On the Goodness-of-Fit Tests for Some Continuous Time Processes

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Abstract

We present a review of several results concerning the construction of the Cramér-von Mises and Kolmogorov-Smirnov type goodness-of-fit tests for continuous time processes. As the models we take a stochastic differential equation with small noise, ergodic diffusion process, Poisson process and self-exciting point processes. For every model we propose the tests which provide the asymptotic size α and discuss the behaviour of the power function under local alternatives. The results of numerical simulations of the tests are presented.

Keywords: Hypotheses testing, diffusion process, Poisson process, self-exciting process, goodness-of-fit tests

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1 Introduction

The goodness-of-fit tests play an important role in the classical mathematical statistics. Particularly, the tests of Cramér-von Mises, Kolmogorov-Smirnov and Chi-Squared are well studied and allow to verify the correspondence of the mathematical models to the observed data (see, for example, Durbin (1973) or Greenwood and Nikulin (1996)). The similar problem, of course, exists for the continuous time stochastic processes. The diffusion and Poisson processes are widely used as mathematical models of many evolution processes in Biology, Medicine, Physics, Financial Mathematics and in many others fields. For example, some theory can propose a diffusion process

$$\mathrm{d}X_t = S_*\left(X_t\right)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T$$

as an appropriate model for description of the real data $\{X_t, 0 \leq t \leq T\}$ and we can try to construct an algorithm to verify if this model corresponds well to these data. The model here is totally defined by the trend coefficient $S_*\left(\cdot\right)$, which is supposed (if the theory is true) to be known. We do not discuss here the problem of verification if the process $\{W_t, 0 \leq t \leq T\}$ is Wiener. This problem is much more complicated and we suppose that the *noise is white Gaussian*. Therefore we have a basic hypothesis defined by the trend coefficient $S_*\left(\cdot\right)$ and we have to test this hypothesis against any other alternative. Any other means that the observations come from stochastic differential equation

$$\mathrm{d}X_t = S\left(X_t\right)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T,$$

where $S\left(\cdot\right) \neq S_*\left(\cdot\right)$. We propose some tests which are in some sense similar to the Cramér-von Mises and Kolmogorov-Smirnov tests. The advantage of classical tests is that they are distribution-free, i.e., the distribution of the underlying statistics do not depend on the basic model and this property allows to choose the *universal thresholds*, which can be used for all models.

For example, if we observe n independent identically distributed random variables $\left(X_1, \ldots, X_n\right) = X^n$ with distribution function $F\left(x\right)$ and the basic hypothesis is simple : $F\left(x\right) \equiv F_*\left(x\right)$, then the Cramér-von Mises W_n^2 and Kolmogorov-Smirnov D_n statistics are

$$W_n^2 = n\int_{-\infty}^{\infty}\left[\hat{F}_n\left(x\right) - F_*\left(x\right)\right]^2\,\mathrm{d}F_*\left(x\right), \qquad D_n = \sup_x\left|\hat{F}_n\left(x\right) - F_*\left(x\right)\right|$$

respectively. Here

$$\hat{F}_n\left(x\right) = \frac{1}{n}\sum_{j=1}^{n} 1_{\{X_j < x\}}$$

is the empirical distribution function. Let us denote by $\{W_0\left(s\right), 0 \leq s \leq 1\}$ a Brownian bridge, i.e., a continuous Gaussian process with

$$\mathbf{E}W_0\left(s\right) = 0, \qquad \mathbf{E}W_0\left(s\right)W_0\left(t\right) = t \wedge s - st.$$

Then the limit behaviour of these statistics can be described with the help of this process as follows

$$W_n^2 \Longrightarrow \int_0^1 W_0(s)^2 \,\mathrm{d}s, \qquad \sqrt{n}D_n \Longrightarrow \sup_{0\le s\le 1} |W_0(s)|.$$

Hence the corresponding Cramér-von Mises and Kolmogorov-Smirnov tests

$$\psi_n(X^n) = 1_{\{W_n^2 > c_\alpha\}}, \qquad \phi_n(X^n) = 1_{\{\sqrt{n}D_n > d_\alpha\}}$$

with constants c_α, d_α defined by the equations

$$\mathbf{P}\left\{\int_0^1 W_0(s)^2 \,\mathrm{d}s > c_\alpha\right\} = \alpha, \qquad \mathbf{P}\left\{\sup_{0\le s\le 1} |W_0(s)| > d_\alpha\right\} = \alpha$$

are of asymptotic size α. It is easy to see that these tests are distribution-free (the limit distributions do not depend of the function $F_*(\cdot)$) and are consistent against any fixed alternative (see, for example, Durbin (1973)).

It is interesting to study these tests for *nondegenerate set of alternatives*, i.e., for alternatives with limit power function less than 1. It can be realized on the close nonparametric alternatives of the special form making this problem asymptotically equivalent to the *signal in Gaussian noise* problem. Let us put

$$F(x) = F_*(x) + \frac{1}{\sqrt{n}} \int_{-\infty}^{x} h(F_*(y))\ \mathrm{d}F_*(y),$$

where the function $h(\cdot)$ describes the alternatives. We suppose that

$$\int_0^1 h(s)\ \mathrm{d}s = 0, \qquad \int_0^1 h(s)^2\ \mathrm{d}s < \infty.$$

Then we have the following convergence (under fixed alternative, given by the function $h(\cdot)$):

$$W_n^2 \Longrightarrow \int_0^1 \left[\int_0^s h(v)\,\mathrm{d}v + W_0(s)\right]^2 \mathrm{d}s,$$

$$\sqrt{n}D_n \Longrightarrow \sup_{0\le s\le 1} \left|\int_0^s h(v)\,\mathrm{d}v + W_0(s)\right|$$

We see that this problem is asymptotically equivalent to the following *signal in Gaussian noise* problem:

$$\mathrm{d}Y_s = h_*(s)\ \mathrm{d}s + \mathrm{d}W_0(s), \quad 0 \le s \le 1. \tag{1}$$

Indeed, if we use the statistics

$$W^2 = \int_0^1 Y_s^2 \,\mathrm{d}s, \qquad D = \sup_{0\leq s\leq 1} |Y_s|$$

then under hypothesis $h(\cdot) \equiv 0$ and alternative $h(\cdot) \neq 0$ the distributions of these statistics coincide with the limit distributions of W_n^2 and $\sqrt{n} D_n$ under hypothesis and alternative respectively.

Our goal is to see how such kind of tests can be constructed in the case of continuous time models of observation and particularly in the cases of some diffusion and point processes. We consider the diffusion processes with small noise, ergodic diffusion processes and Poisson process with Poisson and self-exciting alternatives. For the first two classes we just show how Cramér-von Mises and Kolmogorov-Smirnov - type tests can be realized using some known results and for the last models we discuss this problem in detail.

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2 Diffusion process with small noise

Suppose that the observed process is the solution of the stochastic differential equation

$$\mathrm{d}X_t = S(X_t)\ \mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \qquad X_0 = x_0, \quad 0 \leq t \leq T, \tag{2}$$

where $W_t, 0 \leq t \leq T$ is a Wiener process (see, for example, Liptser and Shiryayev (2001)). We assume that the function $S(x)$ is two times continuously differentiable with bounded derivatives. These are not the minimal conditions for the results presented below, but this assumption simplifies the exposition. We are interested in the statistical inference for this model in the asymptotics of small noise : $\varepsilon \to 0$. The statistical estimation theory (parametric and nonparametric) was developed in Kutoyants (1994).

Recall that the stochastic process $X^\varepsilon = \{X_t, 0 \leq t \leq T\}$ converges uniformly in $t \in [0, T]$ to the deterministic function $\{x_t, 0 \leq t \leq T\}$, which is a solution of the ordinary differential equation

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = S(x_t), \qquad x_0, \quad 0 \leq t \leq T. \tag{3}$$

Suppose that the function $S_*(x) > 0$ for $x \geq x_0$ and consider the following problem of hypotheses testing

$$\begin{aligned} \mathscr{H}_0: \qquad & S(x) = S_*(x), \quad x_0 \leq x \leq x_T^* \\ \mathscr{H}_1: \qquad & S(x) \neq S_*(x), \quad x_0 \leq x \leq x_T^* \end{aligned}$$

where we denoted by x_t^* the solution of the equation (3) under hypothesis $\mathscr{H}_0$:

$$x_t^* = x_0 + \int_0^t S_*(x_v^*)\, \mathrm{d}v, \qquad 0 \leq t \leq T.$$

Hence, we have a simple hypothesis against the composite alternative.

The Cramér-von Mises (W_ε^2) and Kolmogorov-Smirnov (D_ε) type statistics for this model of observations can be

$$W_\varepsilon^2 = \left[\int_0^T \frac{\mathrm{d}t}{S_*(x_t^*)^2}\right]^{-2} \int_0^T \left(\frac{X_t - x_t^*}{\varepsilon\, S_*(x_t^*)^2}\right)^2 \mathrm{d}t,$$

$$D_\varepsilon = \left[\int_0^T \frac{\mathrm{d}t}{S_*(x_t^*)^2}\right]^{-1/2} \sup_{0\leq t\leq T} \left|\frac{X_t - x_t^*}{S_*(x_t^*)}\right|.$$

It can be shown that these two statistics converge (as $\varepsilon \to 0$) to the following functionals

$$W_\varepsilon^2 \Longrightarrow \int_0^1 W(s)^2\, \mathrm{d}s, \qquad \varepsilon^{-1} D_\varepsilon \Longrightarrow \sup_{0\leq s\leq 1} |W(s)|,$$

where $\{W(s), 0 \leq s \leq 1\}$ is a Wiener process (see Kutoyants 1994). Hence the corresponding tests

$$\psi_\varepsilon(X^\varepsilon) = 1_{\{W_\varepsilon^2 > c_\alpha\}}, \qquad \phi_\varepsilon(X^\varepsilon) = 1_{\{\varepsilon^{-1} D_\varepsilon > d_\alpha\}}$$

with the constants c_α, d_α defined by the equations

$$\mathbf{P}\left\{\int_0^1 W(s)^2\, \mathrm{d}s > c_\alpha\right\} = \alpha, \qquad \mathbf{P}\left\{\sup_{0\leq s\leq 1} |W(s)| > d_\alpha\right\} = \alpha \tag{4}$$

are of asymptotic size α. Note that the choice of the thresholds c_α and d_α does not depend on the hypothesis (distribution-free). This situation is quite close to the classical case mentioned above.

It is easy to see that if $S(x) \neq S_*(x)$, then $\sup_{0\leq t\leq T} |x_t - x_t^*| > 0$ and $W_\varepsilon^2 \to \infty$, $\varepsilon^{-1} D_\varepsilon \to \infty$. Hence these tests are consistent against any fixed alternative. It is possible to study the power function of this test for local (contiguous) alternatives of the following form

$$\mathrm{d}X_t = S_*\left(X_t\right)\,\mathrm{d}t + \varepsilon\,\frac{h\left(X_t\right)}{S_*\left(X_t\right)}\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad 0 \le t \le T.$$

We describe the alternatives with the help of the (unknown) function $h\left(\cdot\right)$. The case $h\left(\cdot\right) \equiv 0$ corresponds to the hypothesis $\mathscr{H}_0$. One special class of such nonparametric alternatives for this model was studied in Iacus and Kutoyants (2001).

Let us introduce the composite (nonparametric) alternative

$$\mathscr{H}_1 \quad : \qquad h\left(\cdot\right) \in \mathcal{H}_\rho,$$

where

$$\mathcal{H}_\rho = \left\{ h\left(\cdot\right) \;:\quad \int_{x_0}^{x_T} h\left(x\right)^2\;\mu\left(\mathrm{d}x\right) \ge \rho \right\}.$$

To choose alternative we have to precise the "natural for this problem" distance described by the measure $\mu\left(\cdot\right)$ and the rate of $\rho = \rho_\varepsilon$. We show that the choice

$$\mu\left(\mathrm{d}x\right) = \frac{\mathrm{d}x}{S_*\left(x\right)^3}$$

provides for the test statistic the following limit

$$W_\varepsilon^2 \longrightarrow \int_0^1 \left[\int_0^s h_*\left(v\right)\mathrm{d}v + W\left(s\right)\right]^2 \mathrm{d}s,$$

where we denoted

$$h_*\left(s\right) = u_T^{1/2} h\left(x^*_{u_T s}\right), \qquad u_T = \int_0^T \frac{\mathrm{d}s}{S_*\left(x_s^*\right)^2}$$

We see that this problem is asymptotically equivalent to the *signal in white Gaussian noise* problem:

$$\mathrm{d}Y_s = h_*\left(s\right)\,\mathrm{d}s + \mathrm{d}W\left(s\right), \quad 0 \le s \le 1, \tag{5}$$

with the Wiener process $W\left(\cdot\right)$. It is easy to see that even for fixed $\rho > 0$ without further restrictions on the smoothness of the function $h_*\left(\cdot\right)$ the *uniformly good* testing is impossible. For example, if we put

$$h_n\left(x\right) = c\,S_*\left(x\right)^3\;\cos\left[n\left(x - x_0\right)\right]$$

then for the power function of the test we have

$$\inf_{h(\cdot)\in\mathcal{H}_\rho}\beta\left(\psi_\varepsilon,h\right)\leq\beta\left(\psi_\varepsilon,h_n\right)\longrightarrow\alpha.$$

The details can be found in Kutoyants (2006). The construction of the uniformly consistent tests requires a different approach (see Ingster and Suslina (2003)).

Note as well that if the diffusion process is

$$\mathrm{d}X_t=S\left(X_t\right)\,\mathrm{d}t+\varepsilon\sigma\left(X_t\right)\,\mathrm{d}W_t,\qquad X_0=x_0,\qquad 0\leq t\leq T,$$

then we can put

$$W_\varepsilon^2=\left[\int_0^T\left(\frac{\sigma\left(x_t^*\right)}{S_*\left(x_t^*\right)}\right)^2\mathrm{d}t\right]^{-2}\int_0^T\left(\frac{X_t-x_t^*}{\varepsilon\,S_*\left(x_t^*\right)^2}\right)^2\,\mathrm{d}t$$

and have the same results as above (see Kutoyants (2006)).

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3 Ergodic diffusion processes

Suppose that the observed process is one dimensional diffusion process

$$\mathrm{d}X_t=S\left(X_t\right)\,\mathrm{d}t+\mathrm{d}W_t,\qquad X_0,\qquad 0\leq t\leq T,\tag{6}$$

where the trend coefficient $S\left(x\right)$ satisfies the conditions of the existence and uniqueness of the solution of this equation and this solution has ergodic properties, i.e., there exists an invariant probability distribution $F_S\left(x\right)$, and for any integrable w.r.t. this distribution function $g\left(x\right)$ the law of large numbers holds

$$\frac{1}{T}\int_0^T g\left(X_t\right)\,\mathrm{d}t\longrightarrow\int_{-\infty}^{\infty}g\left(x\right)\,\mathrm{d}F_S\left(x\right).$$

These conditions can be found, for example, in Kutoyants (2004).

Recall that the invariant density function $f_S\left(x\right)$ is defined by the equality

$$f_S\left(x\right)=G\left(S\right)^{-1}\exp\left\{2\int_0^x S\left(y\right)\,\mathrm{d}y\right\},$$

where $G(S)$ is the normalising constant.

We consider two types of tests. The first one is a direct analogue of the classical Cramér-von Mises and Kolmogorov-Smirnov tests based on empirical distribution and density functions and the second follows the considered above (small noise) construction of tests.

The invariant distribution function $F_S(x)$ and this density function can be estimated by the *empirical distribution function* $\hat{F}_T(x)$ and by the *local time type* estimator $\hat{f}_T(x)$ defined by the equalities

$$\hat{F}_T(x) = \frac{1}{T}\int_0^T 1_{\{X_t<x\}}\,\mathrm{d}t, \qquad \hat{f}_T(x) = \frac{2}{T}\int_0^T 1_{\{X_t<x\}}\,\mathrm{d}X_t$$

respectively. Note that both of them are unbiased:

$$\mathbf{E}_S\hat{F}_T(x) = F_S(x), \qquad \mathbf{E}_S\hat{f}_T(x) = f_S(x),$$

admit the representations

$$\eta_T(x) = -\frac{2}{\sqrt{T}}\int_0^T \frac{F_S(X_t\wedge x) - F_S(X_t)F_S(x)}{f_S(X_t)}\,\mathrm{d}W_t + o(1),$$
$$\zeta_T(x) = -\frac{2f_S(x)}{\sqrt{T}}\int_0^T \frac{1_{\{X_t>x\}} - F_S(X_t)}{f_S(X_t)}\,\mathrm{d}W_t + o(1)$$

and are $\sqrt{T}$ asymptotically normal (as $T\to\infty$)

$$\eta_T(x) = \sqrt{T}\left(\hat{F}_T(x) - F_S(x)\right) \Longrightarrow \mathcal{N}\left(0, d_F(S,x)^2\right),$$
$$\zeta_T(x) = \sqrt{T}\left(\hat{f}_T(x) - f_S(x)\right) \Longrightarrow \mathcal{N}\left(0, d_f(S,x)^2\right).$$

Let us fix a simple (basic) hypothesis

$$\mathscr{H}_0 \quad : \qquad S(x) \equiv S_*(x).$$

Then to test this hypothesis we can use these estimators for construction of the Cramér-von Mises and Kolmogorov-Smirnov type test statistics

$$W_T^2 = T\int_{-\infty}^{\infty}\left[\hat{F}_T(x) - F_{S_*}(x)\right]^2\,\mathrm{d}F_{S_*}(x),$$
$$D_T = \sup_x\left|\hat{F}_T(x) - F_{S_*}(x)\right|$$

and

$$V_T^2 = T\int_{-\infty}^{\infty} \left[\hat f_T(x) - f_{S_*}(x)\right]^2\ \mathrm{d}F_{S_*}(x),$$
$$d_T = \sup_x \left|\hat f_T(x) - f_{S_*}(x)\right|$$

respectively. Unfortunately, all these statistics are not distribution-free even asymptotically and the choice of the corresponding thresholds for the tests is much more complicated. Indeed, it was shown that the random functions $(\eta_T(x), x \in R)$ and $(\zeta_T(x), x \in R)$ converge in the space $(\mathscr{C}_0, \mathfrak{B})$ (of continuous functions decreasing to zero at infinity) to the zero mean Gaussian processes $(\eta(x), x \in R)$ and $(\zeta(x), x \in R)$ respectively with the covariance functions (we omit the index S_* of functions $f_{S_*}(x)$ and $F_{S_*}(x)$ below)

$$\begin{aligned}
R_F(x,y) &= \mathbf{E}_{S_*}\left[\eta(x)\,\eta(y)\right] \\
&= 4\mathbf{E}_{S_*}\left(\frac{\left[F(\xi\wedge x) - F(\xi)\,F(x)\right]\left[F(\xi\wedge y) - F(\xi)\,F(y)\right]}{f(\xi)^2}\right) \\
R_f(x,y) &= \mathbf{E}_{S_*}\left[\zeta(x)\,\zeta(y)\right] \\
&= 4f(x)\,f(y)\,\mathbf{E}_{S_*}\left(\frac{\left[1_{\{\xi>x\}} - F(\xi)\right]\left[1_{\{\xi>y\}} - F(\xi)\right]}{f(\xi)^2}\right).
\end{aligned}$$

Here ξ is a random variable with the distribution function $F_{S_*}(x)$. Of course,

$$d_F(S,x)^2 = \mathbf{E}_S\left[\eta(x)^2\right], \qquad d_f(S,x)^2 = \mathbf{E}_S\left[\zeta(x)^2\right].$$

Using this weak convergence it is shown that these statistics converge in distribution (under hypothesis) to the following limits (as $T \to \infty$)

$$W_T^2 \Longrightarrow \int_{-\infty}^{\infty} \eta(x)^2\ \mathrm{d}F_{S_*}(x), \qquad T^{1/2}D_T \Longrightarrow \sup_x |\eta(x)|,$$
$$V_T^2 \Longrightarrow \int_{-\infty}^{\infty} \zeta(x)^2\ \mathrm{d}F_{S_*}(x), \qquad T^{1/2}d_T \Longrightarrow \sup_x |\zeta(x)|.$$

The conditions and the proofs of all these properties can be found in Kutoyants (2004), where essentially different statistical problems were studied, but the calculus are quite close to what we need here.

Note that the Kolmogorov-Smirnov test for ergodic diffusion was studied in Fournie (1992) (see as well Fournie and Kutoyants (1993) for further details), and the weak convergence of the process $\eta_T(\cdot)$ was obtained in Negri (1998).

The Cramér-von Mises and Kolmogorov-Smirnov type tests based on these statistics are

$$\Psi_T\left(X^T\right)=1_{\left\{W_T^2>C_\alpha\right\}},\qquad \Phi_T\left(X^T\right)=1_{\left\{T^{1/2}D_T>D_\alpha\right\}},$$
$$\psi_T\left(X^T\right)=1_{\left\{V_T^2>c_\alpha\right\}},\qquad \phi_T\left(X^T\right)=1_{\left\{T^{1/2}d_T>d_\alpha\right\}}$$

with appropriate constants.

The contiguous alternatives can be introduced by the following way

$$S\left(x\right)=S_*\left(x\right)+\frac{h\left(x\right)}{\sqrt{T}}.$$

Then we obtain for the Cramér-von Mises statistics the limits (see, Kutoyants (2004))

$$W_T^2\Longrightarrow\int_{-\infty}^{\infty}\left[2\mathbf{E}_{S_*}\left(\left[1_{\{\xi<x\}}-F_{S_*}\left(x\varphi\right)\right]\int_0^{\xi}h\left(s\right)\,\mathrm{d}s\right)+\eta\left(x\right)\right]^2\,\mathrm{d}F_{S_*}\left(x\right),$$
$$V_T^2\Longrightarrow\int_{-\infty}^{\infty}\left[2f_{S_*}\left(x\right)\,\mathbf{E}_{S_*}\int_{\xi}^{x}h\left(s\right)\,\mathrm{d}s+\zeta\left(x\right)\right]^2\,\mathrm{d}F_{S_*}\left(x\right).$$

Note that the transformation $Y_t=F_{S_*}\left(X_t\right)$ simplifies the writing, because the diffusion process Y_t satisfies the differential equation

$$\mathrm{d}Y_t=f_{S_*}\left(X_t\right)\left[2S_*\left(X_t\right)\mathrm{d}t+\,\mathrm{d}W_t\right],\qquad Y_0=F_{S_*}\left(X_0\right)$$

with reflecting bounds in 0 and 1 and (under hypothesis) has uniform on $[0,1]$ invariant distribution. Therefore,

$$W_T^2\Longrightarrow\int_0^1 V\left(s\right)^2\mathrm{d}s,\qquad T^{1/2}D_T\Longrightarrow\sup_{0\leq s\leq 1}\left|V\left(s\right)\right|,$$

but the covariance structure of the Gaussian process $\left\{V\left(s\right),0\leq s\leq 1\right\}$ can be quite complicated.

To obtain asymptotically distribution-free Cramér-von Mises type test we can use another statistic, which is similar to that of the preceding section. Let us introduce

$$\tilde W_T^2=\frac{1}{T^2}\int_0^T\left[X_t-X_0-\int_0^t S_*\left(X_v\right)\,\mathrm{d}v\right]^2\mathrm{d}t.$$

Then we have immediately (under hypothesis)

$$\tilde{W}_T^2 = \frac{1}{T^2}\int_0^T W_t^2 \, \mathrm{d}t = \int_0^1 W(s)^2 \, \mathrm{d}s,$$

where we put $t = sT$ and $W(s) = T^{-1/2} W_{sT}$. Under alternative we have

$$\begin{aligned}\tilde{W}_T^2 &= \frac{1}{T^2}\int_0^T \left[W_t + \frac{1}{\sqrt{T}}\int_0^t h(X_v)\,\mathrm{d}v\right]^2 \mathrm{d}t \\ &= \frac{1}{T}\int_0^T \left[\frac{W_t}{\sqrt{T}} + \frac{t}{T}\frac{1}{t}\int_0^t h(X_v)\,\mathrm{d}v\right]^2 \mathrm{d}t.\end{aligned}$$

The stochastic process X_t is ergodic, hence

$$\frac{1}{t}\int_0^t h(X_v)\,\mathrm{d}v \longrightarrow \mathbf{E}_{S_*} h(\xi) = \int_{-\infty}^{\infty} h(x)\, f_{S_*}(x)\,\mathrm{d}x \equiv \rho_h$$

as $t \to \infty$. It can be shown (see section 2.3 in Kutoyants (2004), where we have the similar calculus in another problem) that

$$\tilde{W}_T^2 \Longrightarrow \int_0^1 \left[\rho_h\, s + W(s)\right]^2 \mathrm{d}s.$$

Therefore the power function of the test $\psi\left(X^T\right) = 1_{\left\{\tilde{W}_T^2 > c_\alpha\right\}}$ converges to the function

$$\beta_\psi(\rho_h) = \mathbf{P}\left(\int_0^1 \left[\rho_h\, s + W(s)\right]^2 \mathrm{d}s > c_\alpha\right).$$

Using standard calculus we can show that for the corresponding Kolmogorov-Smirnov type test the limit will be

$$\beta_\phi(\rho_h) = \mathbf{P}\left(\sup_{0\le s\le 1}\left|\rho_h\, s + W(s)\right| > c_\alpha\right).$$

These two limit power functions are the same as in the next section devoted to self-exciting alternatives of the Poisson process. We calculate these functions with the help of simulations in Section 5 below.

Note that if the diffusion process is

$$\mathrm{d}X_t = S(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0, \qquad 0 \le t \le T,$$

but the functions $S(\cdot)$ and $\sigma(\cdot)$ are such that the process is ergodic then we introduce the statistics

$$\hat{W}_T^2 = \frac{1}{T^2\,\mathbf{E}_{S_*}\left[\sigma\left(\xi\right)^2\right]}\int_0^T\left[X_t - X_0 - \int_0^t S_*\left(X_v\right)\,\mathrm{d}v\right]^2\mathrm{d}t.$$

Here ξ is random variable with the invariant density function

$$f_{S_*}\left(x\right) = \frac{1}{G\left(S_*\right)\sigma\left(x\right)^2}\,\exp\left\{2\int_0^x\frac{S_*\left(y\right)}{\sigma\left(y\right)^2}\,\mathrm{d}y\right\}.$$

This statistic under hypothesis is equal to

$$\begin{aligned}\hat{W}_T^2 &= \frac{1}{T^2\,\mathbf{E}_{S_*}\left[\sigma\left(\xi\right)^2\right]}\int_0^T\left[\int_0^t\sigma\left(X_v\right)\mathrm{d}W_v\right]^2\mathrm{d}t\\ &= \frac{1}{T\,\mathbf{E}_{S_*}\left[\sigma\left(\xi\right)^2\right]}\int_0^T\left[\frac{1}{\sqrt{T}}\int_0^t\sigma\left(X_v\right)\mathrm{d}W_v\right]^2\mathrm{d}t.\end{aligned}$$

The stochastic integral by the central limit theorem is asymptotically normal

$$\eta_t = \frac{1}{\sqrt{t\mathbf{E}_{S_*}\left[\sigma\left(\xi\right)^2\right]}}\int_0^t\sigma\left(X_v\right)\mathrm{d}W_v \Longrightarrow \mathcal{N}\left(0,1\right)$$

and moreover it can be shown that the vector of such integrals converges in distribution to the Wiener process

$$\left(\eta_{s_1T},\ldots,\eta_{s_kT}\right) \Longrightarrow \left(W\left(s_1\right),\ldots,W\left(s_k\right)\right)$$

for any finite collection of $0 \leq s_1 < s_2 < \ldots < s_k \leq 1$. Therefore, under mild regularity conditions it can be proved that

$$\hat{W}_T^2 \Longrightarrow \int_0^1 W\left(s\right)^2\,\mathrm{d}s.$$

The power function has the same limit,

$$\beta_\psi\left(\rho_h\right) = \mathbf{P}\left(\int_0^1\left[\rho_h\,s + W\left(s\right)\right]^2\mathrm{d}s > c_\alpha\right).$$

but with

$$\rho_h = \frac{\mathbf{E}_{S_*} h(\xi)}{\sqrt{\mathbf{E}_{S_*}\left[\sigma(\xi)^2\right]}}.$$

The similar consideration can be done for the Kolmogorov-Smirnov type test too.

We see that both tests can not distinguish the alternatives with $h(\cdot)$ such that $\mathbf{E}_{S_*} h(\xi) = 0$. Note that for ergodic processes usually we have $\mathbf{E}_S S(\xi) = 0$ and $\mathbf{E}_{S_*+h/\sqrt{T}}\left[S_*(\xi) + T^{-1/2} h(\xi)\right] = 0$ with corresponding random variables ξ, but this does not imply $\mathbf{E}_{S_*} h(\xi) = 0$.

---

4 Poisson and self-exciting processes

Poisson process is one of the simplest point processes and before taking any other model it is useful first of all to check the hypothesis the observed sequence of events, say, $0 < t_1, \ldots, t_N < T$ corresponds to a Poisson process. It is natural in many problems to suppose that this Poisson process is periodic of known period. For example, many daily events, signal transmission in optical communication, season variations etc. Another model of point processes as well frequently used is self-exciting stationary point process introduced in Hawkes (1972). As any stationary process it can as well describe the periodic changes due to the particular form of its spectral density.

Recall that for the Poisson process $X_t, t \geq 0$ of intensity function $S(t), t \geq 0$ we have (X_t is the counting process)

$$\mathbf{P}\left\{X_t - X_s = k\right\} = (k!)^{-1} \left(\Lambda(t) - \Lambda(s)\right)^k \exp\left\{\Lambda(s) - \Lambda(t)\right\},$$

where we suppose that $s < t$ and put

$$\Lambda(t) = \int_0^t S(v)\, \mathrm{d}v.$$

The self-exciting process $X_t, t \geq 0$ admits the representation

$$X_t = \int_0^t S(s, X)\, \mathrm{d}s + \pi_t,$$

where $\pi_t, t \geq 0$ is local martingale and the intensity function

$$S(t, X) = S + \int_0^t g(t-s)\, \mathrm{d}X_s = S + \sum_{t_i < T} g(t - t_i).$$

It is supposed that

$$\rho = \int_0^\infty g(t)\ \mathrm{d}t < 1.$$

Under this condition the self-exciting process is a stationary point process with the rate

$$\mu = \frac{S}{1-\rho}$$

and the spectral density

$$f(\lambda) = \frac{\mu}{2\pi \left|1 - G(\lambda)\right|^2}, \qquad G(\lambda) = \int_0^\infty e^{\mathrm{i}\lambda t} g(t)\ \mathrm{d}t$$

(see Hawkes (1972) or Daley and Vere-Jones (2003) for details).

We consider two problems: Poisson against another Poisson and Poisson against a close self-exciting point process. The first one is to test the simple (basic) hypothesis

$$\mathscr{H}_0 \qquad : \qquad S(t) \equiv S_*(t), \quad t \geq 0$$

where $S_*(t)$ is known periodic function of period τ, against the composite alternative

$$\mathscr{H}_1 \qquad : \qquad S(t) \neq S_*(t), \quad t \geq 0,$$

but $S(t)$ is always τ-periodic.

Let us denote $X_j(t) = X_{\tau(j-1)+t} - X_{\tau(j-1)}$, $j = 1, \ldots, n$, suppose that $T = n\tau$ and put

$$\hat{\Lambda}_n(t) = \frac{1}{n} \sum_{j=1}^n X_j(t).$$

The corresponding goodness-of-fit tests of Cramér-von Mises and Kolmogorov-Smirnov type can be based on the statistics

$$W_n^2 = \Lambda_*(\tau)^{-2}\, n \int_0^\tau \left[\hat{\Lambda}_n(t) - \Lambda_*(t)\right]^2 \mathrm{d}\Lambda_*(t),$$

$$D_n = \Lambda_*(\tau)^{-1/2} \sup_{0\leq t\leq \tau} \left|\hat{\Lambda}_n(t) - \Lambda_*(t)\right|.$$

It can be shown that

$$W_n^2 \Longrightarrow \int_0^1 W(s)^2\, \mathrm{d}s, \qquad \sqrt{n}\, D_n \Longrightarrow \sup_{0\leq s\leq 1} \left|W(s)\right|$$

where $\{W(s), 0 \leq s \leq 1\}$ is a Wiener process (see Kutoyants (1998)). Hence these statistics are asymptotically distribution-free and the tests

$$\psi_n\left(X^T\right) = 1_{\{W_n^2 > c_\alpha\}}, \qquad \phi_n\left(X^T\right) = 1_{\left\{\sqrt{n}D_n > d_\alpha\right\}}$$

with the constants c_α, d_α taken from the equations (4), are of asymptotic size α.

Let us describe the close contiguous alternatives which reduce asymptotically this problem to *signal in white Gaussian noise* model (5). We put

$$\Lambda(t) = \Lambda_*(t) + \frac{1}{\sqrt{n\Lambda_*(\tau)}} \int_0^t h(u(v))\ \mathrm{d}\Lambda_*(v), \qquad u(v) = \frac{\Lambda_*(v)}{\Lambda_*(\tau)}.$$

Here $h(\cdot)$ is an arbitrary function defining the alternative. Then if $\Lambda(t)$ satisfies this equality we have the convergence

$$W_n^2 \Longrightarrow \int_0^1 \left[\int_0^s h(v)\,\mathrm{d}v + W(s)\right]^2 \mathrm{d}s.$$

This convergence describes the power function of the Cramér-von Mises type test under these alternatives.

The second problem is to test the hypothesis

$$\mathscr{H}_0 \qquad : \qquad S(t) = S_*, \quad t \geq 0$$

against nonparametric close (contiguous) alternative

$$\mathscr{H}_1 \qquad : \qquad S(t) = S_* + \frac{1}{\sqrt{T}} \int_0^t h(t-s)\,\mathrm{d}X_t, \quad t \geq 0,$$

We consider the alternatives with the functions $h(\cdot) \geq 0$ having compact support and bounded.

We have $\Lambda_*(t) = S_*\ t$ and for some fixed $\tau > 0$ we can construct the same statistics

$$W_n^2 = \frac{n}{S_*\tau^2} \int_0^\tau \left[\hat{\Lambda}_n(t) - S_*\ t\right]^2\ \mathrm{d}t, \quad D_n = (S_*\ \tau)^{-1/2} \sup_{0 \leq t \leq \tau} \left|\hat{\Lambda}_n(t) - S_*\ t\right|.$$

Of course, they have the same limits under hypothesis

$$W_n^2 \Longrightarrow \int_0^1 W(s)^2\,\mathrm{d}s, \qquad \sqrt{n}D_n \Longrightarrow \sup_{0 \leq s \leq 1} |W(s)|.$$

To describe their behaviour under any fixed alternative $h(\cdot)$ we have to find the limit distribution of the vector

$$\mathbf{w}_n = \big(w_n(t_1), \ldots, w_n(t_k)\big), \qquad w_n(t_l) = \frac{1}{\sqrt{S_*\tau\, n}} \sum_{j=1}^{n} \left[X_j(t_l) - S_* t_l\right],$$

where $0 \leq t_l \leq \tau$. We know that this vector under hypothesis is asymptotically normal

$$\mathcal{L}_0\left\{\mathbf{w}_n\right\} \Longrightarrow \mathcal{N}(\mathbf{0}, \mathbf{R})$$

with covariance matrix

$$\mathbf{R} = (R_{lm})_{k\times k}, \qquad R_{lm} = \tau^{-1}\min(t_l, t_m).$$

Moreover, it was shown in Dachian and Kutoyants (2006) that for such alternatives the likelihood ratio is locally asymptotically normal, i.e., the likelihood ratio admits the representation

$$Z_n(h) = \exp\left\{\Delta_n(h, X^n) - \frac{1}{2}\,\mathrm{I}(h) + r_n(h, X^n)\right\}$$

where

$$\Delta_n(h, X^n) = \frac{1}{S_*\sqrt{\tau n}} \int_0^{\tau n} \int_0^{t-} h(t-s)\, \mathrm{d}X_s\ \left[\mathrm{d}X_t - S_*\mathrm{d}t\right],$$

$$\mathrm{I}(h) = \int_0^\infty h(t)^2\, \mathrm{d}t + S_*\left(\int_0^\infty h(t)\, \mathrm{d}t\right)^2$$

and

$$\Delta_n(h, X^n) \Longrightarrow \mathcal{N}(0, \mathrm{I}(h)), \qquad r_n(h, X^n) \to 0. \tag{7}$$

To use the Third Le Cam's Lemma we describe the limit behaviour of the vector $(\Delta_n(h, X^n), \mathbf{w}_n)$. For the covariance $\mathbf{Q} = (Q_{lm}), l, m = 0, 1, \ldots, k$ of this vector we have

$$\mathbf{E}_0\Delta_n(h, X^n) = 0, \qquad Q_{00} = \mathbf{E}_0\Delta_n(h, X^n)^2 = \mathrm{I}(h)\,(1 + o(1)).$$

Further, let us denote $\mathrm{d}\pi_t = \mathrm{d}X_t - S_*\mathrm{d}t$ and $H(t) = \int_0^{t-} h(t-s)\, \mathrm{d}X_s$, then we can write

$$
\begin{aligned}
Q_{0l} &= \mathbf{E}_0\left[\Delta_n\left(h,X^n\right)w_n\left(t_l\right)\right]\\
&= \frac{1}{nS_*^{3/2}\tau}\,\mathbf{E}_0\left(\sum_{j=1}^{n}\int_{\tau(j-1)}^{\tau j} H\left(t\right){\rm d}\pi_t\ \sum_{i=1}^{n}\int_{\tau(i-1)}^{\tau(i-1)+t_l}{\rm d}\pi_t\right)\\
&= \frac{1}{n\tau\sqrt{S_*}}\sum_{j=1}^{n}\int_{\tau(j-1)}^{\tau(j-1)+t_l}\mathbf{E}_0 H\left(t\right)\ {\rm d}t = \frac{t_l}{\tau}\,\sqrt{S_*}\ \int_0^\infty h\left(t\right){\rm d}t\ \left(1+o\left(1\right)\right),
\end{aligned}
$$

because

$$
\mathbf{E}_0 H\left(t\right) = S_*\ \int_0^{t-} h\left(t-s\right){\rm d}s = S_*\ \int_0^\infty h\left(s\right){\rm d}s
$$

for the large values of t (such that $[0,t]$ covers the support of $h\left(\cdot\right)$).

Therefore, if we denote

$$
\bar h = \int_0^\infty h\left(s\right){\rm d}s
$$

then

$$
Q_{0l} = Q_{l0} = \frac{t_l}{\tau}\ \sqrt{S_*}\,\bar h.
$$

The proof of the Theorem 1 in Dachian and Kutoyants (2006) can be applied to the linear combination of $\Delta_n\left(h,X^n\right)$ and $w_n\left(t_1\right),\ldots,w_n\left(t_k\right)$ and this yields the asymptotic normality

$$
\mathcal{L}_0\Big(\Delta_n\left(h,X^n\right),\mathbf{w}_n\Big)\Longrightarrow\mathcal{N}\left(\mathbf{0},\mathbf{Q}\right).
$$

Hence by the Third Lemma of Le Cam we obtain the asymptotic normality of the vector $\mathbf{w}_n$

$$
\mathcal{L}_h\Big(\mathbf{w}_n\Big)\Longrightarrow\mathcal{L}\left(W\left(s_1\right)+s_1\sqrt{S_*}\ \bar h,\ldots,W\left(s_k\right)+s_k\sqrt{S_*}\ \bar h\right),
$$

where we put $t_l=\tau\,s_l$. This weak convergence together with the estimates like

$$
\mathbf{E}_h\left|w_n\left(t_1\right)-w_n\left(t_2\right)\right|^2\leq C\left|t_1-t_2\right|
$$

provides the convergence (under alternative)

$$
W_n^2\Longrightarrow\int_0^1\left[\sqrt{S_*}\ \bar h\,s+W\left(s\right)\right]^2{\rm d}s.
$$

We see that the limit experiment is of the type

$$dY_s = \sqrt{S_*}\,\bar{h}\,\mathrm{d}s + \mathrm{d}W(s), \quad Y_0 = 0, \quad 0 \le s \le 1.$$

The power $\beta(\psi_n, h)$ of the Cramer-von Mises type test $\psi_n(X^n) = 1_{\{W_n^2 > c_\alpha\}}$ is a function of the real parameter $\rho_h = \sqrt{S_*}\,\bar{h}$

$$\beta(W_n, h) = \mathbf{P}\left(\int_0^1 [\rho_h\, s + W(s)]^2\,\mathrm{d}s > c_\alpha\right) + o(1) = \beta_\psi(\rho_h) + o(1).$$

Using the arguments of Lemma 6.2 in Kutoyants (1998) it can be shown that for the Kolmogorov-Smirnov type test we have the convergence

$$\sqrt{n}D_n \Longrightarrow \sup_{0\le s\le 1} |\rho_h\, s + W(s)|.$$

The limit power function is

$$\beta_\phi(\rho_h) = \mathbf{P}\left(\sup_{0\le s\le 1} |\rho_h\, s + W(s)| > d_\alpha\right).$$

These two limit power functions will be obtained by simulation in the next section.

---

5 Simulation

First, we present the simulation of the thresholds c_α and d_α of our Cramér-von Mises and Kolmogorov-Smirnov type tests. Since these thresholds are given by the equations (4), we obtain them by simulating 10^7 trajectories of a Wiener process on [0,1] and calculating empirical $1-\alpha$ quantiles of the statistics

$$W^2 = \int_0^1 W(s)^2\,\mathrm{d}s \qquad \text{and} \qquad D = \sup_{0\le s\le 1} |W(s)|$$

respectively. Note that the distribution of W^2 coincides with the distribution of the quadratic form

$$W^2 = \sum_{k=1}^{\infty} \frac{\zeta_k^2}{(\pi k)^2}, \qquad \zeta_k \text{ i.i.d. } \sim \mathcal{N}(0,1)$$

and both distributions are extensively studied (see (1.9.4(1)) and (1.15.4) in Borodin and Salmienen (2002)). The analytical expressions are quite complicated and we would like to compare by simulation c_α and d_α with the real (finite time) thresholds giving the tests of exact size α, that is c_α^T and d_α^T given by equations

$$\mathbf{P}\left\{W_n^2 > c_\alpha^T\right\} = \alpha \qquad \text{and} \qquad \mathbf{P}\left\{\sqrt{n}D_n > d_\alpha^T\right\} = \alpha$$

respectively. We choose $S^* = 1$ and obtain c_α^T and d_α^T by simulating 10^7 trajectories of a Poisson process of intensity 1 on $[0,T]$ and calculating empirical $1-\alpha$ quantiles of the statistics W_n^2 and $\sqrt{n}D_n$. The thresholds simulated for $T = 10$, $T = 100$ and for the limiting case are presented in Fig. 1. The lower curves correspond to the Cramér-von Mises type test, and the upper ones to the Kolmogorov-Smirnov type test. As we can see, for $T = 100$ the real thresholds are already indistinguishable from the limiting ones, especially in the case of the Cramér-von Mises type test.

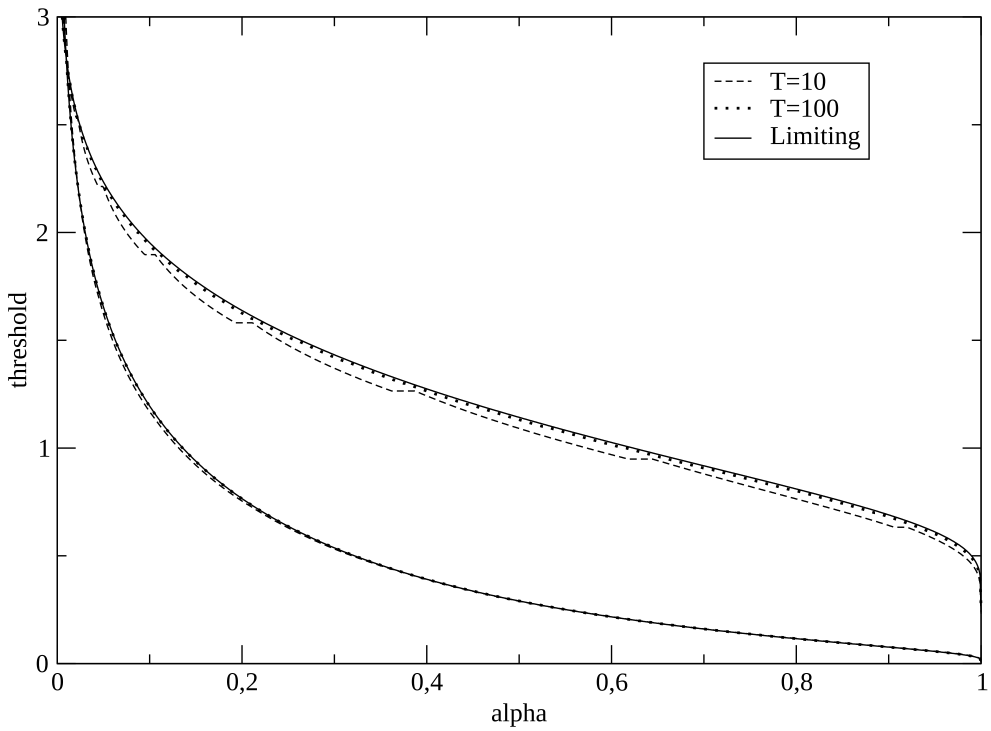

Fig. 1: Threshold choice

It is interesting to compare the asymptotics of the Cramér-von Mises and Kolmogorov-Smirnov type tests with the locally asymptotically uniformly most powerful (LAUMP) test

$$\hat{\phi}_n\left(X^n\right) = 1_{\{\delta_T > z_\alpha\}}, \qquad \delta_T = \frac{X_{n\tau} - S_* n\tau}{\sqrt{S_* n\tau}}$$

proposed for this problem in Dachian and Kutoyants (2006). Here z_α is $1-\alpha$ quantile of the standard Gaussian law, $\mathbf{P}\left(\zeta > z_\alpha\right) = \alpha$, $\zeta \sim \mathcal{N}\left(0,1\right)$. The limit power function of $\hat{\phi}_n$ is

$$\beta_{\hat{\phi}}\left(\rho_h\right) = \mathbf{P}\left(\rho_h + \zeta > z_\alpha\right).$$

In Fig. 2 we compare the limit power functions $\beta_\psi\left(\rho\right), \beta_\phi\left(\rho\right)$ and $\beta_{\hat{\phi}}\left(\rho\right)$. The last one can clearly be calculated directly, and the first two are obtained by simulating 10^7 trajectories of a Wiener process on [0,1] and calculating empirical frequencies of the events

$$\left\{\int_0^1 \left[\rho\, s + W\left(s\right)\right]^2 \mathrm{d}s > c_\alpha\right\} \qquad \text{and} \qquad \left\{\sup_{0\leq s\leq 1} \left|\rho\, s + W\left(s\right)\right| > d_\alpha\right\}$$

respectively.

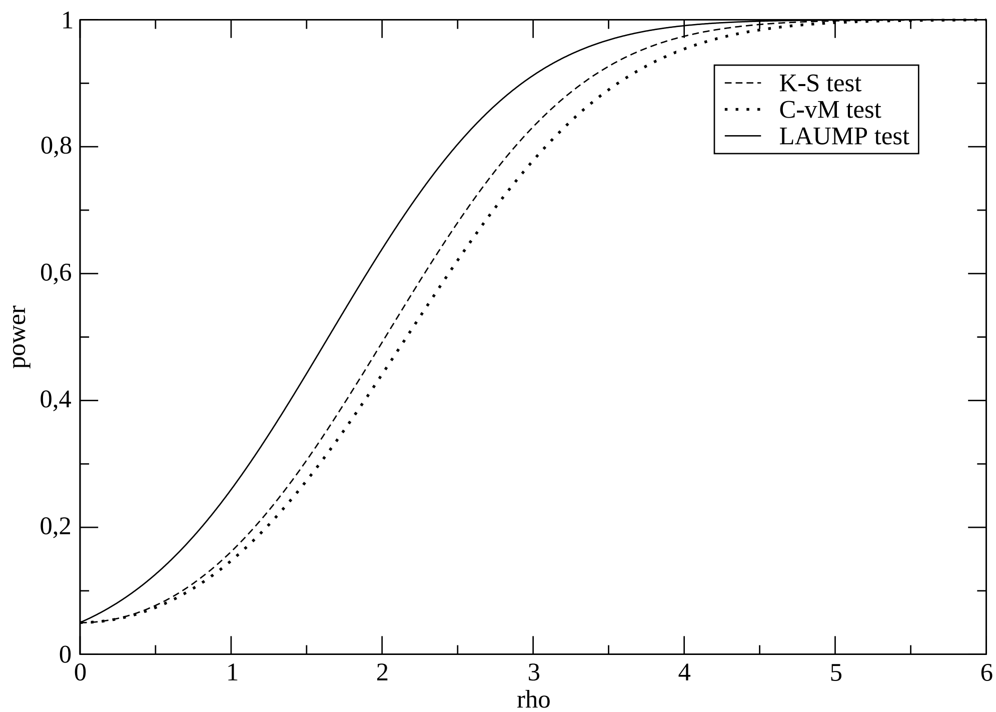

Fig. 2: Limit power functions

The simulation shows the exact (quantitative) comparison of the limit power functions. We see that the power of LAUMP test is higher that the two others and this is of course evident. We see also that the Kolmogorov-Smirnov type test is more powerful that the Cramér-von Mises type test.

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