On the Goodness-of-Fit Testing for Ergodic Diffusion Processes

Yury A. Kutoyants

Laboratoire de Statistique et Processus, Université du Maine

72085 Le Mans, Cédex 9, France

Abstract

We consider the goodness of fit testing problem for ergodic diffusion processes. The basic hypothesis is supposed to be simple. The diffusion coefficient is known and the alternatives are described by the different trend coefficients. We study the asymptotic distribution of the Cramer-von Mises type tests based on the empirical distribution function and local time estimator of the invariant density. At particularly, we propose a transformation which makes these tests asymptotically distribution free. We discuss the modifications of this test in the case of composite basic hypothesis.

MSC 2000 Classification: 62M02, 62G10, 62G20.

Key words: *Cramer-von Mises type tests, diffusion process, goodness of fit, hypotheses testing, ergodic diffusion.*

1 Introduction

The goodness of fit (GoF) tests play a special role in statistics because they form a bridge between mathematical models and the real data. In classical situation of i.i.d. observations $X^n = \{X_1, \ldots, X_n\}$ and the basic hypothesis $\mathscr{H}_0$: *the distribution function of* X_j *is* $F_0(x)$, the traditional solution is to construct a test statistics $\Delta_n = D\left(\hat{F}_n, F_0\right)$ based on some distance between the empirical distribution function $\hat{F}_n(x)$ and the given (known) function $F_0(x)$. Then the test function is defined by $\Psi_n = 1_{\{\Delta_n > c_\varepsilon\}}$, where the constant c_ε is chosen from the condition: $\lim_{n\to\infty} \mathbf{P}_0\{\Delta_n > c_\varepsilon\} = \varepsilon$, $\varepsilon \in (0,1)$.

The diversity of tests comes from the diversity of distances. At particularly, if we take

$$\Delta_n = n\int_{-\infty}^{\infty} H\left(x\right)\left[\hat F_n\left(x\right)-F_0\left(x\right)\right]^2 {\rm d}F_0\left(x\right), \qquad (1)$$

then we obtain the well-known Cramér-von Mises family of statistics [16]. If $H\left(x\right)\equiv 1$, then we have Cramér-von Mises test and if the weight function $H\left(x\right)=\left(F_0\left(x\right)\left[1-F_0\left(x\right)\right]\right)^{-1}$ we obtain the Anderson-Darling test.

In the case of uniform metric

$$\Delta_n = \sup_x \sqrt{n}\left|\hat F_n\left(x\right)-F_0\left(x\right)\right|, \qquad (2)$$

and $\Psi_n = 1_{\{\Delta_n > c_\varepsilon\}}$ we have Kolmogorov-Smirnov test. Remind that the tests based on these statistics are asymptotically distribution free (ADF) and for continuous $F_0\left(x\right)$ under hypothesis $\mathscr{H}_0$ we have the convergence

$$n\int_{-\infty}^{\infty}\left[\hat F_n\left(x\right)-F_0\left(x\right)\right]^2 {\rm d}F_0\left(x\right) \Longrightarrow \int_0^1 W_0\left(s\right)^2\ {\rm d}s,$$

$$n\int_{-\infty}^{\infty}\frac{\left[\hat F_n\left(x\right)-F_0\left(x\right)\right]^2}{F_0\left(x\right)\left[1-F_0\left(x\right)\right]}\ {\rm d}F_0\left(x\right) \Longrightarrow \int_0^1 \frac{W_0\left(s\right)^2}{s\left(1-s\right)}\ {\rm d}s,$$

$$\sup_x \sqrt{n}\left|\hat F_n\left(x\right)-F_0\left(x\right)\right| \Longrightarrow \sup_{0\le s\le 1}\left|W_0\left(s\right)\right|,$$

where $W_0\left(x\right)$ is Brownian bridge. These last property of the statistics allows us to chose once the constant c_ε for all $F_0\left(x\right)$. Note as well that the both statistics tend to ∞ for any fixed alternative $F\left(\cdot\right)\neq F_0\left(\cdot\right)$ and this property provides the consistency of these tests.

The present work is devoted to the similar problem but in the case of continuous time observations $X^T=\left\{X_t, 0\le t\le T\right\}$ of ergodic diffusion process

$${\rm d}X_t = S\left(X_t\right)\ {\rm d}t + \sigma\left(X_t\right)\ {\rm d}W_t, \qquad X_0, \quad 0\le t\le T. \qquad (3)$$

The diffusion coefficient $\sigma\left(\cdot\right)^2$ is supposed to be known and the hypothesis is concern the trend coefficient $S\left(\cdot\right)$ only. That means, that the basic hypothesis is simple:

$\mathscr{H}_0$: *The observed trajectory X^T is solution of the stochastic differential equation*

$${\rm d}X_t = S_0\left(X_t\right)\ {\rm d}t + \sigma\left(X_t\right)\ {\rm d}W_t, \qquad X_0, \quad 0\le t\le T$$

where $S_0\left(x\right)$ is some known function.

The alternative corresponds to the case $S\left(\cdot\right)\neq S_0\left(\cdot\right)$.

We suppose that the trend (under hypothesis and alternative) and diffusion coefficients satisfy the conditions:

$\mathcal{ES}$. *The function $S(\cdot)$ is locally bounded, the function $\sigma(\cdot)^2$ is continuous and positive and for some $A>0$ the inequality*

$$xS(x)+\sigma(x)^2\leq A\left(1+x^2\right)$$

holds.

This condition provides the existence of unique weak solution of this equation (see [3], p. 210).

Moreover, we suppose that the following condition is fulfilled too.

$\mathcal{RP}$. *The function*

$$V(S,x)=\int_0^x \exp\left\{-2\int_0^y \frac{S(z)}{\sigma(z)^2}\,\mathrm{d}z\right\}\,\mathrm{d}y\longrightarrow\pm\infty$$

as $x\to\pm\infty$ and

$$G(S)=\int_{-\infty}^{\infty}\sigma(y)^{-2}\exp\left\{2\int_0^y \frac{S(z)}{\sigma(z)^2}\,\mathrm{d}z\right\}\,\mathrm{d}y<\infty$$

By this condition the observed process is recurrent positive and has ergodic properties with the density of invariant law

$$f(x)=\frac{1}{G(S)\,\sigma(x)^2}\exp\left\{2\int_0^x \frac{S(y)}{\sigma(y)^2}\,\mathrm{d}y\right\}.$$

The corresponding density and distribution function under hypothesis $\mathscr{H}_0$ we denote as $f_0(x)$ and $F_0(x)$ and the mathematical expectation as $\mathbf{E}_0$. Moreover, we suppose that the initial value X_0 is a random variable with this distribution function because this condition simplifies exposition (the observed process is stationary).

Let us fix some $\varepsilon\in(0,1)$ and denote by $\mathcal{K}_\varepsilon$ the class of tests ψ_T of asymptotic size ε, i.e.; $\mathbf{E}_0\psi_T=\varepsilon+o(1)$. We are interested by the GoF tests of asymptotic size ε, which are ADF.

The problem of goodness of fit testing can be considered as follows: let us introduce some statistic δ_T such that its limit distribution $G(x)$ under hypothesis does not depend on the model, then the test $\psi_T=1_{\{\delta_T>c_\varepsilon\}}$, where the constant c_ε is solution of the equation $1-G(c_\varepsilon)=\varepsilon$ is ADF. Moreover, we require as well that for any fixed alternative (defined by the trend coefficient $S(\cdot)$) we have $\mathbf{P}\left\{\delta_T>c_\varepsilon\right\}\to 1$, i.e.; the test is consistent.

Let us remind here some of known tests satisfying these conditions. The first ones $\psi_T\left(X^T\right) = 1_{\{\delta_T > d_\varepsilon\}}$ and $\phi_T\left(X^T\right) = 1_{\{\gamma_T > c_\varepsilon\}}$ are based on the following two statistics

$$\delta_T = \frac{1}{T^2\,\mathbf{E}_0\left[\sigma\left(\xi\right)^2\right]} \int_0^T \left[X_t - X_0 - \int_0^t S_0\left(X_v\right)\,\mathrm{d}v\right]^2 \mathrm{d}t,$$

$$\gamma_T = \frac{1}{\sqrt{T\,\mathbf{E}_0\left[\sigma\left(\xi\right)^2\right]}} \sup_{0\leq t\leq T}\left|X_t - X_0 - \int_0^t S_0\left(X_v\right)\,\mathrm{d}v\right|.$$

Here and in the sequel ξ is the random variable with the density $f_0\left(\cdot\right)$.

It is shown that under hypothesis $\mathscr{H}_0$

$$\delta_T \Longrightarrow \int_0^1 w\left(v\right)^2 \mathrm{d}v, \qquad \gamma_T \Longrightarrow \sup_{0\leq v\leq 1}\left|w\left(v\right)\right|.$$

Hence the constants c_ε and d_ε are defined by the equations

$$\mathbf{P}\left\{\int_0^1 w\left(v\right)^2 \mathrm{d}v > d_\varepsilon\right\} = \varepsilon, \qquad \mathbf{P}\left\{\sup_{0\leq v\leq 1}\left|w\left(v\right)\right| > c_\varepsilon\right\} = \varepsilon. \qquad (4)$$

. These tests belong to $\mathcal{K}_\varepsilon$ and are consistent against any alternative $S\left(\cdot\right) \neq S_0\left(\cdot\right)$ such that $\mathbf{E}_0 S\left(\xi\right) \neq 0$. Moreover, it is shown that the test $\psi_T\left(X^T\right)$ is asymptotically optimal in special sense (see [2] for details).

Another GoF test $\varphi_T = 1_{\left\{\hat\Delta_T > e_\varepsilon\right\}}$ was proposed by Negri and Nishiyama [19]. It is based on the statistic

$$\hat\Delta_T = \frac{1}{\sqrt{T\mathbf{E}_0 \sigma\left(\xi\right)^2}} \sup_x \left|\int_0^T 1_{\{X_t < x\}}\ \left[\mathrm{d}X_t - S_0\left(X_t\right)\mathrm{d}t\right]\right| \qquad (5)$$

which converges to $\hat\Delta_0 = \sup_{0\leq v\leq 1}\left|w\left(v\right)\right|$. The constant c_ε is defined in (4). This test belongs to $\mathcal{K}_\varepsilon$ and is consistent against any fixed alternative satisfying condition: for some x we have $\mathbf{E}_0\left(1_{\{\xi<x\}}\left[S\left(\xi\right) - S_0\left(\xi\right)\right]\right) \neq 0$.

Note that the similar question of the goodness of fit testing for ergodic diffusion processes by discrete time observations was extensively studied (see Chen and Gao [1] and references therein).

The goal of this work is to study the tests which are direct analogues of the classical GoF tests like Anderson-Darling (1) and Kolmogorov-Smirnov (2).

To test the hypothesis $\mathscr{H}_0$ we propose two tests of Cramér-von Mises type. The first one is based on *empirical distribution function* (EDF)

$$\hat F_T\left(x\right) = \frac{1}{T}\int_0^T 1_{\{X_t<x\}}\,\mathrm{d}t$$

and the statistic is similar to (1):

$$\Delta_T = T \int_{-\infty}^{\infty} H(x) \left[\hat{F}_T(x) - F_0(x)\right]^2 \mathrm{d}F_0(x).$$

The second test is based on *local time estimator* (LTE) $\hat{f}_T(x)$ of the invariant density, which can be written as

$$\hat{f}_T(x) = \frac{\Lambda_T(x)}{T\,\sigma(x)^2} = \frac{|X_T - x| - |X_0 - x|}{T\,\sigma(x)^2} - \frac{1}{T\,\sigma(x)^2}\int_0^T \mathrm{sgn}(X_t - x)\ \mathrm{d}X_t.$$

Here $\Lambda_T(x)$ is the local time of the diffusion process. The corresponding statistic is

$$\delta_T = T \int_{-\infty}^{\infty} h(x) \left[\hat{f}_T(x) - f_0(x)\right]^2 \mathrm{d}F_0(x).$$

We discuss as well the Kolmogorov-Smirnov type test with the test statistic

$$\gamma_T = \sup_x \sqrt{T} g(x) \left|\hat{f}_T(x) - f_0(x)\right|. \tag{6}$$

The goodness of fit tests are $\Psi_T = 1_{\{\Delta_T > c_\varepsilon\}}$, and $\psi_T = 1_{\{\delta_T > d_\varepsilon\}}$ and $\hat{\psi}_T = 1_{\{\gamma_T > e_\varepsilon\}}$. These tests with $H(x) \equiv 1, h(x) \equiv 1, g(x) \equiv 1$, were proposed in [2], but unfortunately they are not ADF (see as well [6], [7], where similar test statistics are discussed).

The test φ_T, (5) is ADF. It is interesting to see the relation between the statistics (5) and (6). Suppose that $\sigma(x) \equiv 1$ and remind that

$$\bar{f}_T(x) = \frac{2}{T}\int_0^T 1_{\{X_t < x\}}\,\mathrm{d}X_t, \qquad f_T^*(x) = \frac{2}{T}\int_0^T 1_{\{X_t < x\}} S_0(X_t)\,\mathrm{d}t$$

are unbiased estimators of the invariant density [13]. The first is asymptotically equivalent to the local time estimator $\hat{f}_T(x)$, i.e.;

$$\sqrt{T}\left(\bar{f}_T(x) - \hat{f}_T(x)\right) = o(1)$$

and for the second we have

$$f_T^*(x) \longrightarrow 2\mathbf{E}_0 1_{\{\xi < x\}} S_0(\xi) = f_0(x).$$

Of course, the second estimator is not indeed estimator of the invariant density because it uses the function $S_0(x)$, but this choice of the second term makes the statistic asymptotically distribution free. Therefore

$$\hat{\Delta}_T = \frac{\sqrt{T}}{2}\sup_x \left|\bar{f}_T(x) - f_T^*(x)\right|.$$

The random processes $\{\eta_T(x), x\in\mathcal{R}\}$ and $\{\zeta_T(x), x\in\mathcal{R}\}$ where

$$\eta_T(x)=\sqrt{T}\left[\hat F_T(x)-F_0(x)\right]\quad\text{and}\quad \zeta_T(x)=\sqrt{T}\left[\hat f_T(x)-f_0(x)\right]$$

converge to the Gaussian processes $\{\eta(x), x\in\mathcal{R}\}$ and $\{\zeta(x), x\in\mathcal{R}\}$ with quite complicate structure (for the first convergence see [18] and for the second see [14]). For example,

$$\mathbf{E}_0\zeta(x)\zeta(y)=4f_0(x)f_0(y)\int_{-\infty}^{\infty}\frac{\left(1_{\{z>x\}}-F_0(z)\right)\left(1_{\{z>y\}}-F_0(z)\right)}{\sigma(z)^2f_0(z)}\,\mathrm{d}z$$

The goal of this work is to show that for certain choice of weight functions $H(x)$ and $h(x)$ the tests $\Psi_T=1_{\{\Delta_T>c_\varepsilon\}}$ and $\psi_T=1_{\{\delta_T>d_\varepsilon\}}$ can be ADF and consistent against some alternatives.

We discuss as well some other ADF GoF tests.

2 GoF tests based on LTE

Let us denote by μ the median of the invariant law : $F_0(\mu)=1/2$. We consider a special hypotheses testing problem, when the trend coefficient is fixed for $x\le\mu$ or for $x\ge\mu$. Therefore, we consider *one sided alternatives* only. Suppose that the changes in the trend coefficient are possible for $x\ge\mu$, i.e., the values of $S(x)$ for $x<\mu$ are the same under hypothesis and alternative.

We study first the Cramér-von Mises type test $\psi_T=1_{\{\delta_T>d_\varepsilon\}}$, where the statistic

$$\delta_T=T\int_{\mu}^{\infty}h(x)\left(\hat f_T(x)-f_0(x)\right)^2\mathrm{d}F_0(x)$$

with

$$h(x)=\frac{2F_0(x)-1}{4\Phi(\mu)^2\sigma(x)^2f_0(x)^4}\;e^{-\Phi(x)/\Phi(\mu)}\;1_{\{x\ge\mu\}},$$

and

$$\Phi(x)=\int_{-\infty}^{\infty}\frac{\left(1_{\{y>x\}}-F_0(y)\right)^2}{\sigma(y)^2f_0(y)}\,\mathrm{d}y.$$

The study of this test is based on the asymptotic normality of the LTE:

$$\sqrt{T}\left(\hat f_T(x)-f_0(x)\right)\Longrightarrow\zeta(x)\sim\mathcal{N}\left(0,d_f(x)^2\right),$$

where $d_f(x)^2=4f_0(x)^2\Phi(x)$. But we have two other classes of density estimators, which are consistent and asymptotically normal with the same limit distribution. These are kernel type estimators

$$\bar{f}_T(x) = \frac{1}{\sqrt{T}} \int_0^T K\left(\sqrt{T}\,(X_t - x)\right)\,\mathrm{d}t,$$

and the unbiased estimators

$$\tilde{f}_T(x) = \frac{2}{T}\int_0^T \frac{1_{\{X_t \le x\}}\; h(X_t)}{\sigma(x)^2\, h(x)}\,\mathrm{d}X_t + \frac{1}{T}\int_0^T \frac{1_{\{X_t \le x\}}\; h'(X_t)\;\sigma(X_t)^2}{\sigma(x)^2\, h(x)}\,\mathrm{d}t$$

where $h(\cdot) \in \mathcal{C}'$ is an arbitrary function. Under mild regularity conditions (see [14], Propositions 1.58 and 1.61) we have the same asymptotic normality

$$\sqrt{T}\left(\bar{f}_T(x) - f(x)\right) \Longrightarrow \mathcal{N}\left(0, d_f(x)^2\right)$$

and

$$\sqrt{T}\left(\tilde{f}_T(x) - f(x)\right) \Longrightarrow \mathcal{N}\left(0, d_f(x)^2\right).$$

Therefore, the results obtained for the test ψ_T based on the LTE can be valid for the tests based on the kernel-type or unbiased estimators as well (under strengthen conditions).

The calculation of the statistic δ_T can be slightly simplified. Remind the equalities

$$\frac{1}{T}\int_0^T g(X_t)\,\mathrm{d}t = \int_{-\infty}^{\infty} g(x)\;\hat{f}_T(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\;\mathrm{d}\hat{F}_T(x).$$

Hence

$$\begin{aligned}
\delta_T &= T\int_\mu^\infty h(x)\left(\hat{f}_T(x) - f_0(x)\right)^2 \mathrm{d}\hat{F}_T(x) \\
&\quad + T\int_\mu^\infty h(x)\left(\hat{f}_T(x) - f_0(x)\right)^2 \mathrm{d}\left[F_0(x) - \hat{F}_T(x)\right] \\
&= \int_0^T h(X_t)\left[\hat{f}_T(X_t) - f_0(X_t)\right]^2 \mathrm{d}t - \frac{1}{\sqrt{T}}\int_\mu^\infty h(x)\;\zeta_T(x)^3\,\mathrm{d}x \\
&= \int_0^T h(X_t)\left[\hat{f}_T(X_t) - f_0(X_t)\right]^2 \mathrm{d}t + o(1)
\end{aligned}$$

and we can use the statistic

$$\delta_T^* = \int_0^T h(X_t)\left[\hat{f}_T(X_t) - f_0(X_t)\right]^2 \mathrm{d}t \qquad (7)$$

in the construction of our GoF test. To avoid the difficulties with calculation of stochastic integral related to $\hat{f}_T(X_t)$ we can use here the kernel type estimator of the density.

Note that the LTE of the density is the derivative with probability 1 of the EDF. Indeed, we have the equality ($r > 0$)

$$\frac{\hat{F}_T(x+r) - \hat{F}_T(x)}{r} = \frac{1}{rT}\int_0^T 1_{\{x \le X_t \le x+r\}}\, \mathrm{d}t = \frac{1}{r}\int_x^{x+r} \hat{f}_T(y)\, \mathrm{d}y$$

and, as the local time is continuous with probability one (see [20]), we have

$$\lim_{r\to 0}\frac{1}{r}\int_x^{x+r} \hat{f}_T(y)\, \mathrm{d}y = \hat{f}_T(x).$$

Let us define the constant d_ε as solution of the equation

$$\mathbf{P}\left\{\int_1^\infty w(v)^2\, e^{-v}\, \mathrm{d}v > d_\varepsilon\right\} = \varepsilon,$$

where $w(\cdot)$ is a Wiener process and introduce the following condition.

Condition $\mathcal{A}$. *The function $\Phi(x) < \infty$ for all $x \in (\mu, \infty)$ and*

$$A_1 = \int_\mu^\infty \frac{(2F_0(x) - 1)\,\Phi(x)}{\Phi(\mu)^2\,\sigma(x)^2\, f_0(x)^2}\; e^{-\frac{\Phi(x)}{\Phi(\mu)}}\, \mathrm{d}x < \infty, \tag{8}$$

$$A_2 = \int_\mu^\infty \frac{(2F_0(x) - 1)\; e^{-\frac{\Phi(x)}{\Phi(\mu)}}}{\Phi(\mu)^2\,\sigma(x)^2\, f_0(x)^2}\,\mathbf{E}_0\left(\int_0^\xi \frac{1_{\{v>x\}} - F_0(v)}{\sigma(v)^2\, f_0(v)}\,\mathrm{d}v\right)^2 \mathrm{d}x < \infty. \tag{9}$$

Proposition 1 *Let the conditions $\mathcal{ES}$, $\mathcal{RP}$ and $\mathcal{A}$ be fulfilled, then the GoF test $\psi_T = 1_{\{\delta_T > d_\varepsilon\}}$ is ADF.*

Proof. We have to prove the convergence (under hypothesis $\mathscr{H}_0$)

$$\delta_T \Longrightarrow \delta_0 = \int_1^\infty w(v)^2\, e^{-v}\,\mathrm{d}v.$$

First we remind some properties of the local time estimator of the density (under hypothesis $\mathscr{H}_0$). The following presentation is valid (see [14], p. 29)

$$\zeta_T(x) = \frac{2f_0(x)}{\sqrt{T}}\int_0^T \frac{F_0(X_t) - 1_{\{X_t > x\}}}{\sigma(X_t)\, f_0(X_t)}\,\mathrm{d}W_t + \frac{2f_0(x)}{\sqrt{T}}\int_{X_0}^{X_T} \frac{1_{\{v>x\}} - F_0(v)}{\sigma(v)^2\, f_0(v)}\,\mathrm{d}v = \zeta_T^{(1)}(x) + \zeta_T^{(2)}(x) \tag{10}$$

in obvious notation. We supposed that the process $X_t, t \geq 0$ is stationary, hence

$$\mathbf{E}_0 \left(\zeta_T^{(2)}(x) \right)^2 = \mathbf{E}_0 \left(\frac{2 f_0(x)}{\sqrt{T}} \int_{X_0}^{X_T} \frac{1_{\{v>x\}} - F_0(v)}{\sigma(v)^2 f_0(v)} \, \mathrm{d}v \right)^2$$

$$\leq \frac{16}{T} \mathbf{E}_0 \left(\int_0^{\xi} \frac{1_{\{v>x\}} - F_0(v)}{\sigma(v)^2 f_0(v)} \, \mathrm{d}v \right)^2 .$$

Further, by condition (9) we have

$$4 \int_{\mu}^{\infty} h(x) f_0(x)^2 \mathbf{E}_0 \left(\int_0^{\xi} \frac{1_{\{v>x\}} - F_0(v)}{\sigma(v)^2 f_0(v)} \, \mathrm{d}v \right)^2 \mathrm{d}x$$

$$= \int_{\mu}^{\infty} \frac{(2F_0(x) - 1) \, e^{-\frac{\Phi(x)}{\Phi(\mu)}}}{\Phi(\mu)^2 \sigma(x)^2 f_0(x)^2} \mathbf{E}_0 \left(\int_0^{\xi} \frac{1_{\{v>x\}} - F_0(v)}{\sigma(v)^2 f_0(v)} \, \mathrm{d}v \right)^2 \mathrm{d}x < \infty.$$

Hence

$$\int_{\mu}^{\infty} h(x) \, \mathbf{E}_0 \left(\zeta_T^{(2)} \right)^2 \mathrm{d}F_0(x) \leq \frac{A_2}{T} \longrightarrow 0.$$

Therefore it is sufficient to study the first integral in (10), i.e.,

$$\delta_T = \int_{\mu}^{\infty} \frac{(2F_0(x) - 1) \, e^{-\frac{\Phi(x)}{\Phi(\mu)}}}{T \Phi(\mu)^2 \sigma(x)^2 f_0(x)^2} \left(\int_0^T \frac{1_{\{X_t>x\}} - F_0(X_t)}{\sigma(X_t) f_0(X_t)} \, \mathrm{d}W_t \right)^2 \mathrm{d}x + o(1) .$$

Let us denote

$$\hat{\zeta}_T(x) = \frac{1}{\sqrt{T}} \int_0^T \frac{1_{\{X_t>x\}} - F_0(X_t)}{\sigma(X_t) f_0(X_t)} \, \mathrm{d}W_t.$$

By the law of large numbers we have the convergence

$$\frac{1}{T} \int_0^T \left(\frac{1_{\{X_t>x\}} - F_0(X_t)}{\sigma(X_t) f_0(X_t)} \right)^2 \mathrm{d}t \longrightarrow \Phi(x) .$$

hence we can apply the central limit theorem (see, e.g., [14],Theorem 1.19) and to obtain the convergence of all finite dimensional distributions of $\hat{\zeta}_T(\cdot)$ to the multidimensional Gaussian law:

$$\left(\hat{\zeta}_T(x_1), \ldots, \hat{\zeta}_T(x_k) \right) \Longrightarrow \left(\hat{\zeta}(x_1), \ldots, \hat{\zeta}(x_k) \right), \tag{11}$$

where the Gaussian process $\zeta(x)$ is with zero mean and it's covariance function is

$$R(x, y) = \int_{-\infty}^{\infty} \frac{\left(1_{\{v>x\}} - F_0(v)\right)\left(1_{\{v>y\}} - F_0(v)\right)}{\sigma(v)^2 f_0(v)} \, \mathrm{d}v.$$

The process $\hat{\zeta}(x)$ admits the representation

$$\hat{\zeta}(x) = \int_{-\infty}^{\infty} \frac{1_{\{y>x\}} - F_0(y)}{\sigma(y)\sqrt{f_0(y)}}\, \mathrm{d}W_y.$$

The last integral is with respect to double-sided Wiener process, i.e.; $W_y = W^+(y), y \geq 0$ and $W_y = W^-(-y), y \leq 0$, where $W^-(\cdot)$ and $W^+(\cdot)$ are two independent Wiener processes.

The convergence (11) formally can be obtained as follows (egalities in distribution)

$$\begin{aligned}
\hat{\zeta}_T(x) &= W\left(\frac{1}{T}\int_0^T \left(\frac{1_{\{X_t>x\}} - F_0(X_t)}{\sigma(X_t) f_0(X_t)}\right)^2 \mathrm{d}t\right) + o(1) \\
&= W\left(\int_{-\infty}^{\infty} \left(\frac{1_{\{y>x\}} - F_0(y)}{\sigma(y) f_0(y)}\right)^2 \hat{f}_T(y)\, \mathrm{d}t\right) + o(1) \\
&\Longrightarrow W\left(\int_{-\infty}^{\infty} \left(\frac{1_{\{y>x\}} - F_0(y)}{\sigma(y) f_0(y)}\right)^2 f_0(y)\, \mathrm{d}y\right) \\
&= \int_{-\infty}^{\infty} \frac{1_{\{y>x\}} - F_0(y)}{\sigma(y)\sqrt{f_0(y)}}\, \mathrm{d}W_y = \hat{\zeta}(x)
\end{aligned}$$

where $W(s), s \geq 0$ is some Wiener process. We used above the following property of local time of ergodic diffusion process : for any integrable function $g(\cdot)$

$$\frac{1}{T}\int_0^T g(X_t)\, \mathrm{d}t = \int_{-\infty}^{\infty} g(y)\, \hat{f}_T(y)\, \mathrm{d}y \longrightarrow \int_{-\infty}^{\infty} g(y)\;\, f_0(y)\, \mathrm{d}y. \qquad (12)$$

To verify the convergence

$$\delta_T \Longrightarrow 4\int_{\mu}^{\infty} h(x)\, f_0(x)^2\, \hat{\zeta}(x)^2\, \mathrm{d}F_0(x) = 4\int_{\mu}^{\infty} h(x)\, f_0(x)^3\, W(\Phi(x))^2\, \mathrm{d}x$$

we note that for any $\varepsilon > 0$ there exists $L > \mu$ such that

$$\int_L^{\infty} h(x)\, \mathbf{E}_0\left(\zeta_T^{(1)}(x)\right)^2 \mathrm{d}F_0(x) \leq \varepsilon.$$

Therefore it is sufficient to show that for any $L > \mu$

$$\int_{\mu}^{L} h(x)\, f_0(x)^2\;\, \hat{\zeta}_T(x)^2\, \mathrm{d}F_0(x) \Longrightarrow \int_{\mu}^{L} h(x)\, f_0(x)^2\, \hat{\zeta}(x)^2\, \mathrm{d}F_0(x).$$

This last convergence follows from the the convergence of finite dimensional distributions (11) and the estimate

$$\mathbf{E}_0 \left|\hat{\zeta}_T (x_2)^2 - \hat{\zeta}_T (x_1)^2\right| \leq C_L \; |x_2 - x_1|^{1/2} \tag{13}$$

which is valid for all $|x_i| \leq L$ (see [9], Theorem 9.7.1). The estimate (13) can be obtained as follows. We have

$$\begin{aligned}
&\left(\mathbf{E}_0 \left|\hat{\zeta}_T (x_2)^2 - \hat{\zeta}_T (x_1)^2\right|\right)^2 \\
&\qquad \leq \mathbf{E}_0 \left|\hat{\zeta}_T (x_2) - \hat{\zeta}_T (x_1)\right|^2 \mathbf{E}_0 \left|\hat{\zeta}_T (x_2) + \hat{\zeta}_T (x_1)\right|^2 \\
&\qquad \leq \mathbf{E}_0 \left|\hat{\zeta}_T (x_2) - \hat{\zeta}_T (x_1)\right|^2 \left(2\mathbf{E}_0 \hat{\zeta}_T (x_2)^2 + 2\mathbf{E}_0 \hat{\zeta}_T (x_1)^2\right) \\
&\qquad \leq (2\Phi (x_2) + 2\Phi (x_1))\, \mathbf{E}_0 \left|\hat{\zeta}_T (x_2) - \hat{\zeta}_T (x_1)\right|^2 .
\end{aligned}$$

Further (let $x_2 > x_1$)

$$\begin{aligned}
\mathbf{E}_0 \left|\hat{\zeta}_T (x_2) - \hat{\zeta}_T (x_1)\right|^2 &= \mathbf{E}_0 \left(\frac{1}{\sqrt{T}} \int_0^T \frac{1_{\{x_1 \leq X_t \leq x_2\}}}{\sigma (X_t)\, f_0 (X_t)} \,\mathrm{d}W_t\right)^2 \\
&= \int_{x_1}^{x_2} \frac{\mathrm{d}v}{\sigma (v)^2 f_0 (v)} \leq B_L \,|x_2 - x_1| \,.
\end{aligned}$$

Remind that $h(\cdot)$ is continuous on $[\mu, L]$.

The last step is to verify the equality (in distribution)

$$4 \int_\mu^\infty h (x)\, f_0 (x)^2 \,\hat{\zeta} (x)^2 \,\mathrm{d}F_0 (x) = \int_1^\infty w (v)^2 \,e^{-v} \,\mathrm{d}v.$$

For the function

$$\begin{aligned}
\Phi (x) &= \int_{-\infty}^\infty \frac{\left(1_{\{y>x\}} - F_0 (y)\right)^2}{\sigma (y)^2 f_0 (y)} \,\mathrm{d}y \\
&= \int_{-\infty}^x \frac{F_0 (y)^2}{\sigma (y)^2 f_0 (y)} \,\mathrm{d}y + \int_x^\infty \frac{(1 - F_0 (y))^2}{\sigma (y)^2 f_0 (y)} \,\mathrm{d}y
\end{aligned}$$

we have

$$\Phi' (x) = \frac{2F_0 (x) - 1}{\sigma (x)^2 f_0 (x)} < 0, \quad \text{for} \quad x < \mu$$

and $\Phi' (x) > 0$ for $x > \mu$. Hence the functions $\Phi (x)\,, x \leq \mu$ and $\Phi (x)\,, x \geq \mu$ are strictly monotone (decreasing and increasing respectively). Moreover, $\Phi (\pm\infty) = \infty$.

We can write

$$
\begin{aligned}
4 & \int_{\mu}^{\infty} h(x) f_0(x)^3 W(\Phi(x))^2 \,\mathrm{d}x \\
&= 4 \int_{\mu}^{\infty} \frac{h(x)\,\sigma(x)^2 f_0(x)^4}{2F_0(x)-1} W(\Phi(x))^2 \;\Phi'(x)\;\mathrm{d}x \\
&= \int_{\mu}^{\infty} \frac{W(\Phi(x))^2}{\Phi(\mu)^2}\, e^{-\Phi(x)/\Phi(\mu)}\,\mathrm{d}\Phi(x) \\
&= \int_{\Phi(\mu)}^{\infty} \frac{W(z)^2}{\Phi(\mu)^2}\, e^{-z/\Phi(\mu)}\,\mathrm{d}z = \int_{1}^{\infty} w(v)^2 \; e^{-v}\,\mathrm{d}v.
\end{aligned}
$$

To show the consistency of this test against any fixed alternative

$$
\mathscr{H}_1: \quad S(x) \neq S_0(x)
$$

we just note that

$$
\sqrt{T}\left(\hat{f}_T(x) - f_0(x)\right) = \sqrt{T}\left(\hat{f}_T(x) - f(x)\right) + \sqrt{T}\left(f(x) - f_0(x)\right)
$$

where the first term is asymptotically normal and for the second term we have

$$
T\int_{\mu}^{\infty} h(x)\left(f(x) - f_0(x)\right)^2 \mathrm{d}F_0(x) \longrightarrow \infty.
$$

Here $f(x)$ is the invariant density function (under alternative). Of course, we have to suppose that the corresponding integrals like (8) and (9) are finite.

We now apply the similar arguments to study the Kolmogorov-Smirnov type statistic

$$
\gamma_T = \sup_{x\geq\mu} \sqrt{T} g(x) \left|\hat{f}_T(x) - f_0(x)\right|.
$$

Our goal is to chose such weight function $g(\cdot) \geq 0$ that the GoF test $\hat{\psi}_T = 1_{\{\gamma_T > e_\varepsilon\}}$ based on this statistic be ADF. Let us put $\sigma(x) \equiv 1$ (for simplicity). Remind that the weak convergence

$$
\zeta_T(\cdot) \Longrightarrow \zeta(\cdot)
$$

in the space of continuous functions vanishing at infinity was already proved in [14], Theorem 4.13. This convergence provides as well the convergence of our statistic γ_T with $g(x) \equiv 1$. Suppose that the function $g(\cdot)$ is such that we have the convergence $\gamma_T \Rightarrow \gamma_0 = \sup_x g(x)|\zeta(x)|$ too. Let us put

$$g(x) = \frac{1}{2f_0(x)\sqrt{\Phi(\mu)}}\, e^{-\Phi(x)/\Phi(\mu)}.$$

Then we can write (egalities in distribution)

$$\begin{aligned}\gamma_0 &= \sup_{x\geq\mu} 2g(x)\, f_0(x) \left|\int_{-\infty}^{\infty} \frac{1_{\{y>x\}} - F_0(y)}{\sigma(y)\sqrt{f_0(y)}}\, \mathrm{d}W_y\right| \\ &= \sup_{x\geq\mu} 2g(x)\, f_0(x)\, |W(\Phi(x))| \\ &= \sup_{x\geq\mu} \frac{|W(\Phi(x))|}{\sqrt{\Psi(\mu)}}\, e^{-\Phi(x)/\Phi(\mu)} = \sup_{v\geq 1} |w(v)|\, e^{-v}.\end{aligned}$$

We see that the test $\psi_T = 1_{\{\gamma_T \geq e_\varepsilon\}}$ with e_ε from the equation

$$\mathbf{P}\left\{\sup_{v\geq 1} |w(v)|\, e^{-v} \geq e_\varepsilon\right\} = \varepsilon$$

is ADF and belongs to $\mathcal{K}_\varepsilon$.

Remark. Note that these arguments do not work directly in the case of *double sided alternatives*, i.e., if the function $S(x)$ changes under alternative for the values $x < \mu$ too. It can be shown that the limit

$$\delta_T \Longrightarrow \int_1^{\infty} \left[w_1(v)^2 + w_2(v)^2\right] e^{-v} \mathrm{d}v$$

holds, but the Wiener processes $w_1(\cdot)$ and $w_2(\cdot)$ are correlated and the correlation function depends on the model.

3 GoF test based on EDF

We study the GoF test $\Psi_T = 1_{\{\Delta_T > c_\varepsilon\}}$ with Cramér-von Mises type statistic

$$\Delta_T = T \int_\mu^{\infty} H(x) \left(\hat{F}_T(x) - F_0(x)\right)^2 \mathrm{d}F_0(x)$$

and our goal is to chose such weights $H(\cdot) \geq 0$ that this statistic converges to the *distribution free* limit:

$$\Delta_T \Longrightarrow \int_1^{\infty} w(v)^2\, e^{-v}\, \mathrm{d}v.$$

This statistic can be written in the *empirical form* like (7)

$$\Delta_T^* = \int_0^T H\left(X_t\right)\left(\hat F_T\left(X_t\right) - F_0\left(X_t\right)\right)^2 \,\mathrm{d}t$$

because $\Delta_T = \Delta_T^* + o\left(1\right)$ with the same explication as above.

The properties of this statistic are quite close to that of δ_T, that is why we do not give here all details.

We suppose that the conditions $\mathscr{E}S, RP$ and $\Phi\left(x\right) < \infty$ are fulfilled and

$$\int_\mu^\infty H\left(x\right) f_0\left(x\right) \mathbf{E}_0 \left(\frac{F_0\left(\xi\right) F_0\left(x\right) - F_0\left(\xi\wedge x\right)}{\sigma\left(\xi\right) f_0\left(\xi\right)}\right)^2 \mathrm{d}x < \infty \tag{14}$$

$$\int_\mu^\infty H\left(x\right) f_0\left(x\right) \mathbf{E}_0 \left(\int_\mu^\xi \frac{F_0\left(y\right) F_0\left(x\right) - F_0\left(y\wedge x\right)}{\sigma\left(y\right)^2 f_0\left(y\right)}\mathrm{d}y\right)^2 \mathrm{d}x < \infty \tag{15}$$

The process $\eta_T\left(x\right) = \sqrt{T}\left(\hat F_T\left(x\right) - F_0\left(x\right)\right)$ admits the presentations (see [14], p. 85)

$$\begin{aligned}\eta_T\left(x\right) &= \frac{2}{\sqrt{T}}\int_0^T \frac{F_0\left(X_t\right) F_0\left(x\right) - F_0\left(X_t\wedge x\right)}{\sigma\left(X_t\right) f_0\left(X_t\right)}\,\mathrm{d}W_t \\ &\quad + \frac{2}{\sqrt{T}}\int_{X_0}^{X_T} \frac{F_0\left(v\wedge x\right) - F_0\left(v\right) F_0\left(x\right)}{\sigma\left(v\right)^2 f_0\left(v\right)}\,\mathrm{d}v.\end{aligned} \tag{16}$$

Note that if $\Phi\left(x\right) < \infty$, then (see [14], Remark 1.64)

$$d_F\left(x\right)^2 = 4\mathbf{E}_0\left(\frac{F_0\left(\xi\right) F_0\left(x\right) - F_0\left(\xi\wedge x\right)}{\sigma\left(\xi\right) f_0\left(\xi\right)}\right)^2 < \infty$$

too and

$$\sqrt{T}\left(\hat F_T\left(x\right) - F_0\left(x\right)\right) \Longrightarrow \mathcal{N}\left(0, d_F\left(x\right)^2\right).$$

It can be shown that (under mild regularity conditions)

$$\Delta_T \Longrightarrow \Delta_0 = \int_\mu^\infty H\left(x\right) f_0\left(x\right) \hat\eta\left(x\right)^2 \mathrm{d}x,$$

where

$$\hat\eta\left(x\right) = \left[F_0\left(x\right) - 1\right]\int_{-\infty}^x \frac{F_0\left(y\right)}{\sigma\left(y\right)\sqrt{f_0\left(y\right)}}\mathrm{d}W_y + F_0\left(x\right)\int_x^\infty \frac{F_0\left(y\right) - 1}{\sigma\left(y\right)\sqrt{f_0\left(y\right)}}\mathrm{d}W_y.$$

Hence

$$\Delta_0 = \int_\mu^\infty H\left(x\right) f_0\left(x\right) \left[F_0\left(x\right) - 1\right]^2 W\left(\Psi\left(x\right)\right)^2 \mathrm{d}x,$$

where

$$\Psi\left(x\right)=\int_{-\infty}^{x}\frac{F_0\left(y\right)^2}{\sigma\left(y\right)^2 f_0\left(y\right)}\mathrm{d}y+F_0\left(x\right)^2\int_{x}^{\infty}\left(\frac{F_0\left(y\right)-1}{F_0\left(x\right)-1}\right)^2\frac{\mathrm{d}y}{\sigma\left(y\right)^2 f_0\left(y\right)}.$$

Further

$$\begin{aligned}\Delta_0&=\int_{\mu}^{\infty}\frac{H\left(x\right)f_0\left(x\right)\left[F_0\left(x\right)-1\right]^2}{\Psi'\left(x\right)}W\left(\Psi\left(x\right)\right)^2\,\mathrm{d}\Psi\left(x\right)\\&=\Psi\left(0\right)^{-2}\int_{\mu}^{\infty}W\left(\Psi\left(x\right)\right)^2e^{-\Psi(x)/\Psi(\mu)}\,\mathrm{d}\Psi\left(x\right)\\&=\int_{1}^{\infty}w\left(v\right)^2e^{-v}\,\mathrm{d}v,\end{aligned}$$

where we put $\Psi\left(x\right)=v\Psi\left(\mu\right)$, $w\left(v\right)=\Psi\left(\mu\right)^{-1/2}W\left(v\Psi\left(\mu\right)\right)$ and

$$H\left(x\right)=\frac{\Psi'\left(x\right)}{\Psi\left(\mu\right)^2 f_0\left(x\right)\left[F_0\left(x\right)-1\right]^2}\;e^{-\Psi(x)/\Psi(0)}.$$

Of course, we suppose that the function $\Psi\left(x\right),x\geq\mu$ is strictly monotone and $\Psi\left(\infty\right)=\infty$.

4 Examples

Let us consider two examples. The first one is
Example 1. **Ornstein-Uhlenbeck process.** Suppose that the observed process under the basic hypothesis is

$$\mathrm{d}X_t=-a\left(X_t-b\right)\mathrm{d}t+\sigma\,\mathrm{d}W_t,\quad X_0,\;0\leq t\leq T,$$

where $a>0$. The invariant density is Gaussian $f_0\left(x\right)\sim\mathcal{N}\left(b,\frac{\sigma^2}{2a}\right)$ with median $\mu=b$. To check the conditions (8) and (9) we estimate first the asymptotics of $\Psi\left(x\right)$ as $x\rightarrow\infty$. We have

$$\begin{aligned}\Phi\left(x\right)&=\int_{-\infty}^{\infty}\frac{\left(1_{\{y>x\}}-F_0\left(y\right)\right)^2}{\sigma^2 f_0\left(y\right)}\,\mathrm{d}y=\int_{-\infty}^{x}\frac{1}{\sigma^2 f_0\left(y\right)}\,\mathrm{d}y\;\left(1+o\left(1\right)\right)\\&=\frac{c}{x}\;e^{\frac{ax^2}{\sigma^2}}\;\left(1+o\left(1\right)\right).\end{aligned}$$

Hence the both conditions are fulfilled and the test $\psi_T=1_{\{\delta_T>d_\varepsilon\}}$ with

$$\delta_T=T\int_{b}^{\infty}\frac{2F_0\left(x\right)-1}{4\Phi\left(\mu\right)^2\sigma\left(x\right)^2 f_0\left(x\right)^3}\;e^{-\Phi(x)/\Phi(\mu)}\;\left(\hat f_T\left(x\right)-\frac{\sqrt{a}}{\sigma\sqrt{\pi}}e^{-\frac{a(x-b)^2}{\sigma^2}}\right)^2\mathrm{d}x$$

is ADF.

It is easy to see that the conditions (14) and (15) are fulfilled too and the test $\Psi_T = 1_{\{\Delta_T > d_\varepsilon\}}$ with

$$\Delta_T = T \int_b^\infty \frac{\Psi'(x)}{4\Psi(\mu)^2 [F_0(x) - 1]^2} \; e^{-\Psi(x)/\Psi(0)} \; \left(\hat{F}_T(x) - F_0(x)\right)^2 \, \mathrm{d}x$$

is ADF.

Example 2. **Simple switching process.** Suppose that the observed process under hypothesis is

$$\mathrm{d}X_t = -a \,\mathrm{sgn}\,(X_t - b)\, \mathrm{d}t + \sigma \,\mathrm{d}W_t, \quad X_0, \;\; 0 \le t \le T$$

where $a > 0$. The process is ergodic with invariant density

$$f_0(x) = \frac{a}{\sigma^2} \; \exp\left\{-\frac{2a}{\sigma^2}\,|x - b|\right\}$$

and median $\mu = b$. The function

$$\Psi(x) = \int_{-\infty}^x \frac{1}{\sigma^2 f_0(y)}\, \mathrm{d}y \;\; (1 + o(1)) = a^{-1}\, e^{\frac{2a}{\sigma}x} \;\; (1 + o(1))$$

as $x \to \infty$ and the conditions (8), (9) and (14), (15) are fulfilled. The direct calculation shows that $\Psi(x)$ is strictly monotone function. Therefore the corresponding tests ψ_T and Ψ_T are ADF.

The limit distribution of the test statistic δ_T with $h(x) \equiv 1$ were studied by Gassem [8], who obtained the Karhunen-Loeve expansion for the limit Gaussian process $\zeta(\cdot)$.

5 Composite hypotheses

Suppose that the observed diffusion process (under hypothesis $\mathscr{H}_0$) is

$$\mathrm{d}X_t = S(\vartheta, X_t)\, \mathrm{d}t + \sigma(X_t)\, \mathrm{d}W_t, \quad X_0, \;\; 0 \le t \le T$$

where ϑ is unknown parameter $\vartheta \in (a, b)$. Therefore the basic hypothesis is composite. The test statistic can be

$$\delta_T = T \int_\mu^\infty h\left(\hat{\vartheta}_T, x\right) \left(\hat{f}_T(x) - f_0(\hat{\vartheta}_T, x)\right)^2 \mathrm{d}F_0(\hat{\vartheta}_T, x),$$

where $\hat\vartheta_T$ is some consistent and asymptotically normal estimator of ϑ and $h(\vartheta,x)$ is the same function as before with obvious modification, say, $f_0(x)=f_0(\vartheta,x)$. Unfortunately the test based on this statistic is no more ADF because its limit distribution depends on the distribution of estimator. To compensate this contribution of estimator we can modify this statistic as follows (see, e.g., Koul [12] for similar transformation in time series). Suppose that $\hat\vartheta_T$ is the MLE and the corresponding regularity conditions are fulfilled (see [14], Theorem 2.8). Then we have

$$\sqrt{T}\left(\hat\vartheta_T-\vartheta\right)=\mathrm{I}(\vartheta)^{-1}\,T^{-1/2}\int_0^T\frac{\dot S(\vartheta,X_t)}{\sigma(X_t)}\,\mathrm{d}W_t+o(1),$$

where dot means derivation w.r.t. ϑ.

We want to substitute here the MLE, but in this case the stochastic integral is not well defined, that is why we first rewrite this integral in the following form (Itô formula)

$$\begin{aligned}R_T(\vartheta)&=\int_0^T\frac{\dot S(\vartheta,X_s)}{\sigma(X_s)}\,\mathrm{d}W_s=\int_0^T\frac{\dot S(\vartheta,X_s)}{\sigma(X_s)^2}\left[\mathrm{d}X_s-S(\vartheta,X_s)\,\mathrm{d}s\right]\\&=\int_{x_0}^{X_T}\frac{\dot S(\vartheta,y)}{\sigma(y)^2}\,\mathrm{d}y-\int_0^T\frac{\dot S'(\vartheta,X_s)\,\sigma(X_s)-2\dot S(\vartheta,X_s)\,\sigma'(X_x)}{2\sigma(X_s)}\,\mathrm{d}s\\&\quad-\int_0^T\frac{\dot S(\vartheta,X_s)\;S(\vartheta,X_s)}{\sigma(X_s)^2}\,\mathrm{d}s.\end{aligned}$$

Here prim means derivation w.r.t. x. The last expression contains no stochastic integral and we use it as definition of $R_T(\vartheta)$, where we can put $\hat\vartheta_T$. Now we can introduce the test statistic

$$\begin{aligned}\hat\delta_T=T\int_\mu^\infty h\left(\hat\vartheta_T,x\right)\Big(\hat f_T(x)-f_0(\hat\vartheta_T,x)\\+\dot f_0(\hat\vartheta_T,x)\,\mathrm{I}(\hat\vartheta_T)\,T^{-1}\,R_T(\hat\vartheta_T)\Big)^2\,\mathrm{d}F_0(\hat\vartheta_T,x).\end{aligned}$$

Note that

$$\begin{aligned}\hat f_T(x)-f_0(\hat\vartheta_T,x)&=\hat f_T(x)-f_0(\vartheta,x)+f_0(\vartheta,x)-f_0(\hat\vartheta_T,x)\\&=\hat f_T(x)-f_0(\vartheta,x)-\dot f_0(\hat\vartheta_T,x)(\hat\vartheta_T-\vartheta)\,(1+o(1))\,.\end{aligned}$$

Hence

$$\begin{aligned}\hat\delta_T&=T\int_\mu^\infty h(\vartheta,x)\left(\hat f_T(x)-f_0(\vartheta,x)\right)^2\mathrm{d}F_0(\vartheta,x)+o(1)\\&\Longrightarrow\int_1^\infty w(v)^2\,e^{-v}\,\mathrm{d}v\end{aligned}$$

and the test $\hat{\psi}_T = 1_{\left\{\hat{\delta}_T > d_\varepsilon\right\}}$ is ADF.

We supposed here that the median μ does not depend on ϑ (as in Example 1 with $a = \vartheta$ and $\mu = b$).

In the case of Example 2 the situation is different. Suppose that ϑ is the shift parameter:

$$\mathrm{d}X_t = -a \operatorname{sgn}\left(X_t - \vartheta\right)\, \mathrm{d}t + \sigma\, \mathrm{d}W_t, \quad X_0, \quad 0 \le t \le T.$$

Then we can use the statistic

$$\hat{\delta}_T = T \int_{\hat{\vartheta}_T}^{\infty} h\left(\hat{\vartheta}_T, x\right) \left(\hat{f}_T\left(x\right) - f_0(\hat{\vartheta}_T, x)\right)^2 \mathrm{d}F_0(\hat{\vartheta}_T, x),$$

and it can be shown that

$$\hat{\delta}_T \Longrightarrow \int_1^{\infty} w\left(v\right)^2 e^{-v}\, \mathrm{d}v.$$

Indeed, the MLE $\hat{\vartheta}_T$ converges to ϑ with the rate T (and not $\sqrt{T}$) (see [14], Theorem 3.26) and its contribution to the limit distribution of δ_T is negligible.

Let us see what happens under alternative

$$\mathscr{H}_1 \quad : \qquad S\left(\cdot\right) = S_*\left(\cdot\right), \qquad S_*\left(\cdot\right) \in \mathcal{F}_+$$

where the set

$$\mathcal{F}_+ = \left\{ S\left(\cdot\right) : \inf_{\vartheta \in \Theta} \left\| \frac{S\left(\vartheta, \cdot\right) - S\left(\cdot\right)}{\sigma\left(\cdot\right)} \right\|_* > 0 \right\}$$

and we suppose that the function $S_*\left(\cdot\right)$ satisfies the conditions $\mathcal{ES}$ and $\mathcal{RP}$. Therefore the invariant density is $f_{S_*}\left(\cdot\right)$. Here the norm

$$\left\|h\left(\cdot\right)\right\|_*^2 = \int_{-\infty}^{\infty} h\left(x\right)^2\, f_{S_*}\left(x\right)\, \mathrm{d}x.$$

Note that the MLE in this "misspecified situation" converges to the value ϑ_* which minimizes the Kullback-Leibner distance

$$\left\| \frac{S\left(\vartheta_*, \cdot\right) - S\left(\cdot\right)}{\sigma\left(\cdot\right)} \right\|_* = \inf_{\vartheta \in \Theta} \left\| \frac{S\left(\vartheta, \cdot\right) - S\left(\cdot\right)}{\sigma\left(\cdot\right)} \right\|_* \tag{17}$$

(see [14], Section 2.6.1 for details).

Hence, under regularity conditions we have

$$\hat{f}_T(x) - f_0\left(\hat{\vartheta}_T, x\right) \longrightarrow f_{S_*}(x) - f_0(\vartheta_*, x), \quad \mathrm{I}\left(\hat{\vartheta}_T\right) \to \mathrm{I}(\vartheta_*)$$

and $\dot{f}_0\left(\hat{\vartheta}_T, x\right) \to \dot{f}_0(\vartheta_*, x)$. Further, it can be shown that

$$\begin{aligned}
\frac{R_T\left(\hat{\vartheta}_T\right)}{T} &= \frac{1}{T}\int_0^T \frac{\dot{S}\left(\hat{\vartheta}_T, X_t\right)\left[S_*(X_t) - S\left(\hat{\vartheta}_T, X_t\right)\right]}{\sigma(X_t)^2}\, \mathrm{d}t\,(1 + o(1)) \\
&= \frac{1}{T}\int_0^T \frac{\dot{S}(\vartheta_*, X_t)\left[S_*(X_t) - S(\vartheta_*, X_t)\right]}{\sigma(X_t)^2}\, \mathrm{d}t\,(1 + o(1)) \\
&\longrightarrow \int_{-\infty}^{\infty} \frac{\dot{S}(\vartheta_*, x)\left[S_*(x) - S(\vartheta_*, x)\right]}{\sigma(x)^2}\, f_{S_*}(x)\;\mathrm{d}x = R(\vartheta_*).
\end{aligned}$$

Therefore,

$$\hat{\delta}_T \sim T\int_{-\infty}^{\infty} h(\vartheta_*, x)\left[f_{S_*}(x) - f_0(\vartheta_*, x) + \dot{f}_0(\vartheta_*, x)\,\mathrm{I}(\vartheta_*)\,R(\vartheta_*)\right]^2 \mathrm{d}x$$

and this test can be non consistent against alternatives $S_*(\cdot)$ such that

$$f_{S_*}(x) - f_0(\vartheta_*, x) + \dot{f}_0(\vartheta_*, x)\,\mathrm{I}^{-1}(\vartheta_*)\,R(\vartheta_*) = 0.$$

Suppose that ϑ_* is an interior point of Θ and show that the last equality is impossible. The value ϑ_* defined by the equation (17) is the same time one of the solutions of the equation

$$\int_{-\infty}^{\infty} \frac{\dot{S}(\vartheta_*, x)\left[S_*(x) - S(\vartheta_*, x)\right]}{\sigma(x)^2}\, f_{S_*}(x)\;\mathrm{d}x = 0.$$

Hence $R(\vartheta_*) = 0$. The equality $f_{S_*}(x) = f_0(\vartheta_*, x)$ implies

$$\int_0^x \frac{S_*(y)}{\sigma(y)^2}\,\mathrm{d}y = \int_0^x \frac{S(\vartheta_*, y)}{\sigma(y)^2}\,\mathrm{d}y,$$

which gives us $S_*(x) = S(\vartheta_*, x)$ for almost all x and the last equality contradicts the definition of alternative. Therefore, $\hat{\delta}_T \to \infty$ and the test $\hat{\psi}_T$ is consistent.

6 Discussion

Note, that the similar problems of ADF GoF tests for stochastic differential equations with “small noise” are considered in [15].

The tests $\psi_T = 1_{\{\delta_T > d_\varepsilon\}}$ and $\Psi_T = 1_{\{\Delta_T > c_\varepsilon\}}$ studied in this work are consistent against any fixed alternative and it can be easily shown, that these tests are *uniformly consistent* if the alternatives are separated from hypothesis as follows

$$\mathscr{H}_1 \quad : \qquad S(\cdot) \in \mathcal{F}_r = \left\{ S(\cdot) : \left\| f_S(\cdot) - f_0(\cdot) \right\| \geq r \right\}$$

with some $r > 0$ for ψ_T or

$$\mathscr{H}_1 \quad : \qquad S(\cdot) \in \mathcal{F}_q = \left\{ S(\cdot) : \left\| F_S(\cdot) - F_0(\cdot) \right\| \geq q \right\}$$

for Ψ_T with some $q > 0$. Here the norm

$$\left\| m(\cdot) \right\|^2 = \int_{-\infty}^{\infty} m(x)^2 \, \mathrm{d}F_0(x) .$$

This means that

$$\inf_{S(\cdot)\in\mathcal{F}_r} \mathbf{P}_S \left\{ \delta_T > d_\varepsilon \right\} \longrightarrow 1, \qquad \inf_{S(\cdot)\in\mathcal{F}_q} \mathbf{P}_S \left\{ \Delta_T > c_\varepsilon \right\} \longrightarrow 1.$$

But if the alternative is defined by the Kullback-Leibner distance ($s > 0$)

$$\mathscr{H}_1 \quad : \qquad S(\cdot) \in \mathcal{H}_s = \left\{ S(\cdot) : \left\| \frac{S(\cdot) - S_0(\cdot)}{\sigma(\cdot)} \right\| \geq s \right\},$$

then the both tests are no more uniformly consistent. For example, the functions

$$S_n(x) = S_0(x) + \alpha\sigma(x)^2 \cos(nx), \quad n = 1, 2, \ldots$$

can belong to $\mathcal{H}_s$ but

$$\left\| f_{S_n}(\cdot) - f_0(\cdot) \right\| \to 0 \quad \text{as} \qquad n \to \infty$$

and

$$\inf_{S_n \in \mathcal{H}_s} \mathbf{P}_{S_n} \left\{ \delta_T > d_\varepsilon \right\} \longrightarrow \varepsilon \quad \text{as} \qquad T \to \infty.$$

For such alternatives it is better to use the Chi-squared tests, which can be even asymptotically optimal in minimax sense. The construction of such tests for signals in white Gaussian noise can be found in Ermakov [5] and Ingster and Suslina [11]. For inhomogeneous Poisson processes see [10]. It is interesting to study such tests in the case of ergodic diffusion processes too.

References

[1] Chen S. X., Gao J. (2008) A test for model specification of diffusion processes. *Ann. Statist.* **36**, 1, 167-198.

[2] Dachian, S. and Kutoyants, Yu.A. (2007) On the goodness-of-fit tests for some continuous time processes, in *Statistical Models and Methods for Biomedical and Technical Systems*, F.Vonta *et al.* (Eds), Birkhäuser, Boston, 395-413.

[3] Durett, R. (1996) *Stochastic Calculus: A Practical Introduction*. Boca Raton: CRC Press.

[4] Durbin, (1973) *Distribution Theory for Test Based on the Sample D.F.* SIAM, Philadelphia.

[5] Ermakov, M.S. (1990) Minimax detection of a signal in a Gaussian white noise. *Theory Probab. Appl.*, **35**, 667–679.

[6] Fournie, E. (1992) Un test de type Kolmogorov-Smirnov pour processus de diffusions ergodic. *Rapport de Recherche,* **1696**, INRIA, Sophia-Antipolis.

[7] Fournie, E., Kutoyants, Yu. A. (1993) Estimateur de la distance minimale pour des processus de diffusion ergodiques. *Rapport de Recherche,* **1952**, INRIA, Sophia-Antipolis.

[8] Gassem, A. (2008) Goodness-of-Fit test for switching diffusion, prepublication **08-7**, Université du Maine (http://www.univ-lemans.fr/sciences/statist/download/Gassem/article_anis.pdf).

[9] Gikhman, I.I. and Skorokhod, A.V.. (1969) *Introduction to Theory of Random Processes*. Philadelphia: W.B. Saunders.

[10] Ingster, Yu. I. and Kutoyants Yu. A., (2007) Nonparametric hypothesis testing for intensity of Poisson process, *Mathem. Methods Statist*, **16**, 217-245.

[11] Ingster, Yu.I. and Suslina, I.A. (2003) *Nonparametric Goodness-of-Fit Testing Under Gaussian Models*, Springer, N.Y.

[12] Koul, H. L. (1992). *Weighted Empiricals and Linear Models*. I.M.S. Lecture Notes, **21**. Hayward, California.

[13] Kutoyants, Yu.A. (1997) Some problems of nonparametric estimation by the observations of ergodic diffusion processes. *Statist. Probab. Lett.*, **32**, 311–320.

[14] Kutoyants, Yu.A. (2004) *Statistical Inference for Ergodic Diffusion Processes,* Springer, London.

[15] Kutoyants, Yu.A. (2008) Goodness-of-fit tests for perturbed dynamical systems, prepublication **08-5**, Université du Maine (http://www.univ-lemans.fr/sciences/statist/download//Kutoyants/GoF.pdf).

[16] Lehmann, E.L. and Romano, J.P. (2005) *Testing Statistical Hypotheses.* (3rd ed.) Springer, N.Y.

[17] Liptser, R.S. and Shiryayev, A.N. (2001) *Statistics of Random Processes. I,* (2nd ed.) Springer, N.Y.

[18] Negri, I. (1998) Stationary distribution function estimation for ergodic diffusion process. *Stat. Inference Stoch. Process.,* **1**, 1, 61–84.

[19] Negri, I. and Nishiyama, Y. (2008) Goodness of fit test for ergodic diffusion processes. To appear in *Annals of the Institute of Statistical Mathematics* .

[20] Revuz, D. and Yor, M. (1991) *Continuous Martingales and Brownian Motion.* Springer, N.Y.