathcal{A}), u^2 PD(\alpha,\theta) \in U\}$$

and

$$\inf\left\{ \sup_{x\in\overline{\nabla}_\infty\setminus N} \limsup_{t\to\infty} \frac{1}{t} \log P_x[L_t \in K] \middle| N \subset \overline{\nabla}_\infty, N \text{ is } \mathcal{A} - \text{exceptional} \right\}$$
$$\leq -\inf\{\mathcal{A}(u,u) | u \in D(\mathcal{A}), u^2 PD(\alpha,\theta) \in K\}.$$

Finally, we would like to point out that the infinitely-many-neutral-alleles diffusion model considered in this section can be easily extended to include interactive selection.

Proposition 2.8 *Let $\rho \in L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))$ satisfying $\rho^2 \geq \varepsilon > 0$, $PD(\alpha,\theta)$-a.e., or $\varphi \in D(\mathcal{A})$ and $\rho > 0$, $PD(\alpha,\theta)$-a.e. Then the perturbed bilinear form*

$$\mathcal{A}^\varphi(u,v) = \frac{1}{2}\int_{\overline{\nabla}_\infty} \langle \nabla u, a(x)\nabla v\rangle \rho^2 dPD(\alpha,\theta), \quad u,v \in \mathcal{P}$$

is closable on $L^2(\overline{\nabla}_\infty; \rho^2 PD(\alpha,\theta))$ and its closure $(\mathcal{A}^\rho, D(\mathcal{A}^\rho))$ is a regular local Dirichlet form.

Proof First, we consider the case that $\rho^2 \geq \varepsilon > 0$, $PD(\alpha,\theta)$-a.e. Let $\{u_n \in \mathcal{P}\}_{n\in\mathbf{N}}$ be a sequence satisfying $u_n \to 0$ in $L^2(\overline{\nabla}_\infty; \rho^2 PD(\alpha,\theta))$ as $n \to \infty$ and $\mathcal{A}^\rho(u_n - u_m, u_n - u_m) \to 0$ as $n,m \to \infty$. Then the strict positivity of ρ^2 implies that $\{u_n\}_{n\in\mathbf{N}}$ is an $\mathcal{A}$-Cauchy sequence and $u_n \to 0$ in $L^2(\overline{\nabla}_\infty; PD(\alpha,\theta))$. Hence the closability of $(\mathcal{A},\mathcal{P})$ implies that

$$\lim_{n\to\infty} \int_{\overline{\nabla}_\infty} \langle \nabla u_n, \nabla u_n\rangle dPD(\alpha,\theta) = 0.$$

Thus $\lim_{n\to\infty} \mathcal{A}^\rho(u_n,u_n) = 0$ by Fatou's lemma. Therefore $(\mathcal{A}^\rho,\mathcal{P})$ is closable.

Now we consider the case that $\rho \in D(\mathcal{A})$ and $\rho > 0$, $PD(\alpha,\theta)$-a.e. Let $(X, P_{PD(\alpha,\theta)})$ be the Markov process associated with the Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$. Since $\rho \in D(\mathcal{A})$, it has a quasi-continuous version (cf. [7, Theorem 2.1.7]), which is denoted by $\tilde{\rho}$. For $n \in \mathbf{N}$, we define $\tau_n := \inf\{t > 0 : \tilde{\rho}(X_t) \leq 1/n\}$ and $\tau := \lim_{n\to\infty} \tau_n$. On $\{t < \tau\}$, we define

$$M_t^{[\ln \rho]} := M_t^{[\ln(\rho\vee(1/n))]}, \quad \text{if } t \leq \tau_n,$$

where $M_t^{[\eta]}$ denotes the martingale part of the Fukushima decomposition of the additive functional $\tilde{\eta}(X_t) - \tilde{\eta}(X_0)$ if $\eta \in D(\mathcal{A})$ (cf. [7, Theorem 5.2.2]). We denote by X^ρ the Girsanov transform of X with the multiplicative functional $L_t^{[\rho]} := \exp(M_t^{[\ln\rho]} - \frac{1}{2}\langle M^{[\ln\rho]}\rangle_t)1_{t<\tau}$, where $\langle\cdot\rangle$ denotes the quadratic variation of a martingale. Then X^ρ is associated with a Dirichlet form on $L^2(\overline{\nabla}_\infty; \rho^2 PD(\alpha,\theta))$ that extends $(\mathcal{A}^\rho, \mathcal{P})$. Therefore $(\mathcal{A}^\rho, \mathcal{P})$ is closable. It is easy to check that its closure $(\mathcal{A}^\rho, D(\mathcal{A}^\rho))$ is a regular local Dirichlet form. The proof is complete.

□

3 Labeled Model

In this section, we will construct measure-valued processes associated with the two-parameter Dirichlet process through the study of a general bilinear from. We are successful in two particular cases (cf. Theorems 3.2 and 3.5 below).

Let S be a locally compact, separable metric space and $E := \mathcal{M}_1(S)$ be the space of probability measures on the Borel σ-algebra $\mathcal{B}(S)$ in S. Following (1.2), the two parameter Dirichlet process $\Pi_{\alpha,\theta,\nu_0}$ satisfies

$$\Pi_{\alpha,\theta,\nu_0}(A) = P\left(\sum_{i=1}^{\infty} \rho_i \delta_{\xi_i} \in A\right)$$

for any $A \in \mathcal{B}(E)$, the Borel σ-algebra of E. We denote by E_P the expectation with respect to P. Set

$$\mathcal{F} := \mathrm{Span}\{\langle f_1, \mu\rangle \cdots \langle f_k, \mu\rangle : f_1, \ldots, f_k \in C_b(S), k \in \mathbf{N}\}.$$

Consider the following symmetric bilinear form

$$\mathcal{E}(u,v) = \frac{1}{2}\int_E \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu \Pi_{\alpha,\theta,\nu_0}(d\mu), \quad u, v \in \mathcal{F}. \tag{3.1}$$

Recall that $\nabla u(\mu)$ is the function

$$x \longrightarrow \frac{\partial u}{\partial \mu(x)}(\mu) = \lim_{\varepsilon\to 0+} \frac{u((1-\varepsilon)\mu + \varepsilon\delta_x) - u(\mu)}{\varepsilon}$$

and $\langle f, g\rangle_\mu := \int fg d\mu - (\int f d\mu)(\int g d\mu)$. Note that $\Gamma(u,v) := \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu$ is a square field operator. If $(\mathcal{E}, \mathcal{F})$ is closable on $L^2(E; \Pi_{\alpha,\theta,\nu_0})$, then following the argument of ([21, Lemma 7.5 and Proposition 5.11]), one can show that the closure $(\mathcal{E}, D(\mathcal{E}))$ of $(\mathcal{E}, \mathcal{F})$ is a quasi-regular local Dirichlet form. Hence, there exists an essentially unique diffusion process X which is associated with $(\mathcal{E}, D(\mathcal{E}))$ (cf. [13, Theorems IV.6.4 and V.1.11]). This diffusion process is called the labeled two parameter infinitely-many-neutral-alleles diffusion model. However, quite different from the unlabeled case, we find that the closability problem of

$(\mathcal{E}, \mathcal{F})$ is challenging. To understand this point, let us consider the case that the type space S is finite. This is equivalent to projecting every μ in $\mathcal{M}_1(S)$ to $\{\mu(J_i) : i = 1, 2, \ldots, n\}$ for certain finite partition $\{J_i : i = 1, 2, \ldots, n\}$ of space S.

Let $\{\sigma(t) : t \geq 0, \sigma_0 = 0\}$ be a subordinator with Lévy measure $x^{-(1+\alpha)} e^{-x} d\,x$, $x > 0$, and $\{\gamma(t) : t \geq 0, \gamma_0 = 0\}$ be a gamma subordinator that is independent of $\{\sigma_t : t \geq 0, \sigma_0 = 0\}$ and has Lévy measure $x^{-1} e^{-x} d\,x$, $x > 0$. The next result follows from [18, Proposition 21] and the construction outlined on [17, Page 254].

Proposition 3.1 (Pitman and Yor) *Let*

$$\gamma(\alpha, \theta) = \frac{\alpha \gamma(\frac{\theta}{\alpha})}{\Gamma(1-\alpha)}.$$

For each $n \geq 1$, and each partition $J_i : i = 1, \ldots, n$ of S, let

$$a_i = \nu_0(J_i), \quad i = 1, \ldots, n,$$

and

$$Z_{\alpha,\theta}(t) = \sigma(\gamma(\alpha, \theta) t), \quad t \geq 0.$$

Then the distribution of $(\Xi_{\alpha,\theta,\nu_0}(J_1), \ldots, \Xi_{\alpha,\theta,\nu_0}(J_n))$ is the same as the distribution of

$$\left(\frac{Z_{\alpha,\theta}(a_1)}{Z_{\alpha,\theta}(1)}, \ldots, \frac{Z_{\alpha,\theta}(\sum_{j=1}^{n} a_j) - Z_{\alpha,\theta}(\sum_{j=1}^{n-1} a_j)}{Z_{\alpha,\theta}(1)} \right).$$

In general, the distribution function of $(\Xi_{\alpha,\theta,\nu_0}(J_1), \ldots, \Xi_{\alpha,\theta,\nu_0}(J_n))$ cannot be explicitly identified. The exception is the case that $|S| = 2$, i.e., S contains only two elements.

Theorem 3.2 *Suppose that $|S| = 2$. Then $(\mathcal{E}, \mathcal{F})$ is closable on $L^2(E; \Pi_{\alpha,\theta,\nu_0})$. Moreover, its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular local Dirichlet form, which is associated with a diffusion process on E.*

Proof We assume without loss of generality that $0 < \alpha < 1$ and $\theta > -\alpha$. It is enough to show that $(\mathcal{E}, \mathcal{F})$ is closable on $L^2(E; \Pi_{\alpha,\theta,\nu_0})$. Once this is established, the proof of the last assertion of the theorem is easy. Set $S = \{1, 2\}$, $E = [0, 1]$ and $p = 1 - \bar{p} = \nu_0(1)$. Denote by dx the Lebesgue measure on $[0, 1]$. Then $\mathcal{F}$ is the set of all polynomials restricted to $[0, 1]$ and

$$\mathcal{E}(u, v) = \frac{1}{2} \int_0^1 x(1-x) u'(x) v'(x) \Pi_{\alpha,\theta,\nu_0}(dx), \quad u, v \in \mathcal{F}.$$

First, we consider the case that $\theta = 0$. It is known (cf. [12]) that

$$\Pi_{\alpha,0,\nu_0}(dx) = q_{\alpha,0}(x) dx$$

with

$$q_{\alpha,0}(x) = \frac{p\bar{p}\sin(\alpha\pi)x^{\alpha-1}(1-x)^{\alpha-1}}{\pi[\bar{p}^2x^{2\alpha}+p^2(1-x)^{2\alpha}+2p\bar{p}x^{\alpha}(1-x)^{\alpha}\cos(\alpha\pi)]}, \quad 0 \le x \le 1.$$

Define

$$\begin{aligned} Lu(x) &= \frac{1}{2}x(1-x)u''(x) + \frac{\alpha}{2}u'(x)\left[(1-2x)\right. \\ &\left. - \frac{2\bar{p}^2x^{2\alpha}(1-x) - 2p^2(1-x)^{2\alpha}x + 2p\bar{p}(1-2x)x^{\alpha}(1-x)^{\alpha}\cos(\alpha\pi)}{\bar{p}^2x^{2\alpha}+p^2(1-x)^{2\alpha}+2p\bar{p}x^{\alpha}(1-x)^{\alpha}\cos(\alpha\pi)}\right]. \end{aligned}$$

Then one can check that $Lu \in L^2(E;\Pi_{\alpha,0,\nu_0})$ for any $u \in \mathcal{F}$ and

$$\mathcal{E}(u,v) = -\int_0^1 (Lu)v\,d\Pi_{\alpha,0,\nu_0}, \quad u,v \in \mathcal{F}.$$

Therefore $(\mathcal{E},\mathcal{F})$ is closable on $L^2(E;\Pi_{\alpha,0,\nu_0})$ by ([13, Proposition 3.3]).

We now consider the case that $\theta > 0$. To this end, we need to use a recent result of James et al. By [10, Example 5.1] (cf. also [10, Theorems 3.1 and 5.3]), we have that

$$\Pi_{\alpha,\theta,\nu_0}(dx) = q_{\alpha,\theta}(x)dx, \quad q_{\alpha,\theta}(x) = \theta\int_0^x (x-t)^{\theta-1}\tilde{\Delta}_{\alpha,\theta+1}(t)dt.$$

Here

$$\tilde{\Delta}_{\alpha,\theta+1}(t) = \frac{\gamma_{\alpha-1}(t)\sin(\rho_{\alpha,\theta}(t)) - \zeta_{\alpha-1}(t)\cos(\rho_{\alpha,\theta}(t))}{\pi[\zeta_\alpha^2(t)+\gamma_\alpha^2(t)]^{(\theta+\alpha)/2\alpha}}$$

with

$$\gamma_d(t) = \cos(d\pi)t^d\bar{p} + (1-t)^dp, \quad \zeta_d(t) = \sin(d\pi)t^d\bar{p}, \quad d > -1$$

and

$$\rho_{\alpha,\theta}(t) = \frac{\theta}{\alpha}\arctan\frac{\zeta_\alpha(t)}{\gamma_\alpha(t)} + \frac{\pi\theta}{\alpha}1_{\Gamma_\alpha}(t), \quad \Gamma_\alpha = \{t \in \mathbf{R}^+ : \gamma_\alpha(t) < 0\}.$$

When $\theta > 1$, the expression above can be rewritten as

$$q_{\alpha,\theta}(x) = (\theta-1)\int_0^x (x-t)^{\theta-2}\Delta_{\alpha,\theta}(t)dt$$

with

$$\Delta_{\alpha,\theta}(t) = \frac{\sin\left(\frac{\theta}{\alpha}\arctan\left(\frac{\bar{p}\sin(\alpha\pi)t^{\alpha}}{\bar{p}\cos(\alpha\pi)t^{\alpha}+p(1-t)^{\alpha}}\right) + \frac{\pi\theta}{\alpha}1_{\Gamma_\alpha}(t)\right)}{\pi\{\bar{p}^2t^{2\alpha}+p^2(1-t)^{2\alpha}+2\bar{p}p\cos(\alpha\pi)t^{\alpha}(1-t)^{\alpha}\}^{\theta/2\alpha}},$$

where $\Gamma_\alpha = \emptyset$ if $\alpha \in (0,1/2]$, whereas $\Gamma_\alpha = (0, v_\alpha/(1+v_\alpha))$ with $v_\alpha = (-p/(\bar{p}\cos(\alpha\pi)))^{1/\alpha}$ if $\alpha \in (1/2,1)$.

Define

$$Lu(x)=\frac{1}{2}x(1-x)u^{''}(x)+\frac{1}{2}u'(x)[(1-2x)+x(1-x)q'_{\alpha,\theta}(x)/q_{\alpha,\theta}(x)].$$

Then one can check that $Lu\in L^2(E;\Pi_{\alpha,\theta,\nu_0})$ for any $u\in\mathcal{F}$ and

$$\mathcal{E}(u,v)=-\int_0^1(Lu)vd\Pi_{\alpha,\theta,\nu_0},\quad u,v\in\mathcal{F}.$$

Therefore $(\mathcal{E},\mathcal{F})$ is closable on $L^2(E;\Pi_{\alpha,\theta,\nu_0})$. The proof is complete. $\square$

From Theorem 3.2, one can see that even for the one-dimension case, the generator of the labeled two parameter infinitely-many-neutral-alleles diffusion model is very complicated. This indicates an essential difference between the unlabeled model and the labeled model. More importantly, it explains why it is so difficult to construct the labeled two parameter infinitely-many-neutral-alleles diffusion model only using the ordinary methods that are successful for the one parameter case. So far we have not been able to solve the closability problem for the general case. In what follows, we will give further results on the blinear form $(\mathcal{E},\mathcal{F})$ and hope they can shed some light on the problem.

Set

$$\mathcal{G}:=\{G(\mu)=g(\langle f_1,\mu\rangle,\cdots,\langle f_k,\mu\rangle),g\in C_b^{\infty}(\mathbf{R^k}),f_1,\dots,f_k\in C_b(S)\}.$$

Let $f\in C_b(S)$ satisfying $\nu_0(f)=0$. We introduce the linear functional $B_f:\mathcal{G}\to\mathbf{R}$ defined by

$$B_f(G)=\sum_{s=1}^{\infty}\int G\left(\sum_{i=1}^{\infty}\rho_i\delta_{\xi_i}\right)f(\xi_s)dP,\quad G\in\mathcal{G}. \tag{3.2}$$

Note that (3.2) is well-defined since for $0<\alpha<1$ and $G(\mu)=g(\langle f_1,\mu\rangle,\cdots,\langle f_k,\mu\rangle)\in\mathcal{G}$, we have the following estimate:

$$\begin{aligned}\left|\int G\left(\sum_{i=1}^{\infty}\rho_i\delta_{\xi_i}\right)f(\xi_s)dP\right| &= \left|\int\left\{G\left(\sum_{i=1}^{\infty}\rho_i\delta_{\xi_i}\right)-G\left(\sum_{i\neq s}^{\infty}\rho_i\delta_{\xi_i}\right)\right\}f(\xi_s)dP\right|\\ &\le (\|\partial_1 g\cdot f_1\|_\infty+\cdots+\|\partial_k g\cdot f_k\|_\infty)\|f\|_\infty\int\rho_s dP\\ &\le \frac{c}{s^{1/\alpha}}\end{aligned}$$

by [18, (50)], where $c>0$ is a constant which is independent of s.

Proposition 3.3 *Let $f \in C_b(S)$. Then, for any $v(\mu) = \langle g_1, \mu\rangle \cdots \langle g_l, \mu\rangle$ with $g_1, \ldots, g_l \in C_b(S)$, we have that*

$$\mathcal{E}(\langle f, \mu\rangle, v) = \frac{\theta}{2}\int_E \langle f - \nu_0(f), \mu\rangle \cdot v\Pi_{\alpha,\theta,\nu_0}(d\mu) + \frac{\alpha}{2} B_{f-\nu_0(f)}(v).$$

Proof Let $f \in C_b(S)$ and $v(\mu) = \langle g_1, \mu\rangle \cdots \langle g_l, \mu\rangle$ with $g_1, \ldots, g_l \in C_b(S)$. Without loss of generality we assume that $\nu_0(f) = 0$. Then

$$\begin{aligned}
\mathcal{E}(\langle f, \mu\rangle, v) &= \frac{1}{2}\int_E \langle f, \nabla v(\mu)\rangle_\mu \Pi_{\alpha,\theta,\nu_0}(d\mu) \\
&= \frac{1}{2}\int_E \sum_{i=1}^{l} (\langle f g_i, \mu\rangle \prod_{j\neq i} \langle g_j, \mu\rangle)\Pi_{\alpha,\theta,\nu_0}(d\mu) - \frac{l}{2}\int_E \langle f, \mu\rangle \prod_{j=1}^{l} \langle g_j, \mu\rangle \Pi_{\alpha,\theta,\nu_0}(d\mu) \\
&= \frac{\theta}{2}\int_E \langle f, \mu\rangle \prod_{j=1}^{l} \langle g_j, \mu\rangle \Pi_{\alpha,\theta,\nu_0}(d\mu) + \left\{ \frac{1}{2}\int_E \sum_{i=1}^{l} (\langle f g_i, \mu\rangle \prod_{j\neq i} \langle g_j, \mu\rangle)\Pi_{\alpha,\theta,\nu_0}(d\mu) \right. \\
&\quad \left. - \frac{\theta + l}{2}\int_E \langle f, \mu\rangle \prod_{j=1}^{l} \langle g_j, \mu\rangle \Pi_{\alpha,\theta,\nu_0}(d\mu) \right\}.
\end{aligned} \tag{3.3}$$

Set

$$\mathcal{H} := \{\varphi(\mu) = \langle g, \mu^l\rangle : g \in C_b(S^l), l \in \mathbf{N}\}.$$

For $l \in \mathbf{N}$, let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be an unordered partition of the set $\{1, 2, \ldots, l\}$. We associate each w with β_w boxes, $1 \leq w \leq n$. Assign the integers $1, 2, \ldots, l$ to the l boxes, each box containing exactly one integer. We denote such an arrangement by A. Two arrangements are said to be the same if they have the same partition $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and each w, $1 \leq w \leq n$, is assigned the same (unordered) set of integers. Define a map $\tau : \{1, 2, \ldots, l\} \to \{1, 2, \ldots, n\}$ by $\tau(j) = w$ if j is assigned to w. Then, we introduce a linear functional $C_f : \mathcal{H} \to \mathbf{R}$ defined by

$$\begin{aligned}
&C_f(\langle g, \mu^l\rangle) \\
&\quad = \sum_{\text{distinct } A} \left(\frac{(-\frac{\theta}{\alpha})(-\frac{\theta}{\alpha} - 1)\cdots(-\frac{\theta}{\alpha} - (n-1)) \prod_{w=1}^{n} (-\alpha)(1-\alpha)\cdots(\beta_w - 1 - \alpha)}{\theta(\theta+1)\cdots(\theta + l - 1)} \right. \\
&\qquad \left. \cdot \int_{S^n} g(x_{\tau(1)}, \ldots, x_{\tau(l)}) \sum_{s=1}^{n} f(x_s) \nu_0^n(dx_1 \times \cdots \times dx_n) \right),
\end{aligned} \tag{3.4}$$

where the value of the right hand side is obtained by continuity when $\alpha = 0$ or $\theta = 0$. By (3.3), Pitman's sampling formula, and comparing the arrangements for sizes l and $l + 1$, we find that

$$\mathcal{E}(\langle f, \mu\rangle, v) = \frac{\theta}{2}\int_E \langle f, \mu\rangle \cdot v\Pi_{\alpha,\theta,\nu_0}(d\mu) + \frac{\alpha}{2} C_f(v).$$

Let $g \in C_b(S^l)$. Then by (3.4), the assumption that $\nu_0(f) = 0$ and the dominated convergence theorem, we get

$$\begin{aligned}
C_f(\langle g, \mu^l \rangle) &= E_P \left\{ \sum_{\text{distinct } (i_1, i_2, \ldots, i_l)} \left(\rho_{i_1} \rho_{i_2} \ldots \rho_{i_l} g(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_l}) \sum_{\text{distinct } s \in \{i_1, i_2, \ldots, i_l\}} f(\xi_s) \right) \right\} \\
&= \int \sum_{\text{distinct } (i_1, i_2, \ldots, i_l)} \left(\rho_{i_1} \rho_{i_2} \ldots \rho_{i_l} \sum_{s=1}^{\infty} \int g(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_l}) f(\xi_s) dP^{\xi} \right) dP^{\rho} \\
&= \sum_{s=1}^{\infty} \int \sum_{\text{distinct } (i_1, i_2, \ldots, i_l)} \left(\rho_{i_1} \rho_{i_2} \ldots \rho_{i_l} \int g(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_l}) f(\xi_s) dP^{\xi} \right) dP^{\rho} \\
&= \sum_{s=1}^{\infty} \int \left\langle g, \left(\sum_{i=1}^{\infty} \rho_i \delta_{\xi_i} \right)^l \right\rangle f(\xi_s) dP \\
&= B_f(\langle g, \mu^l \rangle), \qquad (3.5)
\end{aligned}$$

where P^{ξ} and P^{ρ} denote the marginal distributions of P with respect to ξ and ρ, respectively. The proof is complete. □

Remark 3.4 *If one can show that the linear functional $B_{f-\nu_0(f)}$ defined by (3.2) is bounded, then there exists a unique $b_f \in L^2(E; \Pi_{\alpha,\theta,\nu_0})$ such that*

$$B_{f-\nu_0(f)}(G) = \int_E b_f \cdot G d\Pi_{\alpha,\theta,\nu_0}, \quad \forall G \in \mathcal{G}.$$

We define

$$L(\langle f, \cdot \rangle) = -\frac{\theta}{2} \langle f, \cdot \rangle - \frac{\alpha}{2} b_f.$$

Then

$$\mathcal{E}(\langle f, \cdot \rangle, v) = -\int_E (L(\langle f, \cdot \rangle)) v d\Pi_{\alpha,\theta,\nu_0}, \quad \forall v \in \mathcal{F}.$$

In general, we define the operator $L : \mathcal{F} \to L^2(E; \Pi_{\alpha,\theta,\nu_0})$ by induction as follows.

$$\begin{aligned}
L\left(\prod_{i=1}^{k} \langle f_i, \cdot \rangle \right) &= L\left(\prod_{i=1}^{k-1} \langle f_i, \cdot \rangle \right) \cdot \langle f_k, \cdot \rangle + L\left(\langle f_k, \cdot \rangle \right) \cdot \prod_{i=1}^{k-1} \langle f_i, \cdot \rangle \\
&\quad + \left\langle \nabla \left(\prod_{i=1}^{k-1} \langle f_i, \cdot \rangle \right), \nabla \langle f_k, \cdot \rangle \right\rangle.
\end{aligned}$$

Then one can check that

$$\mathcal{E}(u, v) = -\int_E (Lu) v d\Pi_{\alpha,\theta,\nu_0}, \quad \forall u, v \in \mathcal{F}.$$

Therefore, $(\mathcal{E}, \mathcal{F})$ is closable on $L^2(E; \Pi_{\alpha,\theta,\nu_0})$ by ([13, Proposition 3.3]).

If $(\mathcal{E}, \mathcal{F})$ is indeed not closable for the general case, we may consider its relaxation. We refer the reader to [14] for the definition, existence and uniqueness of relaxation. The relaxation of $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form, whose associated Markov process is a good candidate for the labeled two parameter infinitely-many-neutral-alleles diffusion model.

Theorem 3.5 *Let S be a locally compact, separable metric space, $E = \mathcal{M}_1(S)$ and $\nu_0 \in \mathcal{M}_1(S)$. Suppose that $\alpha = -\kappa$ and $\theta = m\kappa$ for some $\kappa > 0$ and $m \in \{2, 3, \ldots\}$. We denote by $\Pi_{\alpha,\theta,\nu_0}$ the finite Poisson-Dirichlet distribution. Then the symmetric bilinear form (3.1) $(\mathcal{E}, \mathcal{F})$ is closable on $L^2(E; \Pi_{\alpha,\theta,\nu_0})$. Moreover, its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local Dirichlet form, which is associated with a diffusion process on E.*

Proof By independence of the random variables $\{\xi_s, s = 1, 2, \ldots\}$, we find that

$$\int G\left(\sum_{i=1}^{\infty} \rho_i \delta_{\xi_i}\right)(f - \nu_0(f))(\xi_s)dP = 0, \quad \forall G \in \mathcal{G} \text{ and } s > m.$$

Then the linear functional $B_{f-\nu_0(f)}$ defined by (3.2) is bounded. Therefore we conclude by Remark 3.4 that $(\mathcal{E}, \mathcal{F})$ is closable on $L^2(E; \Pi_{\alpha,\theta,\nu_0})$. Following the argument of ([21, Lemma 7.5 and Proposition 5.11]), we can further show that the closure $(\mathcal{E}, D(\mathcal{E}))$ of $(\mathcal{E}, \mathcal{F})$ is a quasi-regular local Dirichlet form, which is thus associated with a diffusion process on E. The proof is complete. $\square$

Finally, we present an auxiliary result (cf. Proposition 3.6 below). This result indicates some difficulty of showing the boundedness of the linear functional C_f defined in (3.4). Note that the relation between C_f and B_f is described by (3.5). In order to establish the boundedness of B_f and consequently the closability of $(\mathcal{E}, \mathcal{F})$, a better understanding of the two parameter Poisson-Dirichlet distributions seems to be needed.

Let λ be a partition, i.e., a sequence of the form

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}, 0, 0, \ldots), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)} > 0,$$

where $\lambda_i \in \mathbf{N}$. Denote $|\lambda| := \lambda_1 + \cdots + \lambda_{l(\lambda)}$. We identify partitions with Young diagrams. For $k \in \mathbf{N}$, we denote by $[\lambda : k]$ the number of rows in λ of length k. For $n \in \mathbf{N}$, we set (cf. [16, Page 5])

$$\begin{aligned} M_n(\lambda) \;:=\; & \frac{n!}{\prod_{k=1}^{\infty}[\lambda : k]! \cdot \prod_{i=1}^{l(\lambda)} \lambda_i!} \\ & \cdot \frac{(-\frac{\theta}{\alpha})(-\frac{\theta}{\alpha} - 1)\cdots(-\frac{\theta}{\alpha} - (l(\lambda) - 1)) \prod_{i=1}^{l(\lambda)}(-\alpha)(1-\alpha)\cdots(\lambda_i - 1 - \alpha)}{\theta(\theta+1)\cdots(\theta+n-1)}. \end{aligned}$$

Proposition 3.6 *Let $0 < \alpha < 1$ and $\theta > -\alpha$. Then*

$$\sum_{\lambda:|\lambda|=n} M_n(\lambda)l(\lambda) = O(n^\alpha). \tag{3.6}$$

Proof *We fix an $n \in \mathbf{N}$. Let $u(\mu) = \langle 1, \mu\rangle$, $v(\mu) = \langle 1, \mu\rangle \cdots \langle 1, \mu\rangle$ (n-fold products), $f \equiv 1$ and $g_1 = \cdots = g_n \equiv 1$. By considering (3.3) and (3.4), we get*

$$\begin{aligned} 0 &= \mathcal{E}(u, v) \\ &= \frac{\theta}{2} + \left\{ \frac{1}{2} \int_E \sum_{i=1}^n (\langle f g_i, \mu\rangle \prod_{j \neq i} \langle g_j, \mu\rangle) \Pi_{\alpha,\theta,\nu_0}(d\mu) \right. \\ &\quad \left. - \frac{\theta + n}{2} \int_E \langle f, \mu\rangle \prod_{j=1}^n \langle g_j, \mu\rangle \Pi_{\alpha,\theta,\nu_0}(d\mu) \right\} \\ &= \frac{\theta}{2} + \frac{\alpha}{2} \sum_{\lambda:|\lambda|=n} M_n(\lambda)l(\lambda) - \frac{\theta+n}{2} \int \sum_{i=1}^\infty \rho_i (1-\rho_i)^n \Pi_{\alpha,\theta,\nu_0}(d\mu). \end{aligned}$$

Thus, to prove the desired inequality (3.6), we only need to show that

$$\sup_{n \in \mathbf{N}} \left\{ n^{1-\alpha} \int \sum_{i=1}^\infty \rho_i (1-\rho_i)^n \Pi_{\alpha,\theta,\nu_0}(d\mu) \right\} < \infty.$$

By [18, (6)], we get

$$\begin{aligned} n^{1-\alpha} \int \sum_{i=1}^\infty \rho_i (1-\rho_i)^n \Pi_{\alpha,\theta,\nu_0}(d\mu) &= C_1(\alpha,\theta) n^{1-\alpha} \int_0^1 u^{-\alpha}(1-u)^{\alpha+\theta+n-1} du \\ &= C_1(\alpha,\theta) n^{1-\alpha} \cdot \mathrm{Beta}(1-\alpha, \alpha+\theta+n) \\ &\approx C_1(\alpha,\theta) n^{1-\alpha} \cdot \Gamma(1-\alpha)(1+\theta+n)^{-(1-\alpha)} \\ &\leq C_2(\alpha,\theta), \end{aligned}$$

where $C_1(\alpha,\theta) > 0$ and $C_2(\alpha,\theta) > 0$ are constants depending only on α and θ. The proof is complete. □

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S. Feng
Department of Mathematics & Statistics
McMaster University
1280 Main Street West
Hamilton, L8S 4K1, Canada
E-mail: shuifeng@mcmaster.ca

W. Sun
Department of Mathematics and Statistics
Concordia University
Montreal, H3G 1M8, Canada
E-mail: wsun@mathstat.concordia.ca