Hypotheses Testing: Poisson Versus Self-Exciting

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Abstract

We consider the problem of hypotheses testing with the basic simple hypothesis: observed sequence of points corresponds to stationary Poisson process with known intensity. The alternatives are stationary self-exciting point processes. We consider one-sided parametric and one-sided nonparametric composite alternatives and construct locally asymptotically uniformly most powerful tests. The results of numerical simulations of the tests are presented.

Key words: *hypotheses testing, Poisson process, self-exciting process, uniformly most powerful test*

1 Introduction

Let $\{t_1, t_2, \ldots\}$ be a sequence of events of a stationary point process $X = \{X_t,\ t \geq 0\}$ (X_t is a counting process). The simplest stationary point process is, of course, Poisson process with a constant intensity $S > 0$, i.e., the increments of X on disjoint intervals are independent and distributed according to Poisson law

$$\mathbf{P}\left\{X_t - X_s = k\right\} = \frac{S^k\left(t-s\right)^k}{k!}\, \mathrm{e}^{-S(t-s)}, \quad 0 \leq s < t, \quad k = 0, 1, \ldots$$

Therefore if we have a stationary sequence of events it is interesting to check first of all if this model (Poisson process) corresponds well to the observations.

The importance of this problem was discussed by Cox and Lewis (1966), Section 6.3.

The alternatives close to the basic hypothesis correspond to the case when the non-poissonian behavior is due to small perturbations of the Poisson process and are the most interesting to test. For "far alternatives" any reasonable test has power function close to 1 and the comparison of tests seems less important. Let us consider the problem of small signals detection by the tests of fixed size $\varepsilon \in (0,1)$. Using the terminology of statistical radiothechnics we say that there is at least two types of close alternatives: the first one corresponds to small "signal-noise ratio" (signals of small energy) and the second, when the amplitude of the signal can be small, but the total energy due to the sufficiently long time of observation is comparable with the noise energy (see, e.g., Kutoyants (1976)). For the first class of alternatives the approach of *locally optimal tests*, which provides the optimality of the power function at the small vicinity of the basic hypothesis (the values of the power function are close to ε) was developed (see, e.g. Capon (1961)) and for the second class of *contiguous alternatives* the optimality of the test for a wider class of close alternatives (the values of the power function are in $(\varepsilon, 1)$) is proved (Pitman's (1948) approach, Le Cam's (1956) theory).

For stationary point processes with Poisson hypothesis and stationary alternatives Davies (1977) proposed the *locally optimal (efficient)* or *asymptotically locally efficient* test. This test is based on the comparison of the derivative of the log-likelihood ratio with some threshold. See as well Daley and Vere-Jones (2003), Section 13.1, where the approach of Davies is discussed.

In the present note we suppose that we have observations of the point process $X^T = \{X_t,\ 0 \leq t \leq T\}$ on the interval $[0,T]$ and consider two problems of hypotheses testing in the asymptotics of large samples ($T \to \infty$). In both problems the basic hypothesis is simple: the observed process is standard Poisson with known constant intensity $S_* > 0$. The composite alternatives are: the observed process is a realization of self-exciting point process (sometimes called Hawkes (1972) process) with in the first case intensity function depending on one-dimensional parameter and in the second case the intensity function belonging to a wider (nonparametric) class of functions. We follow the mentioned above Pitman-Le Cam's approach. We start with the *locally asymptotically uniformly most powerful test* (LAUMPT) in the parametric case and the main result of the presented work is the LAUMPT where the optimality is shown for sufficiently large class of local nonparametric alternatives. The similar results for diffusion processes can be found in Iacus, Kutoyants (2001) (small noise asymptotics) and Kutoyants (2003) (ergodic processes).

2 Preliminaries

Remind several facts from the theory of point processes (the details can be found, for example, in Liptser and Shiryaev (2001), Chapter 18). Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space and let $\{\mathfrak{F}_t,\ t \geq 0\}$ be a nondecreasing family of right continuous σ-algebras $\mathfrak{F}_s \subset \mathfrak{F}_t \subset \mathfrak{F}$ for any $0 \leq s < t$. We denote by $t_1, t_2, \ldots$ a sequence of Markov stopping times adapted to $\{\mathfrak{F}_t,\ t \geq 0\}$ (that means $\{\omega : t_i \leq t\} \in \mathfrak{F}_t$ for all $t \geq 0$). Let X_t be the number of *events* t_i up to time t, i.e., $X = \{X_t, \mathfrak{F}_t, t \geq 0\}$ is a random process such that

$$X_t = \sum_{i \geq 1} \chi_{\{t_i < t\}}, \qquad t \geq 0,$$

where $\chi_{\{A\}}$ is the indicator-function of the event A.

We assume that $\mathbf{E} X_t < \infty$ (there is no accumulation points on any bounded interval). The process X admits a unique (up to stochastic equivalence) decomposition (Doob-Meyer decomposition)

$$X_t = \mathcal{A}_t + \mathcal{M}_t, \tag{1}$$

where $\mathcal{M} = \{\mathcal{M}_t, \mathfrak{F}_t, t \geq 0\}$ is a martingale and $\mathcal{A} = \{\mathcal{A}_t, \mathfrak{F}_t, t \geq 0\}$ is predictable increasing process (Liptser and Shiryaeyv (2001), Theorem 18.1). We suppose that the compensator $\mathcal{A}$ is absolutely continuous

$$\mathcal{A}_t = \int_0^t S(v, \omega)\ \mathrm{d}v, \qquad t \geq 0$$

where $\mathcal{S} = \{S(t, \omega), \mathfrak{F}_t, t \geq 0\}$ is called intensity function. We suppose as well that (1) is the *minimal representation* of the point process, i.e., $S(t, \omega)$ is measurable w.r.t. σ-algebra generated by $\{X_s,\ s < t\}$ for any $t > 0$ and we write $S(t, \omega) = S(t, X)$. To describe a point process it is sufficient to specify its intensity function. We study in this work a special class of point processes with intensity functions which can be written as stochastic integrals with respect to the past of the underlying point process.

In the particular case when $\mathcal{S}$ is deterministic, the process X is (inhomogeneous) Poisson process with intensity function $S(v, \omega) = S(v)$. In this case

$$\mathbf{P}\{X_t - X_s = k\} = \frac{\left[\int_s^t S(v)\ \mathrm{d}v\right]^k}{k!}\ \exp\left\{-\int_s^t S(v)\ \mathrm{d}v\right\}$$

for any $t > s \geq 0$ and $k = 0, 1, \ldots$. If the assumption of the independence of increments is no more valid, then $\mathcal{S}$ is no more deterministic and X can be a stationary point process (see Brillinger (1975) and Daley and Vere-Jones (2003) and references therein for wide classes of such processes and their applications in real problems).

Remind that the distribution $\mathbf{P}_S^{(T)}$ of the point process in the space of its realizations $(\mathscr{D}(0,T), \mathfrak{B}_T)$ is entirely characterized by its intensity function $\mathcal{S}$. The likelihood ratio formula (w.r.t. Poisson process of constant intensity S_*) has the following form (see Liptser and Shiriyev (2001), Theorem 19.10)

$$L\left(X^T\right) = \exp\left\{\int_0^T \ln\frac{S(t,\omega)}{S_*}\,\mathrm{d}X_t - \int_0^T \left[S(t,\omega) - S_*\right]\mathrm{d}t\right\},$$

where we suppose that the intensity $S(t,\omega)$ is left continuous function and

$$\mathbf{P}\left\{\int_0^T S(t,\omega)\,\mathrm{d}t < \infty\right\}$$

under all alternatives studied in this work.

3 One-sided parametric alternative

Suppose that we observe a trajectory $X^T = \{X_t, 0 \le t \le T\}$ of point process of intensity function $S_T(\vartheta) = \{S(\vartheta,t,\omega), 0 \le t \le T\}$. If $\vartheta = 0$, then $S(0,t,\omega) \equiv S_*$, i.e., this point process is Poisson process of intensity $S_* > 0$. Under alternative $\vartheta > 0$ and $S_T(\vartheta)$ is the intensity function of self-exciting point process. As usual in such problems, we consider contiguous alternatives (Pitman's (1948) alternatives, Roussas (1972)), hence we change the variable $\vartheta = u/\sqrt{T}$ and test the following two hypotheses

$$\mathscr{H}_0: \qquad u = 0$$
$$\mathscr{H}_1: \qquad u > 0.$$

We denote $\mathbf{E}_0$ the mathematical expectation under the hypothesis $\mathscr{H}_0$, and $\mathbf{E}_u$ under (simple) alternative $\vartheta = u/\sqrt{T}$.

Let us fix $\varepsilon \in (0,1)$ and denote by $\mathscr{K}_\varepsilon$ the class of test functions $\phi_T\left(X^T\right)$ of asymptotic size ε, i.e., for $\phi_T \in \mathscr{K}_\varepsilon$ we have

$$\lim_{T\to\infty} \mathbf{E}_0\, \phi_T\left(X^T\right) = \varepsilon. \tag{2}$$

As usual, $\phi_T\left(X^T\right)$ is the probability to accept the hypothesis $\mathscr{H}_1$ having observations X^T. The corresponding power function is

$$\beta_T\left(u,\phi_T\right) = \mathbf{E}_u\, \phi_T\left(X^T\right), \quad u \ge 0.$$

We introduce the asymptotic optimality of tests with the help of the following definition Le Cam (1956).

Definition 1. A test $\phi_T^*(\cdot)$ is called *locally asymptotically uniformly most powerful in the class* $\mathscr{K}_\varepsilon$ if for any other test $\phi_T(\cdot) \in \mathscr{K}_\varepsilon$ and any constant $K > 0$ we have

$$\lim_{T\to\infty} \inf_{0<u\leq K} \left[\beta_T\left(u, \phi_T^*\right) - \beta_T\left(u, \phi_T\right)\right] \geq 0.$$

Our goal is to construct locally asymptotically uniformly most powerful test in class $\mathscr{K}_\varepsilon$.

Self-exciting type processes were introduced by Hawkes (1972) and defined by intensity function of the following form

$$S(t,\omega) = S_* + \int_0^{t-} g(t-s)\ \mathrm{d}X_s = S_* + \sum_{t_i<t} g(t-t_i), \tag{3}$$

where $S_* > 0$, t_i are the events of the point process and the function $g(\cdot) \geq 0$ satisfies the condition

$$\rho = \int_0^\infty g(t)\ \mathrm{d}t < 1. \tag{4}$$

Remind that according to this representation of the intensity function, the distribution of t_1 is exponential at rate S_* and for all $n \geq 1$

$$\mathbf{P}\left\{t_{n+1} > t \mid t_1, \ldots, t_n\right\} = \exp\left(-S_* t - \int_0^t \sum_{i=1}^{X_s} g(s-t_i)\ \mathrm{d}s\right).$$

Note that $\Lambda(t) = \mathbf{E}X_t$ is solution of the equation

$$\begin{aligned}\Lambda(t) &= \mathbf{E}\int_0^t S(v,\omega)\ \mathrm{d}v = S_* t + \mathbf{E}\int_0^t \int_0^{v-} g(v-s)\ \mathrm{d}X_s\,\mathrm{d}v = \\ &= S_* t + \int_0^t \int_0^v g(v-s)\,\Lambda'(s)\ \mathrm{d}v\ \mathrm{d}s.\end{aligned}$$

In stationary case the intensity $S(t,\omega)$, is a stationary process

$$S(t,\omega) = S_* + \int_{-\infty}^{t-} g(t-s)\ \mathrm{d}X_s,$$

and

$$\Lambda(t) = \frac{S_*}{1-\rho}\, t \equiv \mu\, t.$$

The spectral density of this process is

$$f(\lambda) = \frac{\mu}{2\pi \left|1 - G(\lambda)\right|^2}$$

where

$$G(\lambda) = \int_0^\infty \mathrm{e}^{\mathrm{i}\lambda t} g(t)\ \mathrm{d}t, \qquad \rho = G(0).$$

Example 1. Let $g(t) = \alpha \mathrm{e}^{-\gamma t}$, where $\alpha > 0, \gamma > 0$ and $\alpha/\gamma < 1$. Then the point process X with intensity function

$$S(t,\omega) = S_* + \alpha \sum_{t_i \leq t} \mathrm{e}^{-\gamma(t-t_i)}$$

is self-exciting with the rate

$$\mu = \frac{S_*\,\gamma}{\gamma - \alpha}.$$

Example 2. The function $g(\cdot)$ can be chosen in such a way that the spectral density of the point process will be rational

$$f(\lambda) = \frac{\mu}{2\pi} \frac{\left|Q(\mathrm{i}\lambda)\right|^2}{\left|P(\mathrm{i}\lambda)\right|^2}$$

where $Q(z) = z^p + a_1 z^{p-1} + \ldots + a_p$ and $P(z) = z^p + b_1 z^{p-1} + \ldots + b_p$. It is supposed that $P(\cdot)$ and $Q(\cdot)$ have no zeroes in common and no zeroes in the closed right half plane (see Pham (1981), where the asymptotic properties of the MLE for this model are described).

We assume that the observed process is either Poisson with constant intensity S_* or self-exciting with *contiguous* intensity function

$$S(\vartheta, t, \omega) = S_* + \vartheta_T \int_0^t h(t-s)\ \mathrm{d}X_s.$$

Contiguous means, that the likelihood ratio is *asymptotically non degenerate*. The function $h(\cdot)$ is supposed to be known, bounded and

$$h(\cdot) \in \mathcal{L}_+^1(\mathbb{R}_+) = \left\{ f(\cdot) \geq 0\ :\ \int_0^\infty f(t)\ \mathrm{d}t < \infty \right\}.$$

To have contiguous alternatives we choose, as usual in regular problems, $\vartheta_T = u/\sqrt{T}$, i.e.,

$$S(u,t,\omega) = S_* + \frac{u}{\sqrt{T}} \int_0^t h(t-s)\ \mathrm{d}X_s, \qquad u \geq 0.$$

Note that for any $h(\cdot) \in \mathcal{L}_+^1(\mathbb{R}_+)$ and any $u \leq K$ for sufficiently large T the condition (4) is fulfilled for the corresponding function $g(\cdot) = uT^{-1/2}h(\cdot)$. This leads us to the following one sided hypotheses testing problem:

$$\begin{aligned} \mathscr{H}_0 : &\qquad u = 0, \qquad \text{(Poisson process)} \\ \mathscr{H}_1 : &\qquad u > 0, \qquad \text{(self-exciting process)}. \end{aligned}$$

This model corresponds to *small self-exciting perturbations* of the Poisson process of intensity S_*.

Note that as we use the LAN approach, we study the behavior of the tests statistics under hypothesis only (Poisson process with constant intensity) and do not use the stationarity of the self-exciting processes under alternatives. The limit of the power function is obtained using LAN and Le Cam's Third Lemma.

Let us denote

$$\Delta_T\left(X^T\right) = \frac{1}{S_*\sqrt{T\,\mathrm{I}_h^*}} \int_0^T \int_0^{t-} h(t-s)\ \mathrm{d}X_s\ \left[\mathrm{d}X_t - S_*\ \mathrm{d}t\right].$$

Here

$$\int_0^{t-} h(t-s)\ \mathrm{d}X_s = \sum_{t_i<t} h(t-t_i)$$

(limit from the left of the integral, i.e., the term with $s_i = t$ is excluded) and

$$\mathrm{I}_h^* = \int_0^\infty h(t)^2\ \mathrm{d}t + S_*\ \left(\int_0^\infty h(t)\ \mathrm{d}t\right)^2$$

is the Fisher information of the problem. Throughout this paper we denote by z_ε the $1-\varepsilon$ quantile of the Gaussian law $\mathcal{N}(0,1)$.

Theorem 1 *Let $h(\cdot) \in \mathcal{L}_+^1(\mathbb{R}_+)$ and bounded. Then the test*

$$\hat\phi_T\left(X^T\right) = \chi_{\left\{\Delta_T\left(X^T\right) > z_\varepsilon\right\}}$$

is locally asymptotically uniformly most powerful in the class $\mathscr{K}_\varepsilon$ and for any $u > 0$ its power function

$$\beta_T\left(u, \hat\phi_T\right) \longrightarrow \hat\beta(u) = \mathbf{P}\left\{\zeta > z_\varepsilon - u\ \sqrt{\mathrm{I}_h^*}\right\}, \tag{5}$$

where $\zeta \sim \mathcal{N}(0,1)$.

Proof. First note that the family of measures $\left\{\mathbf{P}_\vartheta^{(T)}, \vartheta > 0\right\}$ under hypothesis $\mathscr{H}_0$ is LAN at the point $\vartheta = 0$, i.e., the random function $Z_T(u) = L\left(u/\sqrt{T}, X^T\right)$ admits the representation (see Kutoyants (1984), Theorem 4.5.3)

$$\begin{aligned} Z_T(u) &= \exp\left\{\int_0^T \ln\left(1 + \frac{u}{S_*\sqrt{T}}\int_0^{t-} h(t-s)\,\mathrm{d}X_s\right)\,\mathrm{d}X_t - \right.\\ &\left. -\frac{u}{\sqrt{T}}\int_0^T\int_0^t h(t-s)\ \mathrm{d}X_s\,\mathrm{d}t\right\} = \\ &= \exp\left\{u\sqrt{\mathrm{I}_h^*}\Delta_T\left(X^T\right) - \frac{u^2}{2}\,\mathrm{I}_h^* + r_T\left(u, X^T\right)\right\} \end{aligned}$$

where

$$\mathcal{L}_0\left\{\Delta_T\left(X^T\right)\right\} \Longrightarrow \mathcal{N}(0,1) \tag{6}$$

and $r_T\left(u_T, X^T\right) \to 0$ for any bounded sequence $\left\{u_T\right\}$.

To verify (6) we check the following two conditions:

- *Lindeberg condition* for stochastic integral: for any $\delta > 0$

$$\lim_{T\to\infty}\frac{1}{T}\,\mathbf{E}_0\int_0^T H_t^2\ \chi_{\left\{|H_t|>\delta\sqrt{T}\right\}}\ \mathrm{d}t = 0,$$

- the *law of large numbers*:

$$\mathbf{P}_0 - \lim_{T\to\infty}\frac{1}{S_*T}\int_0^T H_t^2\ \mathrm{d}t = \mathrm{I}_h^*. \tag{7}$$

Here we denoted

$$H_t = \int_0^{t-} h(t-s)\ \mathrm{d}X_s.$$

By these conditions the stochastic integral $\Delta_T\left(X^T\right)$ is asymptotically normal. The proof of the corresponding central limit theorem can be found, say, in Kutoyants (1984), Theorem 4.5.4 (of course, this theorem is a particular case of general CLT for martingales).

To check these conditions we introduce an independent Poisson process $\{X_t,\ t \le 0\}$ of intensity S_* and replace H_t by

$$H_t^* = \int_{-\infty}^{t-} h(t-s)\ \mathrm{d}X_s.$$

It is easy to see that for the process $H_t^*, t \geq 0$ we have

$$\mathbf{E}_0 \, H_t^* = S_* \int_0^\infty h\left(v\right) \, \mathrm{d}v$$

and

$$\mathbf{E}_0 \left(\left[H_t^* - \mathbf{E}_0 \, H_t^*\right] \left[H_s^* - \mathbf{E}_0 \, H_s^*\right] \right) = S_* \int_{\max(0,s-t)}^\infty h\left(v+t-s\right) \, h\left(v\right) \, \mathrm{d}v.$$

Note as well that

$$\begin{aligned}
&\mathbf{P}_0 \left\{ \frac{1}{\sqrt{T}} \int_0^T \left[H_t^* - H_t\right] \, \left[\mathrm{d}X_t - S_* \, \mathrm{d}t\right] > \nu \right\} \leq \\
&\qquad \leq \frac{S_*}{T\nu^2} \int_0^T \mathbf{E}_0 \left(\int_{-\infty}^0 h\left(t-s\right) \, \mathrm{d}X_s \right)^2 \mathrm{d}t = \\
&\qquad = \frac{S_*^2}{T\nu^2} \int_0^T \left[\int_t^\infty h\left(v\right)^2 \, \mathrm{d}v + S_* \left(\int_t^\infty h\left(v\right) \, \mathrm{d}v \right)^2 \right] \mathrm{d}t \longrightarrow 0
\end{aligned}$$

as $T \to \infty$.

Now the process $H_t^*, t \geq 0$ is second order stationary and

$$\mathbf{E}_0 \left(H_t^*\right)^2 = S_* \int_0^\infty h\left(t\right)^2 \, \mathrm{d}t + S_*^2 \left(\int_0^\infty h\left(t\right) \, \mathrm{d}t \right)^2 = \mathbf{E}_0 \left(H_0^*\right)^2 < \infty.$$

Hence

$$\mathbf{E}_0 \left(H_t^{*2} \, \chi_{\left\{|H_t^*| > \delta\sqrt{T}\right\}} \right) = \mathbf{E}_0 \left(H_0^{*2} \, \chi_{\left\{\left|H_0^*\right| > \delta\sqrt{T}\right\}} \right) \longrightarrow 0$$

as $T \to \infty$ and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{E}_0 \left(H_t^{*2} \, \chi_{\left\{|H_t^*| > \delta\sqrt{T}\right\}} \right) \, \mathrm{d}t = 0.$$

The law of large numbers (7) will follow from the convergence:

$$\begin{aligned}
M_T &= \mathbf{E}_0 \left(\frac{1}{T} \int_0^T H_t^{*2} \, \mathrm{d}t - \mathbf{E}_0 \left(H_0^*\right)^2 \right)^2 = \\
&= \frac{1}{T^2} \int_0^T \int_0^T \mathbf{E}_0 \left(H_t^{*2} - \mathbf{E}_0 \left(H_0^*\right)^2 \right) \left(H_s^{*2} - \mathbf{E}_0 \left(H_0^*\right)^2 \right) \, \mathrm{d}t \, \mathrm{d}s \longrightarrow 0.
\end{aligned}$$

To prove it we need the following elementary result.

Lemma 1 *Let $X = \{X_t,\ t \in A\}$ be a Poisson process of constant intensity $S > 0$ on $A \subset \mathbb{R}$, and let $f(\cdot), g(\cdot) \in \mathcal{L}^k(A) = \left\{ f(\cdot)\ :\ \int_A |f(t)|^k\ \mathrm{d}t < \infty \right\}$, $k = 1, \ldots, 4$. Then*

$$
\begin{aligned}
&\mathbf{Cov}\left(\left(\int_A f(v)\ \mathrm{d}X_v\right)^2, \left(\int_A g(v)\ \mathrm{d}X_v\right)^2\right) = \\
&= 4\int_A f(v)\, S\ \mathrm{d}v \int_A g(v)\, S\ \mathrm{d}v \int_A f(v)\, g(v)\, S\ \mathrm{d}v + \\
&\quad + 2\left(\int_A f(v)\, g(v)\, S\ \mathrm{d}v\right)^2 + \int_A f^2(v)\, g^2(v)\, S\ \mathrm{d}v + \\
&\quad + 2\int_A f(v)\, S\ \mathrm{d}v \int_A f(v)\, g^2(v)\, S\ \mathrm{d}v + 2\int_A g(v)\, S\ \mathrm{d}v \int_A f^2(v)\, g(v)\, S\ \mathrm{d}v.
\end{aligned}
$$

Proof. Using well-known properties of the Poisson processes (see, e.g., Kutoyants (1998), Lemma 1.1), we obtain the moment generating function

$$
\begin{aligned}
\phi(\lambda, \mu) &= \mathbf{E}_0 \exp\left\{\lambda \int_A f(v)\ \mathrm{d}X_v + \mu \int_A g(v)\ \mathrm{d}X_v\right\} = \\
&= \exp\left\{\int_A \left(e^{\lambda f(v) + \mu g(v)} - 1\right)\ S\ \mathrm{d}v\right\}.
\end{aligned}
$$

Remind that

$$
\begin{aligned}
&\mathbf{Cov}\left(\left(\int_A f(v)\ \mathrm{d}X_v\right)^2, \left(\int_A g(v)\ \mathrm{d}X_v\right)^2\right) = \\
&= \left.\frac{\partial^4 \phi(\lambda, \mu)}{\partial \lambda^2\, \partial \mu^2}\right|_{\lambda=0, \mu=0} - \left.\frac{\partial^2 \phi(\lambda, 0)}{\partial \lambda^2}\right|_{\lambda=0} \left.\frac{\partial^2 \phi(0, \mu)}{\partial \mu^2}\right|_{\mu=0}.
\end{aligned}
$$

Therefore the proof of the lemma follows from direct calculations.

Now we can write

$$
\begin{aligned}
R(t, s) &= \mathbf{E}_0 \left((H_t^*)^2 - \mathbf{E}_0 (H_0^*)^2\right)\left((H_s^*)^2 - \mathbf{E}_0 (H_0^*)^2\right) = \\
&= 4a^2 K(t, s) + 2K(t, s)^2 + S_* \int_{-\infty}^{t \wedge s} h(t - v)^2\, h(s - v)^2\, \mathrm{d}v + \\
&\quad + 2aS_* \int_{-\infty}^{t \wedge s} \left[h(t - v)^2\, h(s - v) + h(t - v)\, h(s - v)^2\right] \mathrm{d}v
\end{aligned}
$$

where we put

$$a = S_* \int_0^\infty h(y)\ \mathrm{d}y$$

and (for $\tau = t - s$)

$$K(t,s) = S_* \int_{|\tau|}^\infty h(y)\ h(y - |\tau|)\ \mathrm{d}y = K(\tau).$$

Further, as the function $h(\cdot)$ is bounded, we have the estimate

$$R(t,s) \leq C\ K(\tau).$$

Hence

$$M_T = \frac{1}{T^2}\int_0^T\int_0^T R(t,s)\ \mathrm{d}t\ \mathrm{d}s \leq \frac{C}{T^2}\int_0^T\int_0^T K(t,s)\ \mathrm{d}t\ \mathrm{d}s \leq$$
$$\leq \frac{C}{T}\int_{-T}^T K(\tau)\ \mathrm{d}\tau.$$

For the function $K(\cdot)$ we have

$$\int_{-T}^T K(\tau)\ \mathrm{d}\tau = S_* \int_{-T}^T \int_{|\tau|}^\infty h(y)\ h(y-|\tau|)\ \mathrm{d}y\,\mathrm{d}\tau \leq 2\,S_* \left(\int_0^\infty h(y)\ \mathrm{d}y\right)^2.$$

Hence $M_T \to 0$ and we have the law of large numbers (7).

The property $\hat\phi_T(\cdot) \in \mathscr{K}_\varepsilon$ follows from the mentioned above asymptotic normality of the statistic $\Delta_T\left(X^T\right)$.

Note as well that the convergence (5) follows from

$$\mathcal{L}_u\left\{\Delta_T\left(X^T\right)\right\} \Longrightarrow \mathcal{N}\left(u\,\sqrt{\mathrm{I}_h^*},\ 1\right)$$

(see the Third Lemma of Le Cam (van der Vaart (1998), p. 90)).

The asymptotic optimality of the test follows as well from the general theory (see, e.g., Le Cam (1956) or Roussas (1972)), because if we replace $\mathscr{H}_1$ by any simple alternative $\mathscr{H}_*:\ u = u_*$, then the test

$$\bar\phi_T\left(X^T\right) = \chi_{\left\{L\left(u_*/\sqrt{T},X^T\right) > b_\varepsilon\right\}}$$

is the most powerful. Here

$$b_\varepsilon = \exp\left\{u_* z_\varepsilon \sqrt{\mathrm{I}_h^*} - \frac{1}{2}u_*^2\ \mathrm{I}_h^*\right\}\ (1 + o(1)).$$

It is easy to see that $\bar\phi_T(\cdot)\in\mathscr{K}_\varepsilon$ and the power function

$$\beta_T\left(u_*,\bar\phi_T\right)\rightarrow\hat\beta\left(u_*\right).$$

Therefore the test $\hat\phi_T(\cdot)$ is asymptotically as good as the likelihood ratio test for any simple alternative.

Remark 1. Note that the statistic $\Delta_T\left(X^T\right)$ can be written as follows

$$\Delta_T\left(X^T\right)=\frac{1}{S^*\sqrt{T\,\mathrm{I}_h^*}}\sum_{0\leq t_j\leq T}\sum_{t_i<t_j}h\left(t_j-t_i\right)-\frac{1}{\sqrt{T\,\mathrm{I}_h^*}}\sum_{0\leq t_j\leq T}\int_0^{T-t_j}h\left(v\right)\,\mathrm{d}v,$$

where t_i are the events of the observed process.

Remark 2. By a similar way we can consider the problem of contiguous hypotheses testing when under the hypothesis $\mathscr{H}_0$ the observed process is self-exciting too. For example, let $h\left(\vartheta,t\right)\geq 0, t\geq 0$ be a smooth function of $\vartheta\in\Theta$, such that for all $\vartheta\in\Theta$ the condition

$$\int_0^\infty h\left(\vartheta,t\right)\,\mathrm{d}t<1$$

holds. Then with the help of this function we introduce a family of self-exciting processes with intensity functions

$$S\left(\vartheta,t,\omega\right)=S_*+\int_{-\infty}^t h\left(\vartheta,t-s\right)\,\mathrm{d}X_s.$$

Remind that these are stationary processes.

Now we can test the hypotheses

$$\begin{aligned}\mathscr{H}_0: &\qquad \vartheta=\vartheta_0,\\ \mathscr{H}_1: &\qquad \vartheta>\vartheta_0\end{aligned}$$

by the observations $X^T=\left\{X_t,0\leq t\leq T\right\}$. Suppose as well that the function $h\left(\vartheta,\cdot\right)$ is two times continuously differentiable on ϑ at the point $\vartheta=\vartheta_0$ and the derivatives $\dot h\left(\vartheta,\cdot\right),\ \ddot h\left(\vartheta,\cdot\right)$ satisfy the suitable conditions of integrability. Let us denote

$$\xi_t\left(\vartheta\right)=\int_0^{t-}h\left(\vartheta,t-s\right)\,\mathrm{d}X_s,\quad \dot\xi_t\left(\vartheta\right)=\int_0^{t-}\frac{\partial h\left(\vartheta,t-s\right)}{\partial\vartheta}\,\mathrm{d}X_s,$$

and put

$$\Delta_T\left(\vartheta_0,X^T\right)=\frac{1}{\sqrt{T}}\int_0^T\frac{\dot\xi_t\left(\vartheta_0\right)}{S_*+\xi_t\left(\vartheta_0\right)}\,\left[\mathrm{d}X_t-S_*\mathrm{d}t-\xi_t\left(\vartheta_0\right)\,\mathrm{d}t\right].$$

Then it can be easily shown that the test

$$\hat{\phi}_T\left(X^T\right) = \chi_{\left\{\Delta_T\left(\vartheta_0, X^T\right) > c_\varepsilon\right\}}$$

where $c_\varepsilon = z_\varepsilon \sqrt{\mathrm{I}_h\left(\vartheta_0\right)}$ is chosen from the condition $\hat{\phi}_T \in \mathscr{K}_\varepsilon$ is locally asymptotically uniformly most powerful in the class $\mathscr{K}_\varepsilon$. Here $\mathrm{I}_h\left(\vartheta_0\right)$ is the Fisher information

$$\mathrm{I}_h\left(\vartheta_0\right) = \mathbf{E}_{\vartheta_0}\left(\frac{\dot{\xi}\left(\vartheta_0\right)^2}{S_* + \xi\left(\vartheta_0\right)}\right),$$

where $\dot{\xi}\left(\vartheta_0\right)$ and $\xi\left(\vartheta_0\right)$ are *stationary random variables* related to the limit distribution of the vector $\dot{\xi}_t\left(\vartheta_0\right), \xi_t\left(\vartheta_0\right)$.

4 Testing of dependence

Suppose that we have two sequences of events $0 < t_1 < t_2 < \ldots < t_N < T$ and $0 < s_1 < s_2 < \ldots < s_M < T$ with corresponding counting processes $X^T = \{X_t, 0 \leq t \leq T\}$ and $Y^T = \{Y_t, 0 \leq t \leq T\}$. The first process is Poisson with constant known intensity function $S_X\left(t, \omega\right) = S_X > 0$ and the intensity function of the second process can be written as

$$S_Y\left(t, \omega\right) = S_Y + \int_{-\infty}^{t} r\left(t - s\right)\, \mathrm{d}X_s,$$

where $r\left(\cdot\right) \in \mathcal{L}^1\left(\mathbb{R}_+\right)$. Therefore, if $r\left(t\right) \equiv 0$, then the observed processes are standard (independent) Poisson processes of intensities S_X and S_Y respectively (Hypothesis $\mathscr{H}_0$). For the other values of $r\left(\cdot\right)$ we have dependent point processes.

We suppose that the dependence between these two processes, if exists, is weak, i.e., the function $r\left(\cdot\right)$ is sufficiently small and we can apply the local approach. As before we suppose that $r\left(t\right) = \vartheta_T\, h\left(t\right)$, where $h\left(\cdot\right) \in \mathcal{L}^1\left(\mathbb{R}_+\right)$ and $\vartheta_T = u/\sqrt{T} \rightarrow 0$.

$$\begin{aligned} &\mathscr{H}_0: \qquad u = 0, \qquad \text{(independent Poisson processes)} \\ &\mathscr{H}_1: \qquad u > 0, \qquad \text{(depending processes)}. \end{aligned}$$

Introduce the statistic

$$\begin{aligned} \Delta_T\left(X^T, Y^T\right) &= \frac{1}{S_Y\sqrt{T\,\mathrm{I}_h}} \int_0^T \int_0^{t-} h\left(t - s\right)\, \mathrm{d}X_s\, \left[\mathrm{d}Y_t - S_Y\, \mathrm{d}t\right] = \\ &= \frac{1}{S_Y\sqrt{T\,\mathrm{I}_h}} \sum_{0 \leq s_j \leq T} \sum_{t_j < s_i} h\left(s_j - t_i\right) - \frac{1}{\sqrt{T\,\mathrm{I}_h}} \sum_{0 \leq t_j \leq T} \int_0^{T - t_j} h\left(v\right)\, \mathrm{d}v \end{aligned}$$

where

$$\mathrm{I}_h=\frac{S_X}{S_Y}\;\left(\int_0^\infty h\left(t\right)^2\;\mathrm{d}t+S_X\left(\int_0^\infty h\left(t\right)\;\mathrm{d}t\right)^2\right).$$

Proposition 1 *Let $h\left(\cdot\right)\in\mathcal{L}_+^1\left(\mathbb{R}_+\right)$ and bounded. Then the test*

$$\hat\phi_T\left(X^T,Y^T\right)=\chi_{\left\{\Delta_T\left(X^T,Y^T\right)>z_\varepsilon\right\}}$$

is locally asymptotically uniformly most powerful in the class $\mathscr{K}_\varepsilon$ and for any $u>0$ its power function

$$\beta_T\left(u,\hat\phi_T\right)\longrightarrow\hat\beta\left(u\right)=\mathbf{P}\left\{\zeta>z_\varepsilon-u\,\sqrt{\mathrm{I}_h}\right\},\tag{8}$$

where $\zeta\sim\mathcal{N}\left(0,1\right)$.

Proof. The proof is quite close to the given above proof of the Theorem 1, and hence is omitted.

Remark 3. The similar problem can be considered for the couple of mutually exciting point processes with intensity functions

$$S_X\left(t,\omega\right)=S_X+\int_{-\infty}^t r_{XY}\left(t-s\right)\;\mathrm{d}Y_s,$$
$$S_Y\left(t,\omega\right)=S_Y+\int_{-\infty}^t r_{YX}\left(t-s\right)\;\mathrm{d}X_s,$$

where $r_{XY}\left(\cdot\right),r_{YX}\left(\cdot\right)\in\mathcal{L}^1\left(\mathbb{R}_+\right)$. Therefore, if $r_{XY}\left(t\right)\equiv 0$ and $r_{YX}\left(t\right)\equiv 0$, then the observed processes are standard (independent) Poisson processes of intensities $S_X>0$ and $S_Y>0$ respectively (Hypothesis $\mathscr{H}_0$). Under alternative there exists a weak dependence of these processes through their intensity functions.

5 One-sided nonparametric alternative

In all considered above problems the alternatives are one-sided parametric. It is possible to describe similar asymptotically uniformly most powerful tests even in some nonparametric situations. Using the minimax approach we can consider the least favorable model in the deriving of the upper bound on the powers of all tests, but, of course, for special classes of intensities. This approach sometimes is called semiparametric and the rate of convergence of alternatives is $\sqrt{T}$.

As before, we suppose that under hypothesis $\mathscr{H}_0$ the observed point process $X^T = \{X_t, 0 \leq t \leq T\}$ is standard Poisson with known intensity function $S(t) = S_* > 0$ and under alternative $\mathscr{H}_1$ it is self-exciting point process with intensity function

$$S(t,\omega) = S_* + \int_{-\infty}^{t} g(t-s)\ \mathrm{d}X_s, \qquad 0 \leq t \leq T$$

where $g(\cdot)$ is now *unknown* function. We suppose as well that

$$\int_0^\infty g(t)\ \mathrm{d}t < 1, \tag{9}$$

hence the process X^T is stationary. To describe the class of local nonparametric alternatives we rewrite this intensity function as

$$S(t,\omega) = S_* + \frac{1}{\sqrt{T}} \int_{-\infty}^{t} u(t-s)\ \mathrm{d}X_s, \qquad 0 \leq t \leq T$$

where the function $u(\cdot)$ is from the set $\mathscr{U}_r$ defined below. Let us denote by $\mathcal{C}_+^b$ the set of nonnegative functions bounded by the same constant and introduce the set

$$\mathscr{U}_r = \left\{ u(\cdot) \in \mathcal{C}_+^b : \quad \int_0^\infty u(t)\ \mathrm{d}t = r,\ \mathrm{supp}\, u(\cdot)\ \text{ is bounded} \right\}.$$

Note, that for any $r > 0$ and $T > r^2$ the condition (9) is fulfilled.

Therefore, we consider the following hypotheses testing problem

$$\begin{aligned} \mathscr{H}_0 : \qquad & u(\cdot) \equiv 0, \\ \mathscr{H}_1 : \qquad & u(\cdot) \in \mathscr{U}_r, \quad r > 0. \end{aligned}$$

The power function of a test ϕ_T depends on the function $u(\cdot)$ and we write it as

$$\beta_T\left(u, \phi_T\right) = \mathbf{E}_u\ \phi_T\left(X^T\right).$$

where $u = u(\cdot) \in \mathscr{U}_r$ with some $r > 0$. We want to apply an approach similar to the minimax one in the estimation theory. More precisely, we seek to maximize the minimal power of test on the class $\mathscr{U}_r$. However, for any test $\phi_T \in \mathscr{K}_\varepsilon$ we have

$$\inf_{u(\cdot)\in\mathscr{U}_r} \beta_T\left(u, \phi_T\right) \leq \varepsilon$$

since for any $T > 0$ we can take a function from $\mathscr{U}_r$ equal 0 on $[0, T]$. Hence we introduce the set

$$\mathscr{U}_{r,N} = \left\{u(\cdot) \in \mathscr{U}_r : \quad \operatorname{supp} u(\cdot) \subset [0, N]\right\},$$

denote

$$B_T\left(r, N, \phi_T\right) = \inf_{u(\cdot) \in \mathscr{U}_{r,N}} \beta_T\left(u, \phi_T\right)$$

and give the following

Definition 2. A test $\phi_T^*(\cdot)$ is called *locally asymptotically uniformly most powerful in the class* $\mathscr{K}_\varepsilon$ if for any other test $\phi_T(\cdot) \in \mathscr{K}_\varepsilon$ and any $K > 0$ we have

$$\lim_{N \to \infty} \lim_{T \to \infty} \inf_{0 \le r \le K} \left[B_T\left(r, N, \phi_T^*\right) - B_T\left(r, N, \phi_T\right)\right] \ge 0.$$

Let us introduce the decision function

$$\hat{\phi}_T\left(X^T\right) = \chi_{\left\{\delta_T\left(X^T\right) > z_\varepsilon\right\}}, \qquad \delta_T\left(X^T\right) = \frac{X_T - S_* T}{\sqrt{S_* T}}.$$

Theorem 2 *The test $\hat{\phi}_T$ is locally asymptotically uniformly most powerful in the class $\mathscr{K}_\varepsilon$ and for any $u(\cdot) \in \mathscr{U}_r$ its power function*

$$\beta_T\left(u, \hat{\phi}_T\right) \longrightarrow \hat{\beta}(u) = \mathbf{P}\left\{\zeta > z_\varepsilon - r\sqrt{S_*}\right\}, \tag{10}$$

where $\zeta \sim \mathcal{N}(0, 1)$.

Proof. Let us fix a simple alternative $u(\cdot) \in \mathscr{U}_r$, then the likelihood ratio $L_T\left(\frac{u(\cdot)}{\sqrt{T}}, X^T\right) = Z_T(u(\cdot))$ admits (under hypothesis $\mathscr{H}_0$) the representation (see the proof of the theorem 1)

$$\begin{aligned} Z_T(u(\cdot)) &= \exp\left\{\int_0^T \ln\left(1 + \frac{1}{S_*\sqrt{T}} \int_0^{t-} u(t-s)\, \mathrm{d}X_s\right) \mathrm{d}X_t - \right. \\ &\left. \quad - \frac{1}{\sqrt{T}} \int_0^T \int_0^t u(t-s)\, \mathrm{d}X_s\, \mathrm{d}t\right\} = \\ &= \exp\left\{\Delta_T\left(u, X^T\right) - \frac{1}{2}\,\mathrm{I}(u) + r_T\left(u, X^T\right)\right\} \end{aligned}$$

where

$$\Delta_T\left(u, X^T\right) = \frac{1}{S_*\sqrt{T}} \int_0^T \int_0^{t-} u(t-s)\, \mathrm{d}X_s\, \left[\mathrm{d}X_t - S_*\, \mathrm{d}t\right],$$

$$\mathrm{I}(u) = \int_0^\infty u(t)^2\, \mathrm{d}t + S_* \left(\int_0^\infty u(t)\, \mathrm{d}t\right)^2 = \int_0^\infty u(t)^2\, \mathrm{d}t + S_*\, r^2$$

and

$$\mathcal{L}_0\left\{\Delta_T\left(u,X^T\right)\right\}\Longrightarrow\mathcal{N}\left(0,\mathrm{I}\left(u\right)\right),\qquad r_T\left(u,X^T\right)\to 0.$$

Moreover, these last two convergences are uniform on $u\left(\cdot\right)\in\mathcal{U}_{r,N}$, $0\le r\le K$ for any $K>0$. Hence the likelihood ratio test

$$\bar\phi_T\left(X^T\right)=\chi_{\left\{Z_T(u(\cdot))>d_\varepsilon\right\}}$$

with $d_\varepsilon=\exp\left\{z_\varepsilon\sqrt{\mathrm{I}\left(u\right)}-\mathrm{I}\left(u\right)/2\right\}$ is the most powerful in the class $\mathscr{K}_\varepsilon$ for any two simple hypotheses and its power function

$$\beta\left(u,\bar\phi_T\right)\longrightarrow\mathbf{P}\left\{\zeta>z_\varepsilon-\mathrm{I}\left(u\right)^{1/2}\right\},\qquad\zeta\sim\mathcal{N}\left(0,1\right).$$

It is easy to see that

$$\inf_{u(\cdot)\in\mathscr{U}_{r,N}}\mathrm{I}\left(u\right)=S_*\,r^2+\frac{r^2}{N}$$

because

$$r^2=\left(\int_0^N u\left(t\right)\,\mathrm{d}t\right)^2\le N\int_0^N u\left(t\right)^2\,\mathrm{d}t$$

with equality on the *least favorable alternative* $u^*\left(t\right)=\left(r/N\right)\chi_{\left\{0\le t\le N\right\}}$.

Hence

$$\inf_{u(\cdot)\in\mathscr{U}_{r,N}}\mathbf{P}\left\{\zeta>z_\varepsilon-\mathrm{I}\left(u\right)^{1/2}\right\}=\mathbf{P}\left\{\zeta>z_\varepsilon-r\sqrt{S_*+N^{-1}}\right\}.$$

Now we study the power function of the test $\hat\phi_T$. Let us denote

$$U_t=\int_0^{t-}u\left(t-s\right)\,\mathrm{d}X_s,\qquad\pi_t=X_t-S_*t,$$

then

$$\Delta_T\left(u,X^T\right)=\frac{1}{S_*\sqrt{T}}\int_0^T U_t\,\mathrm{d}\pi_t,\qquad\delta_T\left(X^T\right)=\frac{1}{\sqrt{S_*T}}\int_0^T\,\mathrm{d}\pi_t$$

and

$$\mathbf{E}_0\Delta_T\left(u,X^T\right)=0,\qquad\mathbf{E}_0\Delta_T\left(u,X^T\right)^2=\mathrm{I}\left(u\right),\quad\mathbf{E}_0\delta_T\left(X^T\right)=0,$$
$$\mathbf{E}_0\delta_T\left(X^T\right)^2=1\qquad\mathbf{E}_0\left(\delta_T\left(X^T\right)\Delta_T\left(u,X^T\right)\right)=r\sqrt{S_*}.$$

Hence, under hypothesis $\mathscr{H}_0$, we have

$$\mathcal{L}_0\left\{\Delta_T\left(u,X^T\right),\delta_T\left(X^T\right)\right\}\Longrightarrow\mathcal{N}\left(\mathbf{0},\mathbf{R}\right)$$

where $\mathbf{R}$ is covariance matrix of the vector (Δ_T, δ_T) described above. Therefore $\hat{\phi}_T \in \mathscr{K}_\varepsilon$, and using Le Cam's Third Lemma (van der Vaart (1998)) we obtain that under alternative $u(\cdot) \in \mathscr{U}_r$

$$\delta_T\left(X^T\right) \Longrightarrow \mathcal{N}\left(r\sqrt{S_*}, 1\right).$$

For the power function we have

$$\beta\left(u, \hat{\phi}_T\right) \longrightarrow \mathbf{P}\left\{\zeta > z_\varepsilon - r\sqrt{S_*}\right\}.$$

It can be shown that this convergence is uniform over $u(\cdot) \in \mathscr{U}_{r,N}$, $0 \leq r \leq K$ for any $K > 0$ and this proves the theorem.

6 Simulations

The main results (Theorems 1 and 2) of this work are *asymptotic in nature* and it is interesting to see the properties of the tests for the moderate values of T. This can be done, say, by Monte-Carlo simulations.

6.1 Parametric alternative

To illustrate Theorem 1 we take $S_* = 1$ and $h(t) = \frac{1}{2}\mathrm{e}^{-t/2}$ (see Example 1). This yields

$$S(u, t, \omega) = 1 + \frac{u}{2\sqrt{T}} \sum_{t_i \leq t} \mathrm{e}^{-(t-t_i)/2}, \qquad u \geq 0, \quad 0 \leq t \leq T.$$

In this case

$$\Delta_T\left(X^T\right) = \frac{1}{\sqrt{5\,T}} \sum_{0 \leq t_j \leq T} \sum_{t_i < t_j} \mathrm{e}^{-(t_j - t_i)/2} - \frac{2}{\sqrt{5\,T}} \left(X_T - \sum_{0 \leq t_j \leq T} \mathrm{e}^{-(T-t_j)/2}\right)$$

where t_i are the events of the observed process, and the test $\hat{\phi}_T^\varepsilon$ given by

$$\hat{\phi}_T^\varepsilon = \hat{\phi}_T\left(X^T\right) = \chi_{\left\{\Delta_T\left(X^T\right) > z_\varepsilon\right\}}$$

is locally asymptotically uniformly most powerful in the class $\mathscr{K}_\varepsilon$.

In Figure 1 we represent the size of the test $\hat{\phi}_T^{0.05}$ as a function of $T \in [0, 1000]$. This size is given by

$$\alpha(T) = \mathbf{P}_0\left\{\Delta_T\left(X^T\right) > z_{0.05}\right\}, \qquad 1 \leq T \leq 1000$$

and is obtained by simulating $M = 10^7$ trajectories on $[0,T]$ of Poisson process of constant intensity $S(t,\omega) = 1$ and calculating empirical frequency of accepting the alternative hypothesis.

In Figure 2 we represent the power function of the test $\hat{\phi}_T^{0.05}$ given by

$$\beta_T\left(u, \hat{\phi}_T^{0.05}\right) = \mathbf{P}_u\left\{\Delta_T\left(X^T\right) > z_{0.05}\right\}, \qquad 0 \le u \le 5$$

for $T = 100$, 300 and 1000, as well as the limiting (Gaussian) power function given by

$$\hat{\beta}(u) = \mathbf{P}\left\{\zeta > z_{0.05} - u\sqrt{5}/2\right\} = \frac{1}{\sqrt{2\pi}}\int_{z_{0.05}-u\sqrt{5}/2}^{\infty} e^{-v^2/2}\,\mathrm{d}v, \quad 0 \le u \le 5.$$

The function β_T is obtained by simulating (for each value of u) $M = 10^6$ trajectories on $[0,T]$ of self-exciting process of intensity $S(u,t,\omega)$ and calculating empirical frequency of accepting the alternative hypothesis.

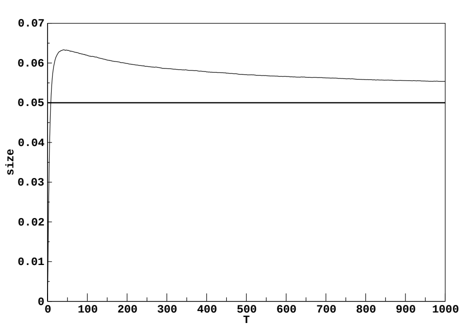

Fig. 1: Test size

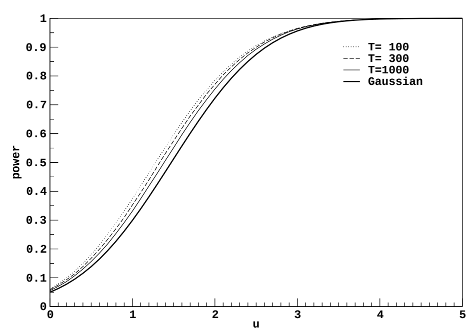

Fig. 2: Test power

Now let us consider the $\tilde{\phi}_T^{\varepsilon}$ given by

$$\tilde{\phi}_T^{\varepsilon} = \tilde{\phi}_T\left(X^T\right) = \chi_{\left\{\Delta_T\left(X^T\right) > z\right\}}$$

where the threshold z is chosen so that this test is of exact size ε. The choice of this threshold z as a function of $\varepsilon \in [0, 0.25]$ is shown in Figures 3 and 4 for $T = 100$, 300 and 1000, as well as the Gaussian threshold z_{ε}. The values of z are obtained by simulating $M = 10^7$ trajectories on $[0,T]$ of Poisson process of constant intensity $S(t,\omega) = 1$ and calculating empirical $1-\varepsilon$ quantiles of

Δ_T.

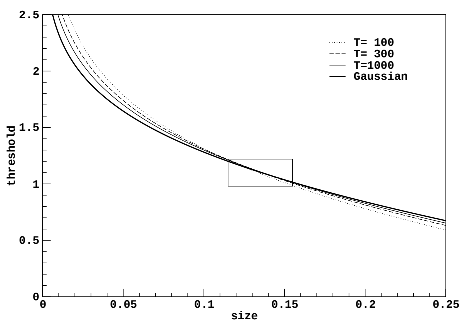

Fig. 3: Threshold choice

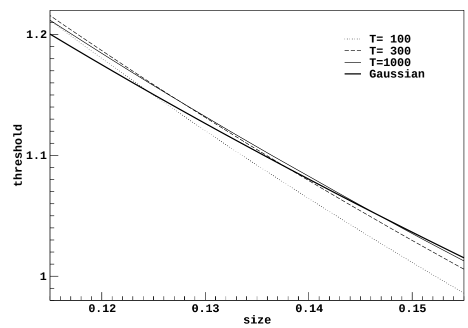

Fig. 4: Threshold choice (zoom)

For example to obtain test of exact size 0.05 one needs take $z \simeq 1.78$ for $T = 100$ ($z \simeq 1.74$ for $T = 300$, $z \simeq 1.70$ for $T = 1000$) against $z_\varepsilon \simeq 1.64$ for Gaussian case.

6.2 Nonparametric alternative

To illustrate the nonparametric alternatives we take intensity functions corresponding to $S_* = 1$ and $u(t) = (r/N)\ \chi_{\{0 \le t \le N\}}$, i.e.,

$$S(t, \omega) = 1 + \frac{r}{N\sqrt{T}} \sum_{t_i < t} \chi_{\{t - t_i \le N\}}, \qquad 0 \le t \le T,$$

where t_i are the events of the observed process. This choice of $u(\cdot)$ allows us to compare the power function of our locally asymptotically uniformly most powerful test

$$\hat{\phi}_T^\varepsilon\left(X^T\right) = \chi_{\left\{X_T > z_\varepsilon \sqrt{T} + T\right\}}$$

with the asymptotic power

$$\beta(r) = \frac{1}{\sqrt{2\pi}} \int_{z_\varepsilon - r}^{\infty} e^{-v^2/2}\, \mathrm{d}v$$

of Neyman-Pearson test for the least favorable alternatives.

Note that under $\mathscr{H}_0$, X_T is Poisson random variable with parameter T, therefore the size of the test $\hat{\phi}_T^\varepsilon$, as well as the threshold giving a test of exact size ε, can be calculated directly (without resort to Monte-Carlo simulations).

We represent the power function of the test $\hat{\phi}_T^{0.05}$ given by

$$\beta_T\left(u, \hat{\phi}_T\right) = \mathbf{P}_u\left\{X_T > z_{0.05}\sqrt{T} + T\right\}, \qquad 0 \le r \le 5$$

for $T = 100$, 300 and 1000 as well as the limiting (Gaussian) function $\beta(r)$, $0 \leq r \leq 5$. In Figures 5 and 6 we take $N = 5$ and in $N = 50$ respectively. The function β_T is obtained by simulating (for each value of r and N) $M = 10^6$ trajectories on $[0, T]$ of self-exciting process of intensity $S(t, \omega)$ and calculating empirical frequency of accepting the alternative hypothesis.

We see that if $1 \ll N \ll T$, then the power function converge to the limiting function (for example, if $N = 50$ and $T = 1000$, the power function almost coincides with the limiting one). If N and T are of the same order (for example, if $N = 50$ and $T = 100$) then the power function of the test can be essentially smaller. This example confirms the importance of use of functions with bounded support and of the order of limits in Definition 2.

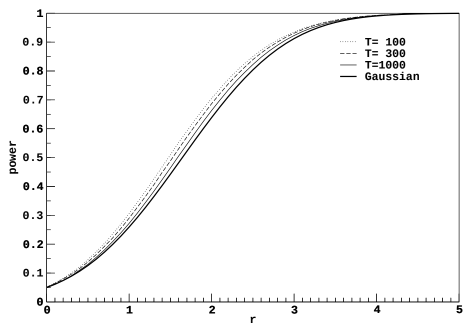

Fig. 5: Test power ($N = 5$)

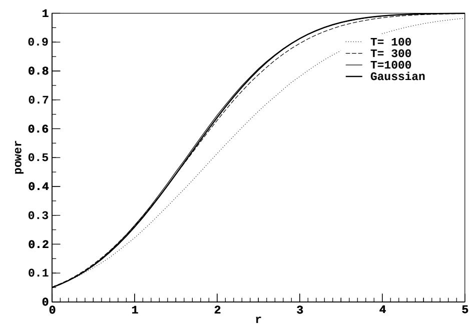

Fig. 6: Test power ($N = 50$)

7 Discussions

The constructed tests are asymptotically optimal for parametric (Section 3) and nonparametric (Section 5) alternatives. It seems that these are just the first results in this field and it is interesting to develope the construction of the asymptotically optimal tests for wider classes of alternatives. Particularly, it is intersting to study *smooth alternatives* like

$$\mathscr{H}_1 : \qquad \int_0^\infty u^{(k)}(t)^2 \, \mathrm{d}t > r,$$

where $r > 0$. Note that the test $\hat{\phi}_T$ is no more uniformly consistent in this situation.

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