# Polymer quantization of the free scalar field and its classical limit

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**Abstract**

Building on prior work, a generally covariant reformulation of free scalar field theory on the flat Lorentzian cylinder is quantized using Loop Quantum Gravity (LQG) type 'polymer' representations. This quantization of the *continuum* classical theory yields a quantum theory which lives on a discrete spacetime lattice. We explicitly construct a state in the polymer Hilbert space which reproduces the standard Fock vacuum- two point functions for long wavelength modes of the scalar field. Our construction indicates that the continuum classical theory emerges under coarse graining. All our considerations are free of the "triangulation" ambiguities which plague attempts to define quantum dynamics in LQG. Our work constitutes the first complete LQG type quantization of a generally covariant field theory together with a semi-classical analysis of the true degrees of freedom and thus provides a perfect infinite dimensional toy model to study open issues in LQG, particularly those pertaining to the definition of quantum dynamics.

## 1 Introduction

Loop Quantum Gravity (LQG) is a non- perturbative approach to quantum gravity which, in its canonical version, attempts to construct a Dirac quantization of a Hamiltonian description of gravity. LQG techniques have yielded many beautiful results such as a satisfactory treatment of spatial diffeomorphisms [1, 2], discrete spatial geometry [3, 4, 5], a calculation of black hole entropy [6, 7] and, in the context of their application to cosmological minisuperspaces, a resolution of the big bang singularity [8, 9]. Nevertheless, significant open issues remain, chief among them being (i) a satisfactory definition of the quantum dynamics of gravity [10, 11, 12] (ii) issues related to

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the extraction of gauge invariant (i.e. spacetime diffeomorphism invariant) physics and (iii) the emergence of general relativity in the classical limit and, specifically, that of flat spacetime and its graviton excitations [13, 14]. While these issues are involved precisely due to the detailed complicated classical dynamics of the gravitational field, one expects that a better understanding of the consequences in quantum theory, of more general feautures of gravity such as its general covariance and field theoretic nature, may be of some use in their elucidation. It is therefore of interest to analyse generally covariant, field theoretic toy models for which appropriate analogs of (i)- (iii) exist and can be resolved. Surprisingly no such toy model exists (at least to our knowledge of the literature) for which such an analysis has been carried out to completion. In this work we continue our investigation [15] of free scalar field theory on flat spacetime (cast in generally covariant disguise) in an LQG type "polymer" representation. We show that this system is indeed a good toy model in the context of our remarks above.

The generally covariant reformulation of free scalar field theory considered here goes by the name of Parametrised Field Theory (PFT)[16]. It offers an elegant description of free scalar field evolution on *arbitrary* (and in general curved) foliations of the background spacetime by treating the 'embedding variables' which describe the foliation as dynamical variables to be varied in the action in addition to the scalar field. Let $X^A = (T, X)$ denote inertial coordinates on 2 dimensional flat spacetime. In PFT, $X^A$ are parametrized by a new set of arbitrary coordinates $x^\alpha = (t, x)$ such that for fixed $t$, the embedding variables $X^A(t, x)$ define a spacelike Cauchy slice of flat spacetime. General covariance of PFT ensues from the arbitrary choice of $x^\alpha$ and implies that in its canonical description, evolution from one slice of an arbitrary foliation to another is generated by constraints. Thus, as in the Hamiltonian formulation of General Relativity, evolution and gauge are intertwined and the theory must be interpreted through gauge invariant Dirac observables. Since the true dynamical content of PFT is identical to that of free scalar field theory the standard mode functions of the latter can be constructed as Dirac observables of the former [16, 15].

In earlier work [15] we used LQG techniques and polymer representations to define and solve the quantum dynamics of PFT, as well as to construct an overcomplete set of Dirac observables as operators on the space of gauge invariant, physical states. We found that this set of Dirac observables is so large that at most only a countable subset thereof could possibly display semiclassical behaviour in any candidate semiclassical state. Since there is no natural choice of a smaller set of observables, further analysis of semiclassical behaviour would necessarily be based on an ad- hoc choice. Here we remedy this defect by starting out with a smaller classical algebra at the kinematic level. The set of functions chosen are still large enough to seperate points in phase space. They are obtained by restricting the embedding and matter charges which multiply the exponents of the holonomy

type functions of Reference [15] to be integer valued rather than real as in [15]. This tighter choice of kinematic algebra has several nice consequences. The charge network states of the quantum theory are now also labelled by integer charges, the spectrum of the analog of the volume operator for LQG becomes discrete, the physical state space becomes much smaller and only a countable subset of the Dirac observables of Reference [15] map the physical state space into itself. Indeed, the fact that integer charges suffice is also true for treatments of scalar field matter in LQG and we anticpate that using them instead of the real valued charges of [17] would have similar nice consequences there.

With this improved choice for PFT, we are able to complete the analysis of [15] and derive the following results. Remarkably, while we started out with a bona- fide quantization of a continuum classical theory, the quantum theory we end up with can be interpreted as the quantum theory of a free scalar field on a *discrete* spacetime lattice. The Poincare group (which for the $S^1 \times R$ spacetime topology consists exclusively of (all) spacetime translations) reduces to that of discrete spacetime translations on this lattice. Thus, only discrete spacetime translations are implemented as unitary operators on the physical Hilbert space. While there are infinitely many discrete-Poincare invariant states, most of them are not semiclassical. We explicitly construct a discrete- Poincare invariant state which is semiclassical in the sense that it reproduces the behaviour of the standard Fock vacuum two point functions for slowly varying modes of the scalar field. thus providing a precise resolution of the analog of issue (iii) in this model.

The layout of the paper is as follows. After a review of classical PFT in section 2, we construct its polymer representation in sections 3- 6 and explore the physical interpretation of this representation in sections 7- 10. Section 11 is devoted to a discussion of our results as well as avenues for further research. As mentioned above, Reference [15] contains a detailed construction of the polymer representation for PFT. In sections 3- 6 the only but *key* difference is that we replace the real labels of the PFT "spin-networks" of Reference [15] by (modulo some technicalities) integer valued ones. While our presentation in sections 3- 6 is self contained, we shall be brief; the interested reader is urged to consult Reference [15] for details. Indeed, our aim in these sections is to introduce our notation to the reader unfamiliar with Reference [15] and to highlight the consequences of the re-placement of the real labels of that work by integer valued ones. In section 3, we construct the quantum kinematics and display the unitary action of finite gauge transformations on the kinematic Hilbert space. In section 4 we construct the Dirac observables of the theory as operators on the kine-matic Hilbert space. In section 5 we construct the physical Hilbert space by Group Averaging techniques [18] and isolate, therein, a physically relevant superselected sector and in section 6 we deal with some subtelities related to the zero mode of the scalar field. In section 7 we demonstrate the existence

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of infinitely many discrete- Poincare invariant states in the physical Hilbert space. In section 8 we show that states in the superselected sector of section 5 can interpreted as quantum excitations of the scalar field on a spacetime lattice. Consequently, a detailed analysis of the physical content of such states requires certain tools from the theory of discrete Fourier transforms which we summarize in section 9. Section 10 is devoted to the construction of a (discrete- Poincare invariant) state which yields Fock- vacuum like behaviour for long (,with respect to the lattice scale,) wavelength excitations of the scalar field. Various technicalities and calculational details are collected in the Appendices.

## 2 Classical PFT on $S^1 \times R$.

We provide a brief review of classical two dimensional PFT. In section 2.1 we review the Lagrangian formalism for two dimensional PFT and summarise the results related to the hamiltonian formulation in sections 2.2-2.4. Once again we recommend [1] to those who are interested in the details.

### 2.1 The Action for PFT.

The action for a free scalar field $f$ on a fixed flat 2 dimensional spacetime $\mathcal{M}$ in terms of global inertial coordinates $X^A$, $A = 0, 1$ is

$$S_0[f] = -\frac{1}{2} \int d^2 X \eta^{AB} \partial_A f \partial_B f, \tag{1}$$

where the Minkowski metric in inertial coordinates, $\eta_{AB}$, is diagonal with entries $(-1, 1)$. If instead, we use coordinates $x^\alpha$, $\alpha = 0, 1$ (so that $X^A$ are 'parameterized' by $x^\alpha$, $X^A = X^A(x^\alpha)$), we have

$$S_0[f] = -\frac{1}{2} \int d^2 x \sqrt{\eta} \eta^{\alpha\beta} \partial_\alpha f \partial_\beta f, \tag{2}$$

where $\eta_{\alpha\beta} = \eta_{AB} \partial_\alpha X^A \partial_\beta X^B$ and $\eta$ denotes the determinant of $\eta_{\alpha\beta}$. The action for PFT is obtained by considering the right hand side of (2) as a functional, not only of $f$, but also of $X^A(x)$ i.e. $X^A(x)$ are considered as 2 new scalar fields to be varied in the action ($\eta_{\alpha\beta}$ is a function of $X^A(x)$). Thus

$$S_{PFT}[f, X^A] = -\frac{1}{2} \int d^2 x \sqrt{\eta(X)} \eta^{\alpha\beta}(X) \partial_\alpha f \partial_\beta f. \tag{3}$$

Note that $S_{PFT}$ is a diffeomorphism invariant functional of the scalar fields $f(x), X^A(x)$. Variation of $f$ yields the equation of motion $\partial_\alpha(\sqrt{\eta} \eta^{\alpha\beta} \partial_\beta f) = 0$, which is just the flat spacetime equation $\eta^{AB} \partial_A \partial_B f = 0$ written in the coordinates $x^\alpha$. On varying $X^A$, one obtains equations which are satisfied if $\eta^{AB} \partial_A \partial_B f = 0$. This implies that $X^A(x)$ are undetermined functions (subject to the condition that determinant of $\partial_\alpha X^A$ is non- vanishing). This 2

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functions- worth of gauge is a reflection of the 2 dimensional diffeomorphism invariance of $S_{PFT}$. Clearly the dynamical content of $S_{PFT}$ is the same as that of $S_0$; it is only that the diffeomorphism invariance of $S_{PFT}$ naturally allows a description of the standard free field dynamics dictated by $S_0$ on *arbitrary* foliations of the fixed flat spacetime.

Note that in PFT, $X^A(x)$ has a dual interpretation - one as dynamical variables to be varied in the action, and the other as inertial coordinates on a flat spacetime. In what follows we shall freely go between these two interpretations.

## 2.2   Canonical formulation

Before we describe the phase-space for the above action, it is useful to keep in mind the following geometric description for $X^A(x)$.

Consider $S^1$ with a "global" angular co-ordinate $x \in [0, 2\pi]$ such that $x = 0 \sim x = 2\pi$. Let $(X^+ = T + X, X^- = T - X)$ be the light-cone co-ordinates on $\mathcal{M} = S^1 \times \mathbf{R}$. whence X is the angular co-ordinate on T=const. slice which we take to be of length $2\pi L$.

Now consider the spacelike embeddings of $(S^1, x)$ in $\mathcal{M}$ via $(X^+(x), X^-(x))$. Due to the cylinder topology of $\mathcal{M}$ we require $(X^+(x), X^-(x))$ to satisfy the following two conditions.

**(i)** $X^\pm(2\pi) - X^\pm(0) = \pm 2\pi L$.

**(ii)** Any two sets of embedding data $(X_1^+(x), X_1^-(x))$ and $(X_2^+(x), X_2^-(x))$ are to be identified if there exists an intger $m$ such that $X_1^+(x) = X_2^+(x) + 2m\pi L \ \forall \ x \in [0, 2\pi]$ and $X_1^-(x) = X_2^-(x) - 2m\pi L \ \forall \ x \in [0, 2\pi]$.

The phase space of PFT splits into two mutually disjoint[1] sectors which are co-ordinatized by, $(X^+, \Pi_+, Y^+)$, $(X^-, \Pi_-, Y^-)$.

$\Pi_\pm$ are the momenta conjugate to the embedding fields $X^\pm$ and $Y^\pm(x) = \pi_f(x) \pm f'(x)$ describe the right(left) moving modes of the scalar field f, whose conjugate momentum is $\pi_f$. $(Y^+, Y^-)$ do not polarize the phase-space and their Poisson brackets are given by,

$$\begin{aligned}
\{Y^\pm(x), \ Y^\pm(y)\} &= \pm (\partial_x \delta(x, y) - \partial_y \delta(y, x)) \\
\{Y^\pm(x), \ Y^\mp(y)\} &= 0
\end{aligned} \tag{4}$$

Due to the general co-variance of the Lagrangian, the $(\pm)$ sectors of the theory contain mutually commuting first-class constraints

$$H^\pm(x) = [\ \Pi_\pm(x) X^{\pm'}(x) \pm \frac{1}{4} Y^\pm(x)^2\ ]. \tag{5}$$

and the constraint algebra is

$$\begin{aligned}
\{H_\pm[N^\pm], H_\pm[M^\pm]\} &= H_\pm[\mathcal{L}_{N^\pm} M^\pm] \\
\{H_\pm[N^\pm], H_\mp[M^\mp]\} &= 0
\end{aligned} \tag{6}$$

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[1]Here we assume that we can consistently "freeze" the zero mode of the scalar field. This is shown in [1]

where $N^\pm$ are the Lagrange multipliers. (Geometrically they are vector fields on $(S^1, x)$).

The action of the constraints $H^\pm[N_\pm]$ on the canonical variables is given by,

$$\begin{aligned}
\{\Phi^\pm(x),\ H_\pm[N^\pm]\ \} &= \mathcal{L}_{N^\pm}\Phi^\pm(x) \\
\{X^\pm(x), H_\pm[N^\pm]\} &= N^\pm(X^\pm)',
\end{aligned} \tag{7}$$

Although the above equations indicate that the constraints generate spatial diffeomorphisms on $S^1$, this interpretation is not quite correct due to the quasi-periodic nature of $X^\pm(x)$. In order to interprete the gauge transformations geometrically, we need to extend domain of the fields (as well as that of $N^\pm$) from $S^1$ to $\mathbf{R}$. Condition **(i)**) above infact suggests the following extension of the domain of $X^\pm$ to the entire real line.

$$X_{ext}^\pm(x + 2m\pi) = X^\pm(x) \pm 2m\pi L \ \forall\ x \in [0, 2\pi) \tag{8}$$

It is straightforward to show that all the remaining fields are periodic in x. Whence the extension of their domain from $[0, 2\pi]$ to $\mathbf{R}$ is rather trivial.

$$\begin{aligned}
\Phi_{ext}^\pm(x + 2m\pi) &= \Phi^\pm(x) \\
N_{ext}^\pm(x + 2m\pi) &= N^\pm(x)
\end{aligned} \tag{9}$$

$\forall x \in [0, 2\pi)$. Here $\Phi^\pm := (\Pi_\pm, Y^\pm)$.

As we showed in [15], the finite gauge transformations generated by the constraints can be interpreted as spatial diffeomorphisms generated by $N_{ext}^\pm$ on $(X_{ext}^\pm, \Phi_{ext}^\pm)$. Let $\Psi^\pm(x) \in (X^\pm(x), \Pi_\pm(x), Y^\pm(x))$ and let its appropriate quasiperiodic/periodic extension on $\mathbf{R}$ be $\Psi_{ext}^\pm$. Then we have that, $\forall x \in [0, 2\pi]$,

$$\begin{aligned}
(\alpha_{\phi^\pm}\Psi^\pm)(x) &= \Psi_{ext}^\pm(\phi^\pm(x)) \\
(\alpha_{\phi^\pm}\Psi^\mp)(x) &= \Psi^\mp(x).
\end{aligned} \tag{10}$$

Where we have labelled every finite gauge transformation by a pair of such diffeomorphisms $(\phi^+, \phi^-)$ so that the Hamiltonian flows generated by $H_\pm$ are denoted by $\alpha_{\phi^\pm}$.

Equations (10) imply a left representation of the group of periodic diffeomorphisms of $\mathbf{R}$ by the Hamiltionian flows corresponding to finite gauge transformations:

$$\alpha_{\phi_1^\pm}\alpha_{\phi_2^\pm} = \alpha_{\phi^\pm \circ \phi_2^\pm} \tag{11}$$

$$\alpha_{\phi_1^\pm}\alpha_{\phi_2^\mp} = \alpha_{\phi_2^\mp}\alpha_{\phi_1^\pm}. \tag{12}$$

### 2.2.1 Mode functions as Dirac observables.

Two dimensional PFT admits a complete set of Dirac observables, which are nothing but the Fourier modes of Klein-Gordon Scalar field.

$$a_{(\pm)n} = \int_{S^1} dx Y^\pm(x) e^{inX^\pm(x)},\ n \in \mathbf{Z},\ n > 0 \tag{13}$$

Together with their complex conjugates, these observables form the (Poisson) algebra,

$$\begin{aligned}
\{a_n, a_m*\} &= -4\pi i n \delta_{n,m}, \\
\{a_n, a_m\} &= 0, \\
\{a_n*, a_m*\} &= 0.
\end{aligned} \tag{14}$$

More generally, since finite gauge transformations act as periodic diffeomorphisms of $\mathbf{R}$, it follows, directly, that the integral over $x \in [0, 2\pi]$ of any periodic scalar density constructed solely from the phase space variables, is an observable. In particular

$$O_{f^\pm} := \int_{S^1} dx\, Y^\pm(x) f^\pm(X^\pm(x)) \tag{15}$$

for all real, periodic functions $f^\pm$ are observables.

### 2.2.2 Conformal Isometries.

Free mass-less scalar field theory in 1+1 dimensions (1) is conformally invariant. As a consequence the generators of conformal isometries in PFT are also Dirac observables (for details, see Reference [16]). Consider the conformal isometry generated by the conformal Killing field $\vec{U}$ on the Minkowskian cylinder. Let $\vec{U}$ have the components $(U^+(X^+), U^-(X^-))$ in the $(X^+, X^-)$ coordinate system. $U^\pm$ are periodic functions of $X^\pm$ by virtue of the fact that $\vec{U}$ is smooth vector field on the flat spacetime $S^1 \times R$. These components of $\vec{U}$ naturally correspond to the functionals $(U^+(X^+), U^-(X^-))$ on the phase space of PFT. The Dirac observable corresponding to the generator of conformal transformations associated with $\vec{U}$ is given by

$$\Pi_\pm[U^\pm] = \int_{S^1} \Pi_\pm(x) U^\pm(X^\pm(x)) \tag{16}$$

These observables generate a Poisson algebra isomorphic to that of the commutator algebra of conformal Killing fields:

$$\begin{aligned}
\{\Pi_\pm[U^\pm], \Pi_\pm[V^\pm]\} &= \Pi[[U, V]^\pm] \\
\{\Pi_\pm[U^\pm], \Pi_\mp[V^\mp]\} &= 0.
\end{aligned} \tag{17}$$

Here $[U, V]^\pm$ refer to the $\pm$ components of the commutator of the spacetime vector fields $\vec{U}, \vec{V}$. $[U, V]^\pm$ define functions of the embedding variables $X^\pm(x)$ in the manner described above.

Note that these observables are weakly equivalent, via the constraints (5) to quadratic combinations of the mode functions [16]. In the standard Fock representation of quantum theory (see for e.g. Reference [19]), these quadratic combinations are nothing but the generators of the Virasoro algebra.

The polymer quantization of PFT given in [15] provides a representation for the finite canonical transformations generated by $\Pi^\pm[U^\pm]$.

$$\alpha_{(\Pi_\pm[U^\pm],t)}X^\pm(x) = (\phi_{(\vec{U},t)}X^\pm)(x). \tag{18}$$

Here $\phi_{(\vec{U},t)}$ denotes the one parameter family of conformal isometries generated by the conformal Killing field $\vec{U}$ on spacetime. $\phi_{(\vec{U},t)}$ maps the spacetime point $(X^+, X^-)$ to $\phi_{(\vec{U},t)}X^\pm$ and hence maps the spatial slice defined by the canonical data $X^\pm(x)$ is mapped to the new slice (and hence the new canonical data) $(\phi_{(\vec{U},t)}X^\pm)(x)$. $\phi_{(\vec{U},t)}$ ranges over all conformal isometries connected to identity. Any such conformal isometry $\phi_c$ is specified by a pair of functions $\phi_c^\pm$ so that $\phi_c(X^+, X^-) := (\phi_c^+(X^+), \phi_c^-(X^-))$. Invertibilty of $\phi_c$ together with connectedness with identity implies that

$$\frac{d\phi_c^\pm}{dX^\pm} > 0, \tag{19}$$

and the cylindrical topology of spacetime implies that

$$\phi_c^\pm(X^\pm \pm 2\pi L) = \phi_c^\pm(X^\pm) \pm 2\pi L. \tag{20}$$

Thus, we may denote the Hamiltonian flows which generate conformal isometries by $\alpha_{\phi_c}$ or, without loss of generality, by $\alpha_{\phi_c^\pm}$ with $\alpha_{\phi_c^\pm}$ acting trivially on the $\mp$ sector.

To summarise: $\alpha_{\phi_c^\pm}$ leave the matter variables untouched, so that

$$\alpha_{\phi_c^\pm}Y^\pm(x) = Y^\pm(x), \quad \alpha_{\phi_c^\pm}Y^\mp(x) = Y^\mp(x), \tag{21}$$

and act on $X^\pm(x)$ as

$$\alpha_{\phi_c^\pm}X^\pm(x) = \phi_c^\pm(X^\pm(x)), \quad \alpha_{\phi_c^\pm}X^\mp(x) = X^\mp(x). \tag{22}$$

Further, since $\Pi_\pm[U^\pm]$ are observables which commute strongly with the constraints, the corresponding Hamiltonian flows are gauge invariant. This translates to the condition that for all

$$\begin{aligned}
\alpha_{\phi_c^\pm} \circ \alpha_{\phi^+} &= \alpha_{\phi^+} \circ \alpha_{\phi_c^\pm} \\
\alpha_{\phi_c^\pm} \circ \alpha_{\phi^-} &= \alpha_{\phi^-} \circ \alpha_{\phi_c^\pm}
\end{aligned} \tag{23}$$

where as before $\phi^\pm$ label finite gauge transformations.

## 3 Polymer quantum kinematics

In [15] we quantized the above theory using techniques of polymer quantization. We quantized the $(\pm)$ sectors separately and within each sector

we quantized the embedding and the matter phase spaces separately. The elementary operators as well as the basis states in the kinematical Hilbert spaces were labelled by charge-networks which were defined as follows.

**Definition 1** : A charge-network $s$ is specified by the labels $(\gamma(s), (j_{e_1}, ..., j_{e_n}))$ consisting of a graph $\gamma(s)$ (by which we mean a finite collection of closed, non-overlapping(except in boundary points) intervals which cover $[0, 2\pi]$) and 'charges' $j_e \in \mathcal{C}$ assigned to each interval e. (Note that $j_e = 0$ is allowed.) $\mathcal{C}$ can be set of reals, or some subset thereof. Equivalence classes of charge- networks are defined as follows. The charge- network $s'$ is said to be finer than $s$ iff (a) every edge of $\gamma(s)$ is identical to, or composed of, edges in $\gamma(s')$ (b) the charge labels of identical edges in $\gamma(s), \gamma(s')$ are identical and the charge labels of the edges of $\gamma(s')$ which compose to yield an edge of $\gamma(s)$ are identical and equal to that of their union in $\gamma(s)$. Two charge-networks are equivalent if there exists a charge- network finer than both. Whenever we refer to charge networks as identical it is understood that we are referring to their equivalence classes. We denote, by $\lambda s$, the charge network $\{\gamma(s), (\lambda j_{e_1}, ..., \lambda j_{e_n})\}$ where $\lambda$ is such that $\lambda j_e \in \mathcal{C} \ \forall \ e \in \gamma(s)$. We denote, by $s + s'$, the charge network obtained by dividing $\gamma(s), \gamma(s')$ into maximal, non-overlapping (upto boundary points) intervals and assigning charge $j_e + j_{e'}$ to $e \cap e'$ where $e \in \gamma(s)$, $e' \in \gamma(s')$. We also define the 'Kronecker delta on charge networks', $\delta_{s,s'}$, by $\delta_{s,s'} := 1$ iff $s_1, s_2$ are identical and $:= 0$ otherwise.

As the aim of this paper is to enjoy the fruits of the labour carried out in [1], we provide a very brief summary of the kinematic structures derived therein using the notion of charge networks defined above. Section 3.1 summarises the embedding sector, section 3.2 the matter sector, section 3.3 the kinematic Hilbert space and section 3.4. the unitary action of finite gauge transformations on the kinematic Hilbert space.

## 3.1  Embedding sector

Charge network : $s^{\pm} = \{\gamma(s^{\pm}), (k^{\pm}_{e^{\pm}_1}, ..., k^{\pm}_{e^{\pm}_n})\}$

Elementary variables : $(T_{s^{\pm}}[\Pi_{\pm}], \ X^{\pm}(x))$
$T_{s^{\pm}}[\Pi_{\pm}] := \exp[i \sum_{e \in \gamma(s^+)} k^{\pm}_{e^{\pm}} \int_{e^{\pm}} \Pi_{\pm}]$

Non- Trivial Poisson brackets:

$$\{X^{\pm}(x), T_{s^{\pm}}[\Pi_{\pm}]\} = -i k^{\pm}_{e^{\pm}} T_{s^{\pm}}[\Pi_{\pm}] \ if \ x \in \text{Interior}(e^{\pm})$$
$$= -\frac{i}{2}(k^{\pm}_{e^{\pm}_{I^{\pm}}} + k^{\pm}_{e^{\pm}_{(I+1)^{\pm}}}) T^{E}_{s^{\pm}}[\Pi_{\pm}] \ \text{if} \ x \in e^{\pm}_{I^{\pm}} \cap e^{\pm}_{(I+1)^{\pm}} \ 1 \le I^{\pm} \le (n^{\pm} - 1)$$
$$\{X^{\pm}(0), T_{s^{\pm}}[\Pi_{\pm}]\} = \{X^{\pm}(2\pi), T_{s^{\pm}}[\Pi_{\pm}]\} = -\frac{i}{2}(k^{\pm}_{e^{\pm}_1} + k^{\pm}_{e^{\pm}_{n^{\pm}}}) T_{s^{\pm}}[\Pi_{\pm}],$$

Elementary Operators: $\hat{X}^{\pm}(x), \hat{T}_{s\pm}$

Charge Network States, Inner Product: $T_{s\pm}$, $\langle T_{s_1^{\pm}}, T_{s_2^{\pm}} \rangle := \delta_{s_1^{\pm}, s_2^{\pm}}$

Representation : $\hat{T}_{s\pm} T_{s_1^{\pm}} = T_{s^{\pm}+s_1^{\pm}}$.

$$\hat{X}^{\pm}(x) T_{s\pm} := \lambda_{x,s\pm} T_{s\pm}, \tag{24}$$

where, for $\gamma(s^{\pm})$ with $n^{\pm}$ edges,

$$\lambda_{x,s\pm} := \hbar k^{\pm}_{e^{\pm}_{I^{\pm}}} \text{ if } x \in \text{Interior}(e^{\pm}_{I^{\pm}}) \; 1 \leq I^{\pm} \leq n^{\pm}$$
$$:= \frac{\hbar}{2}(k^{\pm}_{e^{\pm}_{I^{\pm}}} + k^{\pm}_{e^{\pm}_{(I+1)^{\pm}}}) \text{ if } x \in e^{\pm}_{I^{\pm}} \cap e^{\pm}_{(I^{\pm}+1)} \; 1 \leq I^{\pm} \leq (n^{\pm}-1)$$

$$\tag{25}$$

$$:= \frac{\hbar}{2}(k^{\pm}_{e^{\pm}_{n^{\pm}}} \mp \frac{1}{\hbar}2\pi L + k^{\pm}_{e^{\pm}_1}) \text{ if } x = 0$$
$$:= \frac{\hbar}{2}(k^{\pm}_{e^{\pm}_1} \pm \frac{1}{\hbar}2\pi L + k^{\pm}_{e^{\pm}_{n^{\pm}}}) \text{ if } x = 2\pi \tag{26}$$

The last two equations, (26), implement the boundary condition $X^{\pm}(2\pi) - X^{\pm}(0) = \pm 2\pi L$.

The embedding Hilbert space $\mathcal{H}_E^{\pm}$ is the Cauchy completion of finite linear combination of charge-network states.

**Range of Embedding Charges:** We shall choose $\hbar k_e^{\pm} \in \frac{2\pi L}{A}\mathbf{Z}$. Here A is some fixed positive definite integer. Note that independent of the choice of A, the "holonomy" variables $T_{s\pm}[\Pi_{\pm}]$ seperate points in momentum space by virtue of the fact that the graphs underlying the charge networks can contain arbitrarily small edges. Note also, from (24), that different values of A obtain different spectra for $\hat{X}^{\pm}(x)$. Thus A is the exact analog of the Barbero- Immirizi parameter of LQG [20, 21]: while it does not affect the classical theory, it labels inequivalent representations in quantum theory. The larger the value of A the smaller the seperation between consecutive eigen values of the embedding coordinates. Anticipating our analysis of the classical limit of our quantization, we set A to be an integer much greater than unity. Finally, we remind the reader again that the only difference between our treatment above and the relevant part of [15] is that in that work the embedding charges were in correspondence with the entire real line in contrast to their correspondence with the set of integers here.

## 3.2   Matter sector

**Range of matter charges:** Fix a real parameter $\epsilon$ with dimensions $(ML)^{-\frac{1}{2}}$. Then given any matter charge network, we demand that the *difference* be-

tween the charge labels of any two edges be an integer multiple of $\epsilon$. Thus for any charge network the charge label of any edge is of the form $l_e + \lambda$ where $l_e \in \epsilon \mathbf{Z}$ and $\lambda \in \epsilon[0, 1)$. Note that while $\lambda$ is independent of the edge $e$ for any given charge network, the values of $\lambda$ can vary from charge network to charge network. Thus we denote the charge network label by $s^{\pm}_{\lambda^{\pm}}$. With this choice of range for the matter charges, the holonomies $W_{s^{\pm}_{\lambda^{\pm}}}[Y^{\pm}]$ defined below seperate points in phase space by virtue of the fact that the edges of the charge networks can be arbitrarily small. Indeed, as for the embedding charges, we could also have chosen the matter charges to be integer multiples of a fixed dimension- ful parameter and still obtained a seperating set of functions on the matter phase space. Our slightly more involved choice enables the ensuing Hilbert space to carry a representation of the transformation generated by the zero-mode of the scalar field. This in turn helps us to factor out such transformations in section 6, thus freezing the zero-mode in quantum theory. Once again the only difference from the quantization of the matter sector given in [15] is that here the difference between any two "matter charges" $l_e$ in a given charge network is an integral multiple of a dimensional-full parameter $\epsilon$ whereas, there, each matter charge could independently take values in correspondence with the reals.

Charge-network :

$$s^{\pm}_{\lambda^{\pm}} = \{\gamma(s^{\pm}_{\lambda^{\pm}}), (l^{\pm}_{e^{\pm}_1} + \lambda^{\pm}, ..., l^{\pm}_{e^{\pm}_n} + \lambda^{\pm})\} \ l^{\pm}_{e_I} \in \epsilon \mathbf{Z} \ \forall \ I, \ \lambda^{\pm} \in [0, \epsilon). \quad (27)$$

Elementary variables : $W_{s^{\pm}_{\lambda^{\pm}}}[Y^{\pm}] = \exp[i \sum_{e \in \gamma(s^{\pm})} (l^{\pm}_e + \lambda^{\pm}) \int_e Y^{\pm}]$

Charge-network states, Inner product: $W(s^{\pm}_{\lambda^{\pm}})$ , $\langle W(s^{\pm}_{\lambda^{\pm}}), W(s'^{\pm}_{\lambda'^{\pm}}) \rangle :=$ $\delta_{s^{\pm}, s'^{\pm}} \delta_{\lambda^{\pm}, \lambda'^{\pm}}$

Weyl algebra [2] of operators:
$\hat{W}(s^{\pm}_{\lambda^{\pm}}) \hat{W}(s'^{\pm}_{\lambda'^{\pm}}) = \exp[-i \frac{\hbar}{2} \alpha(s^{\pm}_{\lambda^{\pm}}, s'^{\pm}_{\lambda'^{\pm}})] \hat{W}(s^{\pm} + s'^{\pm})$.
Here the exponent in the phase-factor $\alpha(s^{\pm}_{\lambda^{\pm}}, s'^{\pm}_{\lambda'^{\pm}})$ is given by,

$$\alpha(s^{\pm}_{\lambda^{\pm}}, s'^{\pm}_{\lambda'^{\pm}}) := \sum_{e^{\pm} \in \gamma(s^{\pm}_{\lambda^{\pm}})} \sum_{e'^{\pm} \in \gamma(s'^{\pm}_{\lambda'^{\pm}})} (l^{\pm}_e + \lambda^{\pm})(l^{\pm}_{e'} + \lambda'^{\pm}) \alpha(e^{\pm}, e'^{\pm}), \quad (28)$$

Here $\alpha(e^{\pm}, e'^{\pm}) = (\kappa_{e'^{\pm}}(f(e^{\pm})) - \kappa_{e'^{\pm}}(b(e^{\pm}))) - (\kappa_{e^{\pm}}(f(e^{\pm})) - \kappa_{e^{\pm}}(b(e^{\pm})))$. Here $f(e)$, $b(e)$ are the final and initial points of the edge $e$ respectively. $\kappa_e$ is defined as,

$$\kappa_e(x) = 1 \ \text{if } x \text{ is in the interior of e}$$
$$= \tfrac{1}{2} \ \text{if } x \text{ is a boundary point of e} \quad (29)$$

---

[2]The definition of the Weyl algebra follows in the standard way from the Poisson brackets between $Y^{\pm}(x), Y^{\pm}(y)$ and an application of the Baker- Campbell- Hausdorff Lemma [22]

Representation : $\hat{W}(s_\lambda^{+\pm})W(s'^\pm_{\lambda'\pm}) \; = \; \exp(\frac{-i\hbar}{2}\alpha(s_\lambda^\pm, s'^\pm_{\lambda'\pm}))W(s_{\lambda\pm}^\pm + s'^\pm_{\lambda'\pm}).$

The Cauchy completion of finite linear combinations of charge-network states $W(s_{\lambda\pm}^\pm)$ give $\mathcal{H}_M^\pm$.

## 3.3 The kinematic Hilbert space

The kinematic Hilbert space $\mathcal{H}_{kin}$ is the product of the Hilbert spaces $\mathcal{H}_{kin}^\pm$ with

$$\mathcal{H}_{kin}^\pm \; = \; (\mathcal{H}_E^\pm \; \otimes \; \mathcal{H}_M^\pm) \tag{30}$$

so that

$$\mathcal{H}_{kin} \; = \; (\mathcal{H}_E^+ \; \otimes \; \mathcal{H}_M^+) \; \otimes \; (\mathcal{H}_E^- \; \otimes \; \mathcal{H}_M^-). \tag{31}$$

$\mathcal{H}_{kin}^\pm$ is spanned by an orthonormal basis of equivalence classes of charge network states of the form $T_{s^\pm} \otimes W(s'^\pm_{\lambda'\pm})$ with $s^\pm \; = \; \{\gamma(s^\pm), (k_{e_1^\pm}^\pm, ..., k_{e_{n\pm}^\pm}^\pm)\}$, $s'^\pm_{\lambda'\pm} \; = \; \{\gamma(s'^\pm), (l_{e_1'^\pm}^\pm + \lambda'^\pm, ..., l_{e_{m\pm}'^\pm}^\pm + \lambda'^\pm)\}$.

The equivalence relation between charge networks was defined above in Definition 1. Using this equivalence, it is straightforward to see that we can always choose $s^\pm, s'^\pm_{\lambda'\pm}$ such that $\gamma(s^\pm) = \gamma(s'^\pm_{\lambda'\pm})$. Then each edge $e^\pm$ of $\gamma(s^\pm)$ is labelled by a pair of charges $(k_e^\pm, l_e^\pm + \lambda^\pm)$. Note that such a choice graph and charge pairs is still not unique. However it is easy to see that a unique choice can be made if we require that the pairs of charges, $(k_{e^\pm}^\pm, l_{e^\pm}^\pm)$, are such that no two consecutive edges are labelled by the same pair of charges. We shall denote this unique labelling by $\mathbf{s}_{\lambda\pm}^\pm$ so that

$$\mathbf{s}_{\lambda\pm}^\pm := \{\gamma(\mathbf{s}_{\lambda\pm}^\pm), (k_{e_1^\pm}^\pm, l_{e_1^\pm}^\pm + \lambda^\pm), ..., (k_{e_{n\pm}^\pm}^\pm, l_{e_{n\pm}^\pm}^\pm + \lambda^\pm)\}, \tag{32}$$

with

$$k_{e_{I\pm}^\pm}^\pm \neq k_{e_{(I+1)\pm}^\pm}^\pm \quad \text{or/and } l_{e_{I\pm}^\pm}^\pm \neq l_{e_{(I+1)\pm}^\pm}^\pm. \tag{33}$$

The corresponding charge network state is denoted by $|\mathbf{s}_{\lambda\pm}^\pm\rangle$ so that

$$|\mathbf{s}_{\lambda\pm}^\pm\rangle = T_{s^\pm} \otimes W(s'^\pm_{\lambda\pm}) \tag{34}$$

with $\mathbf{s}_{\lambda\pm}^\pm$ defined from $s^\pm, s'^\pm_{\lambda'\pm}$ in the manner discussed above.

## 3.4 Unitary representation of finite gauge transformations.

The action of finite gauge transformations is most easily specified by introducing the notion of an extension of a charge network $s$ to the real line. Such an extension is labelled by the graph $\gamma(s)_{ext}$ which covers the real line and by charge labels on each edge of $\gamma(s)_{ext}$. Let $T_N(x) \in R$ denote

a rigid translation of the point $x \in [0, 2\pi]$ by $2N\pi$ so that $T_N(\gamma(s))$ spans $[2N\pi, 2(N+1)\pi]$. Then $\gamma(s)_{ext} = \cup_{N \in \mathbf{Z}} \, T_N(\gamma(s))$. For the embedding charge network $s^{\pm}$ we define the quasiperiodic extension $\bar{s}^{\pm}_{ext}$ by specifying the embedding charges on $T_N(\gamma(s))$ by $k^{\pm}_{T_N(e)} := k^{\pm}_e \pm 2N\pi\frac{L}{\hbar}$ for every edge $e \in \gamma(s)$. Similarly, for the matter charge network $s^{\pm}$ we define the periodic extension $s^{\pm}_{ext}$ by setting $l^{\pm}_{T_N(e)} := l_e$.

The action of periodic diffeomorphisms, $\phi$, of the real line on $\bar{s}^{\pm}_{ext}, s^{\pm}_{ext}$ is defined by mapping $\gamma(s)_{ext}$ to $\phi(\gamma(s)_{ext})$ and setting $k^{\pm}_{\phi(e)} = k^{\pm}_e, l^{\pm}_{\phi(e)} = l^{\pm}_e$ for every edge $e \in \gamma(s)_{ext}$.

Then unitary representation of the gauge group is given by,

$$
\begin{aligned}
\hat{U}^{\pm}(\phi^{\pm})T_{s^{\pm}} &:= T_{\phi(\bar{s}^{\pm}_{ext})|_{[0,2\pi]}} \\
\hat{U}^{\mp}(\phi^{\mp})T_{s^{\pm}} &:= T_{s^{\pm}} \\
\hat{U}^{\pm}(\phi^{\pm})W(s'^{\pm}_{\lambda^{\pm}}) &:= W((\phi^{\pm})(s'^{\pm}_{\lambda^{\pm} \, ext})|_{[0,2\pi]}). \\
\hat{U}^{\mp}(\phi^{\mp})W(s'^{\pm}_{\lambda^{\pm}}) &:= W(s'^{\pm}_{\lambda^{\pm}}),
\end{aligned}
\tag{35}
$$

where $s'^{\pm}_{\lambda^{\pm} \, ext}|_{0.2\pi]}$ is the restriction of $s'^{\pm}_{\lambda^{\pm} \, ext}$ to the interval $[0, 2\pi]$. Denoting, $T_{s^{\pm}} \otimes W(s'^{\pm}_{\lambda^{\pm}})$ by $|\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle$ and $T_{\phi(\bar{s}^{\pm}_{ext})|_{[0,2\pi]}} \otimes W((\phi^{\pm})(s'^{\pm}_{\lambda^{\pm} \, ext})|_{[0,2\pi]})$ by $|\mathbf{s}^{\pm}_{\lambda^{\pm} \, \phi^{\pm}}\rangle$, the above equations can be written in a compact form as,

$$
|\mathbf{s}^{\pm}_{\lambda^{\pm} \, \phi^{\pm}}\rangle := \hat{U}^{\pm}(\phi^{\pm})|\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle.
\tag{36}
$$

It was shown in [15] that the above representation is unitary and its action on elementary operators (via conjugation) precisely mimics the action of finite gauge transformations on the corresponding (classical) variables.

# 4 Unitary representation of Dirac observables

## 4.1 Observables of type $e^{iO_{f^{\pm}}}$

In [15] we showed that the (unitary) action of the operator $\widehat{e^{iO_{f^{\pm}}}}$ corresponding to the exponential of the classical observable $O_{f^{\pm}}$ (see equation (15)) on the charge network state $T_{s^{\pm}} \otimes W(s'^{\pm}_{\lambda^{\pm}})$ is:

$$
(\widehat{\exp iO_{f^{\pm}}})T_{s^{\pm}} \otimes W(s'^{\pm}_{\lambda^{\pm}}) := \hat{W}(s^{\pm}_{f^{\pm}})T_{s^{\pm}} \otimes W(s'^{\pm}_{\lambda'^{\pm}}).
\tag{37}
$$

where $s^{\pm}_{f^{\pm}} := \{\gamma(s^{\pm}), (f^{\pm}(\hbar k^{\pm}_{e^{\pm}_1}), ...., f^{\pm}(\hbar k^{\pm}_{e^{\pm}_{n^{\pm}}}))\}$. [3]

**Restriction on $f^{\pm}(X^{\pm})$:** For the real matter charges of Reference [15], the above operator action is well defined for all real, periodic $f^{\pm}(X^{\pm})$. In that

---

[3] A quick way to see this is to expand the exponential in a Taylor series, evaluate the action of the embedding part of each term on (its eigen state) $T_{s^{\pm}}$ and resum the series.

context, we showed [15] that no state exists (whether kinematic or physical) which displays semiclassical behaviour with respect to $(\widehat{\exp iO}_{f^\pm})$ for all real $f^\pm$ and that this lead to the necessity of an ad-hoc choice of a countable subset of such observables for semiclassical analysis. Here, this problem disappears by virtue of the tighter kinematic structure. Specifically, since matter charges are translated by the amount $f^\pm(\hbar k_{e^\pm}^\pm)$ in equation (37), the operator action is only well defined for those $f^\pm(X^\pm)$ such that $f^\pm(\hbar k_{e^\pm}^\pm)$ lie in the range sepcified by (27). Together with the integer valuedness of the embedding charges, this implies that $f^\pm(X^\pm)$ is a (smooth periodic real) function of $X^\pm$ such that

$$f^\pm(\frac{2\pi L}{A}n) \in \mathbf{Z}\epsilon + \lambda \quad \forall\, n \in \mathbf{Z}, \quad \lambda \in \mathbf{R}. \tag{38}$$

As we shall see, this vast reduction in the space of observables allows a semi-classical analysis free from any ad-hoc choice of the type mentioned above.

Note that (37) is a manifestly regularization/triangulation independent definition. Moreover, since $s_{f^\pm}^\pm$ is constructed from the embedding part of the charge network, and since $f^\pm$ are periodic, it is straightforward to check that $e^{i\widehat{O}_{f^\pm}}$ commute with the unitary operators corresponding to finite gauge transformations so that $e^{i\widehat{O}_{f^\pm}}$ are Dirac observables in quantum theory.

## 4.2   Conformal isometries

In [15] we showed how to represent Hamiltonian flows $\alpha_{\phi_c^\pm}$ as gauge invariant unitary operators on $\mathcal{H}_{kin}^\pm$. The resulting operator $\hat{V}[\phi_c^\pm]$ had a trivial action on $\mathcal{H}_M^\pm$ and its action on $T_{s^\pm} \in \mathcal{H}_E^\pm$ was given by,

$$\hat{V}^\pm[\phi_c^\pm]T_{s^\pm} = T_{(\phi_c^\pm)^{-1}(s^\pm)}. \tag{39}$$

Where, $(\phi_c^\pm)^{-1}(s^\pm) = \{\gamma(s^\pm), ((\phi_c^\pm)^{-1}(k_{e_1^\pm}^\pm), ..., (\phi_c^\pm)^{-1}(k_{e_n^\pm}^\pm))\}$.

However as $k_e^\pm$ are now integral multiples of $\frac{2\pi L}{A\hbar}$, generically a monotonically increasing, invertible function $\phi_c^\pm$ will not map $k_e^\pm$ to $k_e'^\pm \in \frac{2\pi L}{A\hbar}\mathbf{Z}$. In fact it is easy to see that only invertible(1-1) monotonic functions which map integers to integers are constant (integer-valued) translations.

Whence in the present case only the Hamilton flows corresponding to discrete translations (which correspond to a discrete subgroup of the Poincare group) can be unitarily represented on $\mathcal{H}_{kin}$.

$$\hat{V}[\tau^\pm]T_{s^\pm} = T_{s_{\tau^\pm}^\pm} \tag{40}$$

Where $\tau^\pm \in \frac{2\pi L}{A\hbar}\mathbf{Z}$ and $s_{\tau^\pm}^\pm = \{\gamma(s^\pm), (k_{e_1^\pm}^\pm + \tau^\pm, ..., k_{e_n^\pm}^\pm + \tau^\pm)\}$. It is straightforward to see that $\hat{V}[\tau^\pm]$ are unitary $\forall \tau^\pm$ and satisfy,

$$\hat{V}^\pm[\tau_1^\pm]\hat{V}^\pm[\tau_2^\pm] = \hat{V}^\pm[\tau_1^\pm + \tau_2^\pm], \tag{41}$$

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so that our definition of $\hat{V}^\pm[\tau^\pm]$ implies an anomaly free representation of a discrete subgroup of the Poincare group.

Note that in the classical theory the conformal group is homomorphic not only to the group of of conformal isometries but also to that of finite gauge transformations [16]. In the quantum theory presented here, we see that while the entire group of gauge transformations is faithfully represented on $\mathcal{H}_{kin}$, the entire group of conformal isometries cannot be represented on $\mathcal{H}_{kin}$; only a discrete Abelian subgroup thereof corresponding to rigid translations is unitarily represented on the Hilbert space. We will see later how this discrete subgroup is tied to the emergence of a lattice in the quantum theory

## 5 Physical state space by group averaging

Only gauge invariant states are physical so that physical states $\Psi$ must satisfy the condition $\hat{U}^\pm(\phi^\pm)\Psi = \Psi$, $\forall \phi^\pm$. A formal solution to this condition is to fix some $|\psi\rangle \in \mathcal{H}_{kin}$ and set $\Psi = \sum |\psi'\rangle$ where the sum is over all distinct $|\psi'\rangle$ which are gauge related to $\psi$. A mathematically precise implementation of this idea places the gauge invariant states in the dual representation (corresponding to a formal sum over bras rather than kets) and goes by the name of Group Averaging. The reader may consult Reference [15] for a quick account of the Group Averaging technique as well its detailed implementation in PFT. The considerations of Reference [15] go through unchanged here with the proviso that both $\mathcal{H}_{kin}$ and the set of Dirac observables are smaller than those used there; $\mathcal{H}_{kin}$ is smaller due to the restriction in the range of the charges (see sections 3.1 and 3.2) and the set of Dirac observables is smaller since they need to be well defined on $\mathcal{H}_{kin}$ (see sections 4.1 and 4.2).

The Group Average of a charge network state $|\mathbf{s}^\pm_{\lambda^\pm}\rangle$ yields the physical, gauge invariant distribution $\eta^\pm(|\mathbf{s}^\pm_{\lambda^\pm}\rangle)$ with

$$
\begin{aligned}
\eta^\pm(|\mathbf{s}^\pm_{\lambda^\pm}\rangle) &= \eta^\pm_{[\mathbf{s}^\pm_{\lambda^\pm}]} \sum_{\mathbf{s}'^\pm_{\lambda^\pm} \in [\mathbf{s}^\pm_{\lambda^\pm}]} < \mathbf{s}'^\pm_{\lambda^\pm}| \\
&= \eta^\pm_{[\mathbf{s}^\pm_{\lambda^\pm}]} \sum_{\phi^\pm \in Diff^P_{[\mathbf{s}^\pm_{\lambda^\pm}]}\mathbf{R}} < \mathbf{s}^\pm_{\lambda^\pm \phi^\pm}|.
\end{aligned} \tag{42}
$$

Here $\eta^\pm$ denotes the (antilinear) Group Averaging map from the finite span of charge network states, $\mathcal{D}^\pm$ into the space of distributions [4] $\mathcal{D}^{\pm*}$, $[\mathbf{s}^\pm_{\lambda^\pm}]$ is the set of charge networks defined by $[\mathbf{s}^\pm_{\lambda^\pm}] = \{\mathbf{s}'^\pm_{\lambda^\pm} | \mathbf{s}'^\pm_{\lambda^\pm} = \mathbf{s}^\pm_{\lambda^\pm \phi^\pm}$ for some $\phi^\pm\}$, $Diff^P_{[\mathbf{s}^\pm_{\lambda^\pm}]}\mathbf{R}$ is a set of gauge transformations such that for each $\mathbf{s}'^\pm_{\lambda^\pm} \in [\mathbf{s}^\pm_{\lambda^\pm}]$ there is precisely one gauge transformation in the set which maps $\mathbf{s}^\pm_{\lambda^\pm}$ to

---

[4]Distributions are linear maps from $\mathcal{D}^\pm$ to the complex numbers; only finitely many terms in the formal sum (42) contribute to its action on any spin network state thus ensuring that the left hand side is indeed a distribution.

$\mathbf{s}'^{\pm}_{\lambda\pm}$ and $\eta_{[\mathbf{s}^{\pm}_{\lambda\pm}]}$ is a positive real number depending only on the gauge orbit $[\mathbf{s}^{\pm}_{\lambda\pm}]$. The space of such gauge invariant distributions comes equipped with the inner product

$$< \eta^{\pm}(|\mathbf{s}^{(1)\pm}_{\lambda\pm}\rangle), \eta^{\pm}(|\mathbf{s}^{(2)\pm}_{\lambda'\pm}\rangle) >_{phys} = \eta^{\pm}(|\mathbf{s}^{(1)\pm}_{\lambda\pm}\rangle)[|\mathbf{s}^{(2)\pm}_{\lambda'\pm}\rangle]. \qquad (43)$$

As in Reference [15] we shall focus on a superselected sector of $\eta^{\pm}(\mathcal{D}^{\pm})$ which is obtained by Group Averaging the superselected sector, $\mathcal{D}^{\pm}_{ss}$ of $\mathcal{D}^{\pm}$. This superselected sector is of physical interest because, as is obvious below, it captures (as well as possible) the classical non-degeneracy property $\pm X^{\pm\prime}(x) > 0$ in the context of quantum theory wherein $\hat{X}^{\pm}(x)$ has the discrete spectrum given by (24).

$\mathcal{D}^{\pm}_{ss}$ is defined as follows. Fix a pair of graphs $\gamma^{\pm}$ with A edges. We will denote the edges by $e^{\pm}_{I}$ with $0 \leq I \leq (A-1)$. Place the embedding charges $\vec{k}^{\pm}$ such that $k^{\pm}_{e^{\pm}_{I}} - k^{\pm}_{e^{\pm}_{I-1}} = \frac{2\pi L}{A\hbar} \ \forall \ 1 \leq I \leq A-1$. Consider the set of all charge-network states
$\{|\mathbf{s}^{\pm}_{\lambda\pm}\rangle = |\gamma^{\pm}, \vec{k}^{\pm}, (l^{\pm}_{e^{\pm}_{0}}+\lambda^{\pm}, ..., l^{\pm}_{e^{\pm}_{A-1}}+\lambda^{\pm}))\}$, where $l^{\pm}_{e^{\pm}_{I}} \in \mathbf{Z}\epsilon$ and $\lambda^{\pm} \in [0, \epsilon)$ are allowed to take all possible values. Let $\mathcal{D}^{\pm}_{ss}$ be finite span of charge network states of the type $\{|\mathbf{s}^{\pm}_{\lambda\pm \ \phi\pm}\rangle \ \forall \ \phi^{\pm}\}$.

It is straightforward to check that $\mathcal{D}^{\pm}_{ss}$ is invariant under the action of observables as well as gauge transformations. Whence $\mathcal{D}^{+}_{ss} \otimes \mathcal{D}^{-}_{ss}$ constitutes a super-selected (kinematical) sector of the theory.

Next we note that the enormous "$\eta^{\pm}_{[\mathbf{s}^{\pm}_{\lambda\pm}]}$" worth of ambiguity in the Group Averaging map can be reduced, as in Reference [15] by requiring that the Group Averaging map commutes with the Dirac observables of the quantum theory.

We now provide a brief sketch of how this happens in the '+' sector; the '−' sector can be dealt with in a similar manner. From [15], commutatitivity with the Dirac observables of section 4.1 implies that $\eta^{+}_{[\mathbf{s}^{+}_{\lambda}]} = \eta^{+}_{[\widetilde{\mathbf{s}^{+}}]}$, where $\widetilde{\mathbf{s}^{+}}$ is the embedding charge network given by

$$\widetilde{\mathbf{s}^{+}} = \{\gamma(\mathbf{s}^{+}), (k^{+}_{e^{+}_{0}}, ..., k^{+}_{e^{+}_{A-1}})\}$$

and the equivalence class $[\widetilde{\mathbf{s}^{+}}]$ is the the set of all $\widetilde{\mathbf{s}^{+}}'$ such that $\widetilde{\mathbf{s}^{+}}' = \hat{U}^{+}(\phi^{+})\widetilde{\mathbf{s}^{+}}$ for some $\phi^{+}$. This is due to the fact (see [15] for a proof) that given any two charge networks $\mathbf{s}^{(1)+}_{\lambda+}$, $\mathbf{s}^{(2)+}_{\lambda'+}$ such that $\widetilde{\mathbf{s}^{(1)+}} = \widetilde{\mathbf{s}^{(2)+}}$, one can always find a function f such that $(l^{(1)+}_{e^{+}_{I}}+\lambda^{(1)+})+f(\hbar k^{(1)+}_{e^{+}_{I}}) = (l^{(2)+}_{e^{+}_{I}}+\lambda^{(2)+})$ $\forall I \in \{0, ..., A-1\}$. Here $\vec{l}^{(1)+}+\lambda^{(1)+}$, $\vec{l}^{(2)+}+\lambda^{(2)+}$ are the matter charges associated to $\mathbf{s}^{(1)+}_{\lambda+}$, $\mathbf{s}^{(2)+}_{\lambda'+}$ respectively. Again, from [15], the commutativity of $\eta$ with the observables of section 4.2 implies that

$$\eta^{+}_{[\widetilde{\mathbf{s}^{+}}]} = \eta^{+}_{[\gamma^{+}]} \qquad (44)$$

16

where $[\gamma^+]$ is the equivalence class of all the graphs which are related to $\gamma^+$ by a periodic diffeomorphism. That is, we say that $\gamma_1^+ \sim \gamma^+$ if $\exists \, \phi^+ \in Diff^P\mathbf{R}$ such that $(\phi^+ \cdot \gamma_{ext}^+)|_{[0,2\pi]} = \gamma_1^+$, where $(\phi^+ \cdot \gamma_{ext}^+)|_{[0,2\pi]}$ is the restriction of $(\phi^+ \cdot \gamma_{ext}^+)$ to $[0, 2\pi]$.

Once again this is due to the fact that given $\widetilde{\mathbf{s}^{(1)+}}, \widetilde{\mathbf{s}^{(2)+}}$ such that $\gamma(\widetilde{\mathbf{s}^{(1)+}}) = \gamma(\widetilde{\mathbf{s}^{(2)+}})$, one can always find $\tau^+ \in \frac{2\pi L}{A\hbar}\mathbf{Z}$, such that $k_{e_I^+}^{(1)+} + \tau^+ = k_{e_I^+}^{(2)+} \; \forall I$. (Recall that we are in the super-selected sector, and whence $(\triangle k)_{e_I^+}^{(1)+} = (\triangle k)_{e_I^+}^{(2)+} = \frac{2\pi L}{A\hbar}$).

Thus we obtain that $\eta^+(\mathcal{D}_{ss}^+)$ is the finite span of states of the form

$$\eta^+(|\mathbf{s}_{\lambda^+}^+\rangle) \;=\; \eta_{[\gamma^+]}^+ \sum_{\phi \in Diff^P_{[\mathbf{s}_{\lambda^+}^+]}\mathbf{R}} \langle \phi \cdot \mathbf{s}_{\lambda^+}^+| \tag{45}$$

The physical Hilbert space for right moving sector $\mathcal{H}_{phy}^{ss+}$ is the Cauchy completion of of $\eta^+(\mathcal{D}_{ss}^+)$ An anologous construction yields $\mathcal{H}_{phy}^{ss-}$.

$$\mathcal{H}_{phy}^{ss} = \mathcal{H}_{phy}^{ss+} \otimes \mathcal{H}_{phy}^{ss-}$$

# 6 Taking care of the Zero-mode constraint

The zero mode constraint is $p = 0$. As $p = \int_{S^1} Y^+ = \int_{S^1} Y^-$, we can impose $\int_{S^1} Y^\pm \approx 0$ by Group averaging with respect to the one-parameter family of unitaries $\hat{W}(s_{0,\mu^\pm})$ where $s_{0,\mu^\pm} = \{\gamma = [0, 2\pi], l_{S^1}^\pm = \mu^\pm \in \mathbf{R}\}$ (i.e. $\gamma$ consists of a single edge which covers the circle and which is labelled by the real charge $\mu^\pm$). As explained in [15], one could solve the zero-mode constraint before or after solving the $(H^+, \; H^-)$ constraints, as

$$\hat{U}^\pm(\phi^\pm)\hat{W}(s_{0,\mu^\pm})\hat{U}^\pm(\phi^\pm)^{-1} = \hat{W}(s_{0,\mu^\pm}) \tag{46}$$

$\forall \mu^\pm \in \mathbf{R}$.

As we have already averaged over the gauge group, we seek to solve the zero-mode constraint by defining a Group averaging map

$$\overline{\eta}^\pm : \eta^\pm(\mathcal{D}_{ss}^\pm) \to \eta^\pm(\mathcal{D}_{ss}^\pm)^*. \tag{47}$$

where $\mathcal{D}_{ss}^\pm$ be finite span of charge network states of the type $\{|\mathbf{s}_{\lambda^\pm \; \phi^\pm}^\pm\rangle \; \forall \, \phi^\pm\}$ as defined in the previous section.

Before defining $\overline{\eta}^\pm$, note that,

$$\hat{W}(s_{0,\mu^\pm})|\mathbf{s}_{\lambda^\pm}^\pm\rangle =: |\mathbf{s}_{\mu^\pm+\lambda^\pm}^\pm\rangle \tag{48}$$

where $\mathbf{s}^{\pm}_{\mu^{\pm}+\lambda^{\pm}}$ is obtained from $\mathbf{s}^{\pm}_{\lambda^{\pm}} = \{\gamma(\mathbf{s})^{\pm}, \vec{k}^{\pm}, \vec{l}^{\pm} + \lambda^{\pm}\}$ by adding $\mu^{\pm}$ to all the matter charges.

We now define,

$$\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle)) = \overline{\eta}^{\pm}_{[[\mathbf{s}^{\pm}]]_0} \eta^{\pm}_{[\gamma(\mathbf{s}^{\pm}_{\lambda^{\pm}})]} \bigoplus_{\mu^{\pm}\in\mathbf{R}} \sum_{\phi^{\pm}\in Diff^{P}_{[\gamma(\mathbf{s}^{\pm}_{\lambda^{\pm}})]}\mathbf{R}} \langle(\mathbf{s}^{\pm}_{\phi^{\pm}})_{\mu^{\pm}+\lambda^{\pm}}|$$
(49)

The equivalence class $[[\mathbf{s}^{\pm}]]_0$ is defined via following relation.

$[\mathbf{s}^{\pm}_{\lambda^{\pm}}] \sim [\mathbf{s}^{(1)\pm}_{\lambda^{\pm}_1}]$ iff for any $\{\gamma(\mathbf{s}^{\pm}), \vec{k}^{\pm}, \vec{l}^{\pm} + \lambda^{\pm}\} \in [\mathbf{s}^{\pm}_{\lambda^{\pm}}]$, $\exists$ $(\{\gamma(\mathbf{s}^{\pm}), \vec{k}^{\pm}, \vec{l}^{\pm} + \lambda^{\pm} + \mu^{\pm}\} \in [\mathbf{s}^{(1)\pm}_{\lambda^{\pm}_1}]$ for some $\mu_{\pm} \in \mathbf{R}$.

However in light of (46) the sums in (49) can be interchanged as,

$$\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle)) = \overline{\eta}_{[[\mathbf{s}^{\pm}]]_0} \eta_{[\gamma(\mathbf{s}^{\pm}_{\lambda^{\pm}})]} \sum_{\phi^{\pm}\in Diff^{P}_{[\mathbf{s}^{\pm}_{\lambda^{\pm}}]}\mathbf{R}} \bigoplus_{\mu^{\pm}\in\mathbf{R}} \langle(\mathbf{s}^{\pm}_{\mu^{\pm}+\lambda^{\pm}})_{\phi^{\pm}}|$$
(50)

We can define an invariant characterization of the distribution $\bigoplus_{\mu^{\pm}\in\mathbf{R}}\langle(\mathbf{s}^{\pm}_{\mu^{\pm}+\lambda^{\pm}})|$ as follows.

Given $\vec{l}^{\pm} + \lambda^{\pm}$ , $\vec{l}^{\pm}_{(1)} + \lambda^{\pm}_{(1)}$ they will both give rise to the same distribution $\bigoplus_{\mu^{\pm}\in\mathbf{R}}\langle(\mathbf{s}^{\pm}_{\mu^{\pm}+\lambda^{\pm}})|$ iff $l^{\pm}_{e^{\pm}_I}+\lambda^{\pm}+\mu^{\pm} = l^{\pm}_{(1),e^{\pm}_I}+\lambda^{\pm}_{(1)} \; \forall \; 0 \leq I \leq A-1$. Which is equivalent to $l^{\pm}_{e^{\pm}_{I+1}} - l^{\pm}_{e^{\pm}_I} = l^{\pm}_{(1),e^{\pm}_{I+1}} - l^{\pm}_{(1),e^{\pm}_I} \; \forall \; 0 \leq I \leq A-2$.

Whence,

$$\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle)) = \overline{\eta}^{\pm}_{[[\mathbf{s}^{\pm}]]_0} \eta^{\pm}_{[\gamma(\mathbf{s}^{\pm}_{\lambda^{\pm}})]} \sum_{\phi^{\pm}\in Diff^{P}_{[\mathbf{s}^{\pm}_{\lambda^{\pm}}]}\mathbf{R}} ((\gamma(s^{\pm}), \vec{k}^{\pm}, \triangle\vec{l}^{\pm})_{\phi^{\pm}}|$$
(51)

where $\triangle\vec{l}^{\pm} := \{l^{\pm}_{e^{\pm}_0} - l^{\pm}_{e^{\pm}_{A-1}}, l^{\pm}_{e^{\pm}_1} - l^{\pm}_{e^{\pm}_0}, ..., l^{\pm}_{e^{\pm}_{A-1}} - l^{\pm}_{e^{\pm}_{A-2}}\}$, $(\gamma(s^{\pm}), \vec{k}^{\pm}, \triangle\vec{l}^{\pm}|$ is the distribution $\bigoplus_{\mu^{\pm}\in\mathbf{R}}\langle(\mathbf{s}^{\pm}_{\mu^{\pm}+\lambda^{\pm}})|$ on $\mathcal{D}^{\pm}_{ss}$.

Henceforth we will denote the corresponding charge network $\{\gamma(s^{\pm}), \vec{k}^{\pm}, \triangle\vec{l}^{\pm}\}$ by $\mathbf{s}^{\pm}_0$.

The unitary action of gauge group gives rise to a dual action on such distributions.

$$\hat{U}^{\pm}(\phi^{\pm})|\mathbf{s}^{\pm}_0\rangle =: |(\mathbf{s}_0)^{\pm}_{\phi^{\pm}}\rangle$$
(52)

In light of (46), this action can be understood by picking any charge-network $\mathbf{s}^{\pm}$ in the equivalence class $\mathbf{s}^{\pm}_0$, applying $\hat{U}^{\pm}(\phi^{\pm})$ to $|\mathbf{s}^{\pm}\rangle$ and denoting the corresponding equivalence class by $(\mathbf{s}^{\pm}_0)_{\phi^{\pm}}$.

It is also easy to see that, $Diff^{P}_{[\mathbf{s}^{\pm}_{\lambda^{\pm}}]}\mathbf{R} = Diff^{P}_{[\mathbf{s}^{\pm}_{\lambda^{\pm}=0}]}\mathbf{R} \; \forall \; \lambda^{\pm}$.

Whence finally the solution to the zero-mode constraints are given by,

$$\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle)) = \overline{\eta}^{\pm}_{[[\mathbf{s}^{\pm}]]_0} \eta^{\pm}_{[\gamma(\mathbf{s}^{\pm}_{\lambda^{\pm}})]} \sum_{\phi^{\pm}\in Diff^{P}_{[\mathbf{s}^{\pm}]}\mathbf{R}} \rangle(\mathbf{s}^{\pm}_0)_{\phi^{\pm}}|.$$
(53)

Since the equivalence classes of charge network under the equivalence defined above are characterized by $\mathbf{s}^{\pm}_0$, we denote $\overline{\eta}_{[[\mathbf{s}^{\pm}]]_0}$ by $\overline{\eta}_{\mathbf{s}^{\pm}_0}$. Once again, the

requirement that the Group averaging map $\overline{\eta}$ commutes with the action of Dirac observables and (discrete) isometries reduces the above ambiguity to $\overline{\eta}_{\gamma(\mathbf{s}_0^{\pm})}$.

We will denote the final physical Hilbert space (after solving the zero-mode constraint) by $\mathcal{H}^{ss}_{(0)phy} := \mathcal{H}^{ss+}_{(0)phy} \otimes \mathcal{H}^{ss-}_{(0)phy}$.

# 7 Existence of Infinitely many Poincare-invariant states

In this section we show that $\mathcal{H}^{ss}_{(0)phy}$ admits infinitely many Poincare invariant states. As the unitaries corresponding to the Poincare group commute with the zero mode constraint[5], we ignore the averaging w.r.t the zero mode constraint defined in (45) in this section. (i.e. we will show the existence of Poincare invariant states in $\mathcal{H}^{ss}_{phy}$.) Once again, we restrict the analysis to the left-moving sector.

Consider a charge-network state $|\mathbf{s}^+\rangle$ in $\mathcal{D}^+_{ss}$ with

$\mathbf{s}^+ = \{\gamma^+, \left(k^+_{e_0^+} = 0, k^+_{e_1^+} = \frac{2\pi L}{A\hbar}, ...k^+_{e_I^+} = \frac{2\pi LI}{A\hbar}, ..., k^+_{e_{A-1}^+} = \frac{2\pi L(A-1)}{A\hbar}\right), \vec{l}^+ = \left(l^+_{e_0^+}, ..., l^+_{e_{A-1}^+}\right)\}$[6]

Also let $\mathbf{s}^+_{cyc} = \{\gamma^+, \left(k^+_{e_0^+} = 0, k^+_{e_1^+} = \frac{2\pi L}{A\hbar}, ...k^+_{e_I^+} = \frac{2\pi LI}{A\hbar}, ..., k^+_{e_{A-1}^+} = \frac{2\pi L(A-1)}{A\hbar}\right), \vec{l}^+ = \left(l^+_{e_{A-1}^+}, l^+_{e_0^+}, ..., l^+_{e_{A-2}^+}\right)\}$.

Let us also denote the (dual) representation of Poincare group on $\mathcal{H}^{ss}_{phy}$ by $\hat{V}[\tau^+]'$.

**Lemma 1**: Show that $\hat{V}[\tau^+ = \frac{2\pi L}{A\hbar}]'\eta^+(|\mathbf{s}^+\rangle) = \eta^+(|\mathbf{s}^+_{cyc}\rangle)$

**Proof**: Note that,

$$\hat{V}[\tau^+ = \frac{2\pi L}{A\hbar}]'\eta^+(|\mathbf{s}^+\rangle) = \eta^+(|\mathbf{s}^+_{\tau^+}\rangle) \tag{54}$$

where $\mathbf{s}^+_{\tau^+} = \{\gamma^+, \left(k^+_{e_0^+} + \tau^+, k^+_{e_1^+} + \tau^+, ..., k^+_{e_{A-1}^+} + \tau^+\right), \vec{l}^+ = \left(l^+_{e_0^+}, ..., l^+_{e_{A-1}^+}\right)\}$.

Notice that $k^+_{e_I^+} + \tau^+ = k^+_{e_{I+1}^+} \forall\ 0 \leq I \leq (A-2)$ and $k^+_{e_{A-1}^+} + \tau^+ = 2\pi L$. Whence,

$$\mathbf{s}^+_{\tau^+} = \{\gamma^+, \left(k^+_{e_1^+}, k^+_{e_2^+}, ..., k^+_{e_{A-1}^+}, 2\pi L\right), \vec{l}^+\} \tag{55}$$

---

[5]$\hat{W}(s_{0,\mu\pm})\hat{V}[\tau^{\pm}]\hat{W}(s_{0,\mu\pm})^{-1} = \hat{V}[\tau^{\pm}]$.

[6]We set $\lambda^+$ to zero.

Whence we want to show that

$$\eta^+(|\mathbf{s}^+_{\tau^+}\rangle) \;=\; \eta^+(|\mathbf{s}^+_{cyc}\rangle) \tag{56}$$

The definition of the Group averaging map as given in the first equation in (42) makes it clear that to prove (56), it suffices to show that $\exists\, \phi^+$ such that

$$\phi^+ \cdot \mathbf{s}^+_{\tau^+} \;=\; \mathbf{s}^+_{cyc} \tag{57}$$

Note that the quasi-periodic extension $\mathbf{s}^+_{\tau^+ext}$ of $\mathbf{s}^+_{\tau^+}$ as defined in section 3.4, is given by

$$\mathbf{s}^+_{\tau^+ext} \;=\; \{\gamma^+_{ext}, \left(..., k^+_{e^+_0}, (k^+_{e^+_1}, ..., \frac{2\pi L}{\hbar}), k^+_{e^+_1} + \frac{2\pi L}{\hbar}, ...\right), \left(..., l^+_{e^+_{A-1}}, (l^+_{e^+_0}, ..., l^+_{e^+_{A-1}}), l^+_{e^+_0}, ...\right)\} \tag{58}$$

e.g. the embedding charges on $T_{-1}(\gamma^+)$ are $(k_{e^+_1} - \frac{2\pi L}{\hbar}, ..., 0 = k^+_{e^+_0})$ and the embedding charges on $T_1(\gamma^+)$ are $(k^+_{e^+_1} + \frac{2\pi L}{\hbar}, ..., \frac{4\pi L}{\hbar})$.

As $\phi^+ \cdot \mathbf{s}^+_{\tau^+} \;=\; \phi^+(\mathbf{s}^+_{\tau^+ext})|_{[0,2\pi]}$. It is easy to envisage a $\phi^+$ which is such that

$$\phi^+(\mathbf{s}^+_{\tau^+ext})|_{[0,2\pi]} \;=\; \mathbf{s}^+_{cyc} \tag{59}$$

This finishes the proof.

However the above result implies that any state in $\mathcal{D}_{ss}$ which is invariant under the cyclic permutations of the matter charges, yields, on Group averaging, a Poincare invariant Physical state.

Consider any state in (the closure of) $\mathcal{D}_{ss}$ of the form,

$$|\Phi^+\rangle \;=\; \sum_{\vec{l}^+} c(\vec{l}^+)|\gamma^+, \vec{k}^+, \vec{l}^+\rangle \tag{60}$$

where each charge-network state in the sum has the same underlying graph $\gamma^+$ and same embedding charges $\vec{k}^+$. $|\Phi^+\rangle$ is clearly invariant under cyclic permutations of matter charges, if the function $c(\vec{l}^+)$ is symmetric. As there are infinitely many functions of $\{l^+_{e^+_0}, ..., l^+_{e^+_{A-1}}\}$ which are invariant under cyclic permutations, each corresponding $|\Phi^+\rangle$ on Group averaging will yield a Poincare invariant Physical state. Thus there are infinite number of Poincare invariant states in the theory.

# 8  Emergence of spacetime lattice

In section 8.1 we show that the Group average of a (suitably defined) charge-network state is associated with a discrete spacetime [7]. In section 8.2 we

---

[7]We note here that similar arguments in [15], while incomplete, yielded the same picture

show in a precise sense that the Dirac observables of the theory cannot resolve the spacetime at scales finer then the minimum embedding charge separation $a = \frac{2\pi L}{A}$. Sections 8.1 and 8.2 together imply that the quantum theory of the true degrees of freedom is a lattice field theory.

## 8.1 Discrete Cauchy slices and discrete spacetime

The polymer quantization of the embedding variables replaces the classical (flat) spacetime continuum with a discrete structure consisting of a countable set of points. This can be seen as follows.

Consider the classical canonical data $(X^+(x), X^-(x), Y^+(x), Y^-(x))$. The data $(X^+(x), X^-(x))$ is a map from $S^1$ into the flat spacetime $(S^1 \times R, \eta)$ and embeds the former into the latter as a spatial Cauchy slice with coordinate $x$ on which $(Y^+(x), Y^-(x))$ serve as initial data for scalar field evolution. Any gauge transformation generated by the constraints maps this data to new data which, in turn, defines matter data on a new Cauchy slice in the flat spacetime. In particular, the action of the one parameter family of gauge transformations generated by smearing the constraints with some choice of "lapse-shift" type functions $N^A$ (see section 2) generates a foliation of $(S^1 \times R, \eta)$. The matter data on each slice of this foliaton together define a single solution $f$ to the flat spacetime wave equation, and are related to this solution through the equation $Y^\pm(x) = \pm 2 X^{\pm\prime}(x)\frac{\partial f}{\partial X^\pm}$ [16]. We shall refer to this property (that the same flat spacetime and the same solution arise independent of the choice of $N^A$) as spacetime covariance. Spacetime covariance is guaranteed by the detailed nature of the constraints and their algebra [16].

Next, consider the corresponding quantum structures. Consider a charge network state with n edges, $|\mathbf{s}_{\lambda+}^+\rangle \otimes |\mathbf{s}_{\lambda-}^-\rangle$ with $|\mathbf{s}_{\lambda\pm}^\pm\rangle = T_{s^\pm} \otimes W_{s_{\lambda\pm}^{\prime\pm}}$ , where $\mathbf{s}_{\lambda\pm}^\pm$ is such that
(a) the embedding labels of successive edges are unequal and montonically increasing (or decreasing) for the left (or right) moving sectors, and
(b) $|\hbar k_{e_n^\pm}^\pm - \hbar k_{e_0^\pm}^\pm| < 2\pi L$.
From [15] it follows that the Group Average of such charge networks span a superselected subspace of physical states (Clearly, the sector $\mathcal{H}_{phys}^{ss\pm}$ (as well as $\mathcal{H}_{(0)phy}^{ss\pm}$) obtained by averaging states in $\mathcal{D}_{ss}^\pm$ is of this type). The action of the Dirac Observables of section 4.1 contains the physical information in the charge network state. This action depends on the pairs $(k_{e\pm}^\pm, l_{e\pm}^\pm + \lambda^\pm)$ of embedding and matter charges for each edge $e^\pm \in \gamma(\mathbf{s}_{\lambda\pm}^\pm)$. Consider the coarsest graph, $\gamma(\mathbf{s}_{\lambda+}^+, \mathbf{s}_{\lambda-}^-)$, which is finer than $\gamma(\mathbf{s}_{\lambda+}^+), \gamma(\mathbf{s}_{\lambda-}^-)$. Then the pairs $(k_e^+, k_e^-)$ for each edge of this graph define a point $(X^+, X^-) = (\hbar k_e^+, \hbar k_e^-)$ in the flat spacetime by virtue of the embedding charges being eigen values of the embedding coordinate operators. Hence, we may place on each such spacetime point, the matter charges $(l_e^+ + \lambda^+, l_e^- + \lambda^-)$. The action of any

gauge transformation on such a charge network state yields another charge network state for which a similar association may be made. Note that the set of spacetime points associated with each such charge network are either spacelike or null related and constitute the quantum version of a classical Cauchy slice. The gauge invariant state obtained by group averaging a charge network state is a sum over all distinct gauge equivalent states. It is straightforward to see that the set of labels $(k_e^+, k_e^-, l_e^+ + \lambda^+, l_e^- + \lambda^-)$ for all the edges of all the charge networks in the sum yield a *consistent* spacetime association. In other words, the different sets of labels coming from different gauge related states all fit into a *single* labelling of a countable set of points in flat spacetime each point being labelled by a single pair of matter charges. This property is the analog of classical spacetime covariance.

It is easy to see that for states in $\mathcal{H}_{phy}^{ss}$ (as well as in $\mathcal{H}_{(0)phy}^{ss}$) the countable set of points correspond to those of of a regular spacetime lattice with lattice spacing $\frac{2\pi L}{A}$ and that no finer discrete structure is available for states which satisfy the "nondegeneracy conditions" **(a),(b)** above. The existence of this *finest lattice* is tied to the integer- valuedness of the embedding charges used in this work; no such finest lattice exists for the real charges of Reference [15].

## 8.2 Dirac observables and the emergence of the spacetime lattice

Consider the Weyl algebra generated by observables of the type $e^{iO_{f^+}} = \exp(i \int_x f^+(X^+(x))Y^+(x))$ where $f^+$ is a continuous periodic function (in the $X^+$ variable) such that $f^+(Ia) \in \mathbf{Z}\epsilon \ \forall I$. The abstract Weyl algebra generated by such observables is

$$\widehat{e^{iO_{f^+}}} \widehat{e^{iO_{g^+}}} = e^{-\frac{i\hbar}{2}\alpha(\hat{f}^+, \hat{g}^+)} \widehat{e^{iO_{f^+ + g^+}}} \tag{61}$$

where $\alpha(\hat{f}^+, \hat{g}^+) = \int_x \left( f^+(\hat{X}^+)(x)\partial_x g^+(\hat{X}^+)(x) - g^+(\hat{X}^+)(x)\partial_x f^+(\hat{X}^+)(x) \right)$.

Note that action of the phase factor $e^{-\frac{i\hbar}{2}\alpha(\hat{f}^+, \hat{g}^+)}$ on a charge-network state $|\mathbf{s}_{\lambda^+}^+\rangle = |\gamma(\mathbf{s}_{\lambda^+}^+), \vec{k}^+, \vec{l}^+ + \lambda^+\rangle$ is given by,

$$e^{-\frac{i\hbar}{2}\sum_{e^+ \in \gamma(\mathbf{s}_{\lambda^+}^+)} \left( f^+(\hbar k_{e^+}^+) \left( g^+(\frac{\hbar k_{e^+}^+ + \hbar k_{(e+1)^+}^+}{2}) - g^+(\frac{\hbar k_{e^+}^+ + \hbar k_{(e-1)^+}^+}{2}) \right) - f^+ \leftrightarrow g^+ \right)} |\mathbf{s}_{\lambda^+}^+\rangle \tag{62}$$

Let $|\mathbf{s}_{\lambda^+}^+\rangle \in \mathcal{D}_{ss}^+$.

A simple computation shows that action of $\widehat{e^{iO_{f^+}}} \widehat{e^{iO_{g^+}}}$ on $|\mathbf{s}_{\lambda^+}^+\rangle$ is given by,

$$e^{-\frac{i\hbar}{2}\sum_{I=0}^{A-1} \left( f^+(\hbar k_{e_I^+}^+) \left( \frac{g^+(\hbar k_{e_I^+}^+) + g^+(\hbar k_{e_{I+1}^+}^+)}{2} - \frac{g^+(\hbar k_{e_I^+}^+) + g^+(\hbar k_{e_{I-1}^+}^+)}{2} \right) - f^+ \leftrightarrow g^+ \right)} \widehat{e^{iO_{f^+ + g^+}}} |\mathbf{s}_{\lambda^+}^+\rangle \tag{63}$$

where we have identified $e^+_{-1}$ with $e^+_{A-1}$ and $e^+_A$ with $e^+_0$.

Using (62) and (63) it is straightforward to see that the Weyl-algebra of observables is faithfully represented on $\mathcal{D}^+_{ss}$ iff the test functions satisfy a set of discrete consistency conditions,

$$
\begin{aligned}
\frac{f^+(\hbar k^+_{e^+_I})+f^+(\hbar k^+_{e^+_{I+1}})}{2} &= f^+(\frac{\hbar k^+_{e^+_I}+\hbar k^+_{e^+_{I+1}}}{2}) \ \forall \ 0 \le I \le (A-2) \\
\frac{f^+(\hbar k^+_{e^+_0})+f^+(\hbar k^+_{e^+_{A-1}})}{2} &= f^+(\frac{\hbar k^+_{e^+_0}+\hbar k^+_{e^+_{A-1}}}{2})
\end{aligned}
\tag{64}
$$

Thus the class of functions for which the Weyl algebra generated by $\widehat{e^{iO}_{f^+}}$ can be represented on $\mathcal{D}^+_{ss}$ is tied to the structure of the finest lattice.

As the Weyl algebra strongly commutes with the Group averaging map, the above consistency conditions continue to hold at the Physical state space level in the following sense. A basis state in $\mathcal{H}^{+ss}_{phy}$ is given by, [8]

$$
\eta^+(|\mathbf{s}^+_{\lambda^+}\rangle) = \eta^+_{[\gamma(\mathbf{s}^+_{\lambda^+})]} \sum_{\phi^+} \langle \mathbf{s}^+_{\lambda^+ \phi^+}|
\tag{65}
$$

Due to periodicity of $f^+$, conditions given in (64) are satisfied for all $\mathbf{s}^+_{\lambda^+ \phi^+}$ if they are satisfied for $\mathbf{s}^+_{\lambda^+}$.

Now let us look at the commutator of two Dirac observables.

$$
[\widehat{e^{iO}_{f^+}}, \widehat{e^{iO}_{g^+}}] = \sin(\frac{\hbar}{2}\alpha(f^+,g^+))\widehat{e^{iO}_{f^++g^+}}
\tag{66}
$$

Let us assume that there exist a semi-classical state in which expectation value of $\widehat{e^{iO}_{f^+}}$ equals its classical value (at certain point in phase-space) plus corrections.

To the leading order in $\hbar$ the above commutator will equal $i\hbar$ times the corresponding Poisson bracket iff the quantum phase factor $\alpha(f^+,g^+)$ equals the classical phase factor $\int_{X^+}( f(X^+)\partial_{X^+}g(X^+) - g(X^+)\partial_{X^+}f(X^+) )$. From (62) and (64),

$$
\alpha(f^+,g^+) = \sum_{I=0}^{A-1} \left[ f(\hbar k^+_{e^+_I}) \left( g(\hbar k^+_{e^+_{I+1}}) - g(\hbar k^+_{e^+_{I-1}}) \right) - f \leftrightarrow g \right].
$$

(Where $e^+_{-1} = e^+_{A-1}$ and $e^+_A = e^+_0$.)

It is straightforward to show that $\alpha(f^+,g^+)$ equals the classical phase factor iff $f^+$ and $g^+$ are piecewise constant functions i.e.

$$
f^+(X^+) = f^+(\hbar k^+_{e^+_I}) \ \forall \ X^+ \text{ such that } \frac{\hbar k^+_{e^+_I}+\hbar k^+_{e^+_{I-1}}}{2} < X^+ < \frac{\hbar k^+_{e^+_I}+\hbar k^+_{e^+_{I+1}}}{2}.
$$

---

[8] We ignore the zero-mode constraint in this section as the corresponding Group averaging map does not effect the graph or the embedding charges. The results described here can be easily lifted to $\mathcal{H}^{ss}_{(0)phy}$.

(Here once again $e^+_{-1} = e^+_{A-1}$ and $e^+_A = e^+_0$.)

The upshot of the above arguements is that the classical symplectic structure (on the reduced phase space) can only emerge for (periodic) functions satisfy following two conditions.

**a** Given the set of embedding data $\{k^+_{e^+_0} = 0, ..., k^+_{e^+_{A-1}} = \frac{2\pi(A-1)L}{A\hbar}\}$ $f^+$ has to satisfy $\frac{f^+(\hbar k^+_{e^+_I}) + f^+(\hbar k^+_{e^+_{I+1}})}{2} = f^+(\frac{\hbar k^+_{e^+_I} + \hbar k^+_{e^+_{I+1}}}{2}) \ \forall \ I$.

**b** $f^+(X^+)$ has to be piecewise constant. In other words it cannot probe the underlying continuum spacetime at scales finer then $\frac{2\pi L}{A}$.

The above two conditions describe the precise sense in which the quantum theory is a lattice field theory.[9]

# 9  Tool-kit for the lattice theory

As we saw in the last section, the polymer quantization of parametrized field theory leads to a lattice field theory. Detailed analysis of this theory requires certain tools from Discrete Fourier transforms which we summarise in this section.

Let $F(X^\pm)$ be a real-valued periodic function with periodicity $L$.

$$\tilde{F}(n) = \frac{1}{2\pi L} \int_0^{2\pi L} F(X^+) e^{\frac{inX^+}{L}} \ \forall n \in \mathbf{Z} \tag{67}$$

Discrete Fourier transform is an approximation of the above integral by Riemann sum.

$$\tilde{F}_D(n) := \frac{1}{2\pi L} \sum_{I=0}^{A-1} F(k^+_I) e^{\frac{2\pi i n I}{A}} a \tag{68}$$

where $k^+_I = \frac{2\pi I L}{A}$ and $a = \frac{2\pi L}{A}$ is the lattice spacing. The subscript D in $\tilde{F}_D(n)$ stands for discrete.

Notice that whereas in the case of continuous Fourier transform , $n \in \mathbf{Z}$, in the discrete case we have $-(A-1) \leq n \leq (A-1)$. Finally note that, using

$$\sum_{m=0}^{A-1} e^{\frac{2\pi i n(I-J)}{A}} = A\delta_{I,J} \tag{69}$$

We can write inverse (discrete) Fourier transform as,

$$F(k^+_J) = \sum_{n=0}^{A-1} \tilde{F}_D(n) e^{-\frac{2\pi i n J}{A}} \tag{70}$$

---

[9]Also notice that $\hat{V}[\tau^+ = \frac{2\pi L}{A\hbar}]e^{\widehat{iO_{F_I^+}}} \hat{V}[\tau^+ = \frac{2\pi L}{A\hbar}]^{-1} = e^{\widehat{iO_{F_{I+1}^+}}} \ \forall \ I$ where I=(A+1) is identified with I=0. Whence $e^{\widehat{iO_{F_I}}}$ also satisfy discrete evolution equations under time translations. This is yet another indication that we are dealing with a lattice field theory.

Finally if the function F doesnot have a zero-mode, i.e.

$$\int_{X^+} F(X^+) \;=\; 0 \tag{71}$$

then $\tilde{F}(0) \;=\; \tilde{F}_D(0) \;=\; 0$.

We will use these equations extensively in the subsequent sections.

## 10 Fock vacuum as a polymer state

This section is devoted to the construction of a state in the polymer Hilbert space which approximates the Fock vacuum in the behaviour of its (1 and) 2 point functions. The nature of the approximation is precisely defined as follows. Recall that the two Barbero- Immirizi parameters, $a = \frac{2\pi L}{A}, \epsilon$ dictate the spacing of the embedding and matter charges, the former also being the lattice spacing of the spacetime lattice associated with states in $\mathcal{H}^{ss}_{(0)phy}$ (see section 8). We define the *continuum limit* to be the limiting behaviour as $a, \epsilon \to 0$. Thus, the continuum limit is a limiting property of a 2 parameter family of quantum theories. Accordingly, our aim is to construct a 2 parameter set of states (each state in an inequivalent quantum theory) and hence a 2 parameter family of 2 point functions which approach the Fock vacuum 2 point function in the continuum limit.

We are interested here in the Fourier transform of the two point function. This quantity is determined by the expectation values of all operators quadratic in the creation-annihilation modes. An added complication in the polymer quantization is that these modes themselves have to be approximated by suitable polymer operators (recall [15] that only exponentials of the modes are well defined operators not the modes themselves). As we shall see, in the spirit of lattice field theory, the approximants depend on $\epsilon, a$.

Specifically we aim to construct a 2 parameter family of states $|\Psi^{\pm}\rangle$ and a corresponding 2 parameter family of suitable approximants $\hat{a}^{poly}_n$ to $\hat{a}_n$ such that

$$\lim_{continuum} ||\hat{a}^{poly}_{(\pm)n} \frac{|\Psi^{\pm}\rangle}{||\Psi^+||}|| = 0 \ \forall \ 0 \ n < \infty$$
$$\lim_{continuum} \frac{1}{||\Psi^{\pm}||^2} \langle \Psi^{\pm} |[\hat{a}^{poly}_{(\pm)n}, \hat{a}^{\dagger poly}_{(\pm)m}]|\Psi^{\pm}\rangle \;=\; \hbar n \delta_{n,m} \tag{72}$$

That it suffices to satisfy the above equations to obtain the correct Fock (1 and) 2 point function follows straightforwardly from a judicious use of the Cauchy Schwarz inequality. More in detail, the vanishing of the one point function follows from the first equation above and various quadratic combinations of $\hat{a}_{(\pm)n}, \hat{a}^{\dagger}_{(\pm)m}$ obtain the correct expectation values in the continuum limit using both the above equations and the Cauchy- Schwarz inequality.

Note that in the Fock representation the analog of the first equation constitutes the definition of the Fock vacuum and the second equation is a trivial consequence of the canonical commutation relations. In contrast, the polymer representation (as we shall see explicitly in section 10.3) supports the canonical commutation relations in expectation value only in the continuum limit. Note also that in the Fock representation the analog of the first equation together with the canonical commutation relations, generates all the $n$- point functions. This gives us hope that the computations here will help in establishing whether or not the behaviour of all $n$- point functions can be adequately approximated in the polymer representation. Finally, note that obtaining the 2 point function is more ambitious than obtaining semiclassical behaviour peaked at trival classical data.

In section 10.1 we summarise our results. In section 10.2 we discuss the polymer approximant to the mode function. In section 10.3 we outline a heuristic set of calculations which allow us to make an educated guess for the solution of section 10.1. We show in section 10.4 that the candidate solution guessed in section 10.3 and displayed in section 10.1 actually solves the annihilation operator condition (i.e the first equation of (72)), and in 10.5 that it solves the commutation relation condition (i.e. the second equation of (72)).

With out loss of generality, we work with the solution to the zero-mode constraint $|\gamma^{\pm}, \vec{k}^{\pm}, \vec{\triangle l}^{\pm})$ instead of the working with states in $\mathcal{H}^{ss}_{(0)phy}$. This is only in the interest of pedagogy. Towards the end of the section we describe how to generalise these constructions to physical states (which are solutions to all the constraints.)

## 10.1 Summary of main results

Fix the graph $\gamma^{+}$ and the embedding data $\vec{k}^{+} = \left( k^{+}_{e^{+}_{0}} = 0, ..., k^{+}_{e^{+}_{A-1}} = \frac{2\pi(A-1)L}{\hbar} \right)$ once and for all.
Consider a state in the left moving sector

$$|\Psi^{+}\rangle = \sum_{\vec{\triangle l}^{+}} c(\vec{\triangle l}^{+})|\gamma^{+}, \vec{k}^{+}, \vec{\triangle l}^{+}) \tag{73}$$

where

$$c(\vec{\triangle l}^{+}) = \exp[-\frac{\hbar 2\pi L^{2}\epsilon^{2}}{a^{2}} \sum_{n>0} \frac{|\tilde{\triangle l}^{+}(n)|^{2}}{f(n)}] \tag{74}$$

where $f(n)$ is given by

$$\begin{aligned} f(n) &= \frac{\tan(\frac{n\pi}{A})}{\frac{n\pi}{A}} \ \forall n \leq N_{0} \\ f(n) &= n \ \forall n \geq N_{0}, \end{aligned} \tag{75}$$

and $N_0$ is a fixed integer independent of $\epsilon, a$. This state(as shown in appendix A) is normalizable.

In the next section we show that a suitable choice of approximants to the mode functions which can be quantized on $\mathcal{H}^{ss}_{(0)phy}$ are given by,

$$\hat{a}_n^{poly} \;=\; \frac{1}{A\epsilon} \sum_{I=0}^{A-1} a(e^{\widehat{iO}_{F_I^+}} - e^{-\widehat{iO}_{F_I^+}}) e^{in\hbar k_{e_I^+}^+} \tag{76}$$

with the functions $F_I^+$ defined as follows.

$$\begin{aligned}
F_I^+(X^+) &= \epsilon \ \text{if} \ \frac{\hbar k_{e_I^+}^+ + \hbar k_{e_{I-1}^+}^+}{2} < X^+ < \frac{\hbar k_{e_I^+}^+ + \hbar k_{e_{I+1}^+}^+}{2} \\
&= \tfrac{\epsilon}{2} \ \text{if} \ X^+(x) = \frac{\hbar k_{e_I^+}^+ + \hbar k_{e_{I-1}^+}^+}{2} \\
&= 0 \ \text{otherwise}
\end{aligned} \tag{77}$$

$\forall \ 0 \leq I \leq A - 1$. Here $\hbar k_{e_{-1}^+}^+ := -\frac{-2\pi L}{A}$ and $\hbar k_{e_{A+1}^+}^+ = \frac{2\pi L}{A}$.

The precise statement about the continuum limit and emergence of Fock vacuum (in the left moving sector) is the following.

$$\begin{aligned}
\hat{a}_n^{poly} \frac{|\Psi^+\rangle}{||\Psi^+||} &= 0 \;+\; |\delta\Psi^+\rangle \ \forall \ 0 \;<\; n \;< N_0 \\
\frac{1}{||\Psi^+||^2} \langle \Psi^+ | \hat{a}_n^{poly} \hat{a}_m^{poly\,\dagger} | \Psi^+ \rangle &= \frac{\hbar}{\pi L^2} n \delta_{n,m} + \text{error-terms}
\end{aligned} \tag{78}$$

where $n, \ m \; > \; 0$,
and $||\delta\Psi^+||$ as well as the error-terms in (78) vanish in the limit

$$\begin{aligned}
a &\to \; 0 \\
\epsilon &= C_0 a^\triangle \ \text{with} \ \triangle \; > \; 4.
\end{aligned} \tag{79}$$

Henceforth we shall assume that the continuum limit (i.e. $\epsilon, \ a \to \; 0$) is governed by (79).

We note that even though the form of $|\Psi^+\rangle$ is reminiscent of a complexifier coherent state [23], we have not been able to obtain it via the complexifier techniques, the reason being that we do not know of any function whose operator correspondent possesses the matter charges $\triangle l_I^+$ as its eigen values.

Another note-worthy feature of $|\Psi^+\rangle$ is its invariance under cyclic permutations of $(l_{e_0^+}^+, ..., l_{e_{A-1}^+}^+)$, which implies that it is (discrete) Poincare invariant.

## 10.2 Polymer approximants to the mode functions.

Consider the classical Fourier modes of K-G scalar field $\Phi(X^+, X^-)$[10]

$$a_{(\pm)n} \;=\; \pm \frac{1}{2\pi L} \int_0^{2\pi L} \frac{\partial \Phi}{\partial X^\pm} e^{\frac{inX^\pm}{L}} \tag{80}$$

where $n > 0$.

Let us restrict our attention to the $(+)$-sector.

Whence $a_n$ is just Fourier transform of $\frac{\partial \Phi}{\partial X^+}$. The corresponding discrete Fourier transform is (68),

$$a_{n,D} \;=\; \frac{1}{A} \sum_{I=0}^{A-1} (\frac{\partial \Phi}{\partial X^+})(k_{e_I^+}^+) e^{\frac{in\hbar k_{e_I^+}^+}{L}} \tag{81}$$

Where $\hbar k_{e_I^+}^+ = \frac{2\pi I L}{A}$. As the computations performed subsequently do not change the graph or the embedding charges, we will supress the edge label on the embedding data. Whence we will denote $\hbar k_{e_I^+}^+$ by $k_I^+$ from now on.

Further more, as the only combination in which $k_I^+$ will come up in all the subsequent calculations, is $\frac{k_I^+}{L} = \frac{2\pi I}{A}$, *we will abuse the notation slightly more and denote* $\frac{2\pi I}{A}$ *by* $k_I^+$.

Whence the suitable approximant to $a_n$ which can be quantized on $\mathcal{H}_{phy}^+$ is given by,

$$a_n^{poly} \;:=\; \frac{1}{A} \sum_{I=0}^{A-1} (\frac{e^{iO_{F_I^+}} - e^{-iO_{F_I^+}}}{2ia\epsilon}) e^{ink_I^+} \tag{82}$$

where we have approximated $\frac{\partial \Phi}{\partial X^+}$ by its lattice approximant

$$(\frac{\partial \Phi}{\partial X^+})^{poly} := \frac{e^{iO_{F_I^+}} - e^{-iO_{F_I^+}}}{2ia\epsilon} \tag{83}$$

The corresponding quantum operator is

$$\hat{a}_n^{poly} \;:=\; \frac{1}{A} \sum_{I=0}^{A-1} (\frac{e^{\widehat{iO_{F_I^+}}} - e^{\widehat{-iO_{F_I^+}}}}{2ia\epsilon}) e^{ink_I^+} \tag{84}$$

## 10.3 Heuristic Guesswork

This section could be ommitted without affecting the logic and completeness of the paper. We include it in the hope that similar heuristics might prove useful for constructions in LQG. In this section our aim is to provide an

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[10]In section 2 we denoted the spacetime scalar field by $f$. Henceforth we shall denote it by $\Phi$.

educated guess for a solution to equations (78). Since the argumentation is heuristic and intuitive, we do not attempt to summarise it precisely and the reader is advised to temporarily relax standards of rigor and simply "go with the flow". The argumentation uses an intuitive idea of a continuum limit wherein discrete structures are summarily replaced by intuitive continuum analogs. This transforms the first equation of (78) into a functional differential equation for the continuum analog of the state; this equation can easily be solved and its discrete correspondent provides an educated guess for the polymer state.

**As we are working in $\overline{\eta}^+(\mathcal{D}_{ss}^+)$ with a fixed graph $\gamma^+$ and the fixed embedding data $\left(k_0^+ = 0, ..., k_{A-1}^+ = \frac{2\pi(A-1)}{A}\right)$, we will supress the (+)-label on the $\vec{k}^+$, $\vec{\triangle l}^+$ charges in all our subsequent calculations.**

It is easy to show that the action of $\hat{a}_n^{poly}$ on an arbitrary state $|\Psi^+\rangle = \sum_{\vec{\triangle l}} c(\vec{\triangle l})|\gamma, \vec{k}, \vec{\triangle l})$ in $\overline{\eta}(\mathcal{D}_{ss}^+)$ is given by,

$$\hat{a}_n^{poly}|\Psi^+\rangle =$$
$$\frac{1}{2i\epsilon aA} \sum_{I=0}^{A-1} e^{ink_I} \sum_{\vec{\triangle l}} \left[ e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})} c(\vec{\triangle l} - \vec{\triangle \epsilon}^{(I)}) - \right.$$
$$\left. e^{\frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})} c(\vec{\triangle l} + \vec{\triangle \epsilon}^{(I)}) \right] \quad (85)$$
$$|\gamma, \vec{k}, \vec{\triangle l})$$

where $(\triangle \epsilon)_J^{(I)} := \epsilon(\delta_J^I - \delta_{J-1}^I)$.

Now as, $a \to 0$, the embedding charges behave as,

$$k_I \, L = \frac{2\pi I L}{A} = Ia$$
$$\to x_I \quad (86)$$

where $x_I$ is a continuous variable whose range is $[0, 2\pi L]$.

The exponents in the phase factors in (85) are $\pm\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})$ which can be written as, $\pm\frac{\hbar\epsilon a}{2}(\frac{\epsilon\triangle l_I}{a} + \frac{\epsilon\triangle l_{I+1}}{a})$.

Hence the phase factor can be written as a power-series in $\epsilon$, $a$ only if $\frac{\epsilon\triangle l_I}{a} = \frac{\epsilon(l_I - l_{I-1})}{a}$ remains finite as $\epsilon, a \to 0$.
Whence let us assume that,

$$\lim_{\epsilon, a \to 0} \left(\frac{\epsilon\triangle l_I}{a}\right) = O(1) \quad (87)$$

More precisely the above assumption hints at $\lim_{\epsilon,a\to 0}(\frac{\epsilon\triangle l_I}{a}) = g(x_I)$ where g is a continuous function of the (continuous) variable $x_I$. As we want to try the simplest possible scenarios under which (85) admits a solution, we also assume that $g(x)$ is a smooth function. Whence in the limit $\epsilon$, $a \to 0$,

$$\frac{\epsilon\triangle l_{I+1}}{a} \;=\; \frac{\epsilon\triangle l_I}{a} + \text{error} - \text{terms} \tag{88}$$

Where the error-terms vanish in the limit $a \to 0$. Another quantity, whose continuum limit will be crucial to subsequent calculations is the following.

$$\begin{aligned}
\lim_{a\to 0} \epsilon_J^{(I)} \;&=\; \lim_{a\to 0} \epsilon(\delta_{I,J} - \delta_{I,J-1}) \\
&= \epsilon\tfrac{d}{dx_I}\delta(x_I, x_J)
\end{aligned} \tag{89}$$

where we have used $\lim_{a\to 0} \frac{\delta_{I,J}}{a} \;=\; \delta(x_I, x_J)$ and $\lim_{a\to 0} \frac{\delta(x_I,x_J)\,-\,\delta(x_I,x_J-a)}{a} \;=\; \frac{d}{dx_J}\delta(x_I, x_J)$.

In order to ensure that in the continuum limit we can also taylor expand $c(\vec{\triangle l})$ in terms of (a certain combination of) $\epsilon$ and a, we take,

$$c(\vec{\triangle l}) \;=\; c(\frac{\epsilon\vec{\triangle l}}{a}) \tag{90}$$

Whence in the continuum limit,

$$\begin{aligned}
c(\vec{\triangle l} \pm \vec{\triangle\epsilon}^{(I)}) \;&=\; c(\frac{\epsilon\vec{\triangle l}}{a} \pm \frac{\vec{\triangle\epsilon}^{(I)}}{a}) \\
&\to c(g(x) \pm a\epsilon\tfrac{d}{dx_I}\delta(x, x_I)) \\
&= c(g(x)) \pm a\epsilon\tfrac{d}{dx_I}\delta(x, x_I) \int dy \frac{\delta c(g(x))}{\delta c(g(y))}
\end{aligned} \tag{91}$$

The continuum limit of $\hat{a}_n^{poly}|\Psi^+\rangle$ (i.e. the continuum limit of (85)) is given by,

$$\begin{aligned}
\lim_{\epsilon,a\to 0} &\hat{a}_n^{poly}|\Psi^+\rangle \;= \\
&= \lim_C \tfrac{1}{2i\epsilon(2\pi L)} \sum_{I=0}^{A-1} \sum_{\triangle l} e^{ink_I} \left[ e^{\frac{-i\hbar\epsilon a}{2}(\frac{\epsilon\triangle l_I}{a} + \frac{\epsilon\triangle l_{I+1}}{a})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) - \right. \\
&\qquad\qquad\qquad\qquad \left. e^{\frac{i\hbar\epsilon a}{2}(\frac{\epsilon\triangle l_I}{a} + \frac{\epsilon\triangle l_{I+1}}{a})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)}) \right]|\vec{\triangle l}\rangle \\
&= \lim_C \tfrac{1}{2i\epsilon(2\pi L)} \sum_{I=0}^{A-1} \sum_{\triangle l} e^{ink_I} \left[ \left( c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) - c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)}) \right) \right. \\
&\qquad\qquad \left. - i\hbar\epsilon a\tfrac{\epsilon\triangle l_I}{a} \left( c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) + c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)}) \right) \right]|\vec{\triangle l}\rangle
\end{aligned} \tag{92}$$

where $\lim_C$ is an abbreviation for the limit $\epsilon$, $a \to 0$.

We have expanded the phase factor in $\epsilon a$ and kept only the leading order terms. $|\vec{\triangle l}\rangle$ is an abbreviation for the charge network state $|\gamma, \vec{k}, \vec{\triangle l}\rangle$. Whence the equation $\lim_C \hat{a}_n^{poly}|\Psi^+\rangle = 0$ implies

$$\lim_C \frac{1}{2i\epsilon(2\pi L)} \sum_{I=0}^{A-1} \sum_{\triangle l} e^{ink_I} \left[ \left( c(\vec{\triangle l} - \vec{\triangle \epsilon}^{(I)}) - c(\vec{\triangle l} + \vec{\triangle \epsilon}^{(I)}) \right) \right.$$
$$\left. - i\hbar \epsilon a \frac{\epsilon \triangle l_I}{a} \left( c(\vec{\triangle l} - \vec{\triangle \epsilon}^{(I)}) + c(\vec{\triangle l} + \vec{\triangle \epsilon}^{(I)}) \right) \right] |\vec{\triangle l}\rangle$$
$$= 0$$

(93)

We can now use (87), (91) ,(86) and, (89) to turn the above discrete equation into an integral equation.

$$\lim_C \frac{1}{2i\epsilon a 2\pi (2\pi L)} \int dx_I e^{\frac{inx_I}{L}} \left[ c(g(x) - \epsilon a \frac{d}{dx_I} \delta(x, x_I)) - c(g(x) + \epsilon a \frac{d}{dx_I} \delta(x, x_I)) \right.$$
$$\left. - i\hbar \epsilon a g(x_I)(c(g(x) - \epsilon a \frac{d}{dx_I} \delta(x, x_I)) + c(g(x) + \epsilon a \frac{d}{dx_I} \delta(x, x_I))) \right] = 0$$

(94)

Using (91) above equation further simplifies to,

$$\frac{1}{2\pi i L} \int dx_I [ \ e^{in\frac{x_I}{L}} \left( \frac{d}{dx_I} \frac{\delta c}{\delta g(x_I)} - i\hbar g(x_I) c(g(x)) \right) \ ] = 0 \qquad (95)$$

Using integration by parts for the first term and $\frac{1}{2\pi L} \int dx_I g(x_I) e^{\frac{inx_I}{L}} =: \tilde{g}(n)$, we get,

$$- \int dx_I (in\frac{1}{L}) e^{in\frac{x_I}{L}} (\frac{\delta c}{\delta g(x_I)}) - i\hbar(2\pi L)\tilde{g}(n) c(g) = 0 \qquad (96)$$

Now notice the following,

$$\frac{\delta c}{\delta g(x_I)} = \sum_{m \in \mathbf{Z}} \frac{\delta c}{\delta \tilde{g}(m)} \frac{\delta \tilde{g}(m)}{\delta g(x_I)}$$
$$= \sum_m \frac{\delta c}{\delta \tilde{g}(m)} \frac{1}{2\pi L} e^{im\frac{x_I}{L}} \qquad (97)$$

Above equation is satisfied by,

$$c(g) = \mathcal{A} \exp[-\hbar 2\pi L^2 \sum_{n>0} \frac{|\tilde{g}(n)|^2}{n}] \qquad (98)$$

Where $|\tilde{g}(n)|^2 = \tilde{g}(n)\tilde{g}^*(n)$ and $\mathcal{A}$ is an arbitrary constant. As $\tilde{g}(n) = \frac{1}{2\pi L} \int dx g(x) e^{in\frac{x}{L}}$, The corresponding discrete Fourier transform is given by,

$$\tilde{g}_D(n) = \frac{1}{2\pi L} \sum_{I=0}^{A-1} a(\frac{\epsilon \triangle l_I}{a}) e^{ink_I} \qquad (99)$$

But as the (discrete) Fourier transform of $\vec{\triangle}l$ is given by,

$$\tilde{\triangle}l(n) \;=\; \frac{a}{2\pi L}\sum_{I}\triangle l_{I}e^{ink_{I}} \tag{100}$$

This implies that $\tilde{g}_{D}(n) \;=\; \frac{\epsilon}{a}\tilde{\triangle}l(n)$. This suggests that we consider $c(\vec{\triangle}l)$ of the form,

$$c(\vec{\triangle}l) \;=\; \exp[-\frac{\hbar 2\pi L^{2}\epsilon^{2}}{a^{2}}\sum_{n>0}\frac{|\tilde{\triangle}l(n)|^{2}}{n}] \tag{101}$$

In fact there is more freedom in the choice of these weights. We can consider $c(\vec{\triangle}l)$ of the form,

$$c(\vec{\triangle}l) \;=\; \exp[-\frac{\hbar 2\pi L^{2}\epsilon^{2}}{a^{2}}\sum_{n>0}\frac{|\tilde{\triangle}l(n)|^{2}}{f(n)}] \tag{102}$$

where $f(n)$ should be such that $\lim_{a\to 0}f(n) = n\ \forall n$. One such choice of $f(n)$ is given by,

$$\begin{aligned} f(n) &\;=\; \frac{\tan(\frac{n\pi}{A})}{\frac{n\pi}{A}}\ \forall n \le N_{0}\\ f(n) &\;=\; n\ \forall n \;>\; N_{0} \end{aligned} \tag{103}$$

Where $N_{0} << A$ but is otherwise arbitrary. Thus choice of f(n) will be justified aposteriori in the subsequent section. The corresponding approximant to the Fock-vacuum is given by,

$$|\Psi^{+}\rangle \;=\; \sum_{\vec{\triangle}l}c(\vec{\triangle}l)|\vec{\triangle}l\rangle \tag{104}$$

The naive, intuitive continuum calculation suggests that the only non-trivial contributions to $\hat{a}_{n}^{poly}|\Psi^{+}\rangle$ come from the matter charges which are such that $\frac{\epsilon\triangle l_{I}}{a} \sim O(1)\ \forall I$ as $\epsilon, a \to 0$. As we will see in the next section, certain highly delicate cancellations between such contributions will contrive to ensure that $\hat{a}_{n}^{poly}\frac{|\Psi^{+}\rangle}{||\Psi^{+}||} \;=\; 0 \;+\; |\delta\Psi^{+}\rangle$ in the polymer Hilbert space.

## 10.4 The annihilation operator condition

In this section we compute the action of $\hat{a}_{n}^{poly}$ on $|\Psi^{+}\rangle \;=\; \sum_{\vec{\triangle}l}c(\vec{\triangle}l)|\gamma, \vec{k}, \vec{\triangle}l\rangle$ with $c(\vec{\triangle}l)$ given in (74). Our aim is to show that

$$\hat{a}_{n}^{poly}\frac{|\Psi^{+}\rangle}{||\Psi^{+}||} \;=\; 0 + |\delta\Psi^{+}\rangle\ \forall n \le N_{0} \tag{105}$$

and the $||\delta\Psi^+||$ vanishes in the continuum limit.

Using (84) we have,

$$\hat{a}_n^{poly}|\Psi^+\rangle = \frac{1}{2i\epsilon(2\pi L)} \sum_{I=0}^{A-1} e^{ink_I} \sum_{\triangle l}$$
$$\left[ e^{-\frac{i\hbar\epsilon a}{2}(\frac{\epsilon\triangle l_I}{a} + \frac{\epsilon\triangle l_{I+1}}{a})} \exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2} \sum_{m>0} \frac{|\tilde{\triangle}l(m)- \tilde{\triangle}\epsilon^{(I)}(m)|^2}{f(m)}] \right.$$
$$\left. - e^{\frac{i\hbar\epsilon a}{2}(\frac{\epsilon\triangle l_I}{a} + \frac{\epsilon\triangle l_{I+1}}{a})} \exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2} \sum_{m>0} \frac{|\tilde{\triangle}l(m)+\tilde{\triangle}\epsilon^{(I)}(m)|^2}{f(m)}] \right]$$
$$|\vec{\triangle l}\rangle$$

(106)

Note that $\triangle\epsilon_J^{(I)} := \epsilon(\delta_{I,J} - \delta_{I,J-1})$. However in this and all the subsequent sections, we will slightly abuse the notation and denote $(\delta_{I,J} - \delta_{I,J-1})$ by $\triangle\epsilon_J^{(I)}$.

(106) can be easily re-written as,

$$\hat{a}_n^{poly}|\Psi^+\rangle = \frac{1}{2i\epsilon(2\pi L)} \sum_{I=0}^{A-1} e^{ink_I} \sum_{\vec{\triangle l}} c(\vec{\triangle l}) \exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2} \sum_{m>0} \frac{|\tilde{\triangle}\epsilon^{(I)}(m)|^2}{f(m)}|]$$
$$\left[ e^{-\frac{i\hbar\epsilon a}{2}(\frac{\epsilon\triangle l_I}{a} + \frac{\epsilon\triangle l_{I+1}}{a})} \exp[+\frac{\hbar 2\pi L^2\epsilon^2}{a^2} \sum_{m>0} [\frac{\tilde{\triangle}l(m)\triangle\epsilon^{(I)}(-m)+c.c.}{f(m)}] \right.$$
$$\left. - e^{\frac{i\hbar\epsilon a}{2}(\frac{\epsilon\triangle l_I}{a} + \frac{\epsilon\triangle l_{I+1}}{a})} \exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2} \sum_{m>0} [\frac{\tilde{\triangle}l(m)\triangle\tilde{\epsilon}^{(I)}(-m)+c.c.}{f(m)}] \right]$$
$$|\vec{\triangle l}\rangle$$

(107)

where, using $(\triangle\epsilon)_J^{(I)} := (\delta_J^I - \delta_{J-1}^I)$ and (70)

$$\tilde{\triangle}\epsilon^{(I)}(n) = \frac{1}{A} \sum_{J=0}^{A-1} e^{ink_J}(\delta_{I,J} - \delta_{I,J-1})$$
$$= \frac{1}{A}e^{ink_I}(1 - e^{in\frac{2\pi}{A}})$$

(108)

Whence,

$$\exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2} \sum_{m>0} \frac{|\tilde{\triangle}\epsilon^{(I)}(m)|^2}{f(m)}|] =$$
$$\exp[-\frac{\hbar\pi L^2\epsilon^2}{a^2} \sum_{m>0} |\frac{1}{A}\frac{e^{imk_I}(1-e^{im\frac{2\pi}{A}})}{m}|^2] = \exp[-\frac{\hbar\pi L^2\epsilon^2}{a^2} \sum_{m>0} \frac{1}{A^2}4\frac{\sin^2(\frac{m\pi}{A})}{m}]$$

(109)

Before proceeding, we state a lemma whose proof is provided in Appendix-C. This lemma makes the computations a lot easier.

Consider a disjoint union of the set of all matter charges into two mutually exclusive subsets, $S_1(\epsilon, a) := \{\vec{\triangle l}|\frac{\epsilon^2}{a^2} \sum_I a(\triangle l_I)^2 < \frac{C}{a^\delta}\}$, $S_2(\epsilon, a) = S_1(\epsilon, a)^c$,[11] where C is a (dimensionful) constant which is positive and is

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[11] $S_2(\epsilon, a)$ is the compliment of $S_1(\epsilon, a)$ in the set of all $\vec{\triangle l}$.

independent of $\epsilon$ and $a$. We also choose $2 > \delta > 1$. [12]

(107) can be written in a schematic form as,

$$
\begin{aligned}
\hat{a}_n^{poly}|\Psi^+\rangle &= \tfrac{1}{2i\epsilon(2\pi L)} \sum_{\vec{\triangle l}} \sum_I e^{ink_I} [e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})} c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) \\
&\qquad - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})} c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})]|\vec{\triangle l}\rangle \\
&= [\sum_{\vec{\triangle l}\in S_1} + \sum_{\vec{\triangle l}\in S_2}](\sum_I e^{ink_I} [e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})} c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) \\
&\qquad - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})} c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})]|\vec{\triangle l}\rangle \\
&= |\Psi_1\rangle + |\Psi_2\rangle
\end{aligned}
\tag{110}
$$

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[12]As shown in appendix-A, choosing $\delta > 1$ ensures that contribution of $S_2(\epsilon, a)$ to the norm of the state is finite and tends to zero as $\epsilon$ and $a$ tend to zero.

**Lemma:**Show that, $||\Psi_2|| \rightarrow 0$ rapidly as $a \rightarrow 0$
**Proof**:
Given in Appendix C.

Whence it suffices to restrict the attention to $|\Psi_1\rangle$. Whence from now on, *we will assume that we are summing over* $\vec{\triangle l} \in S_1(\epsilon, a)$.
We will indicate this explicitly only where we think its necessary. Because $\frac{\epsilon \triangle l_I}{a}$ are bounded from above, to leading order in $\epsilon a$ we can expand the phase factors $e^{\mp \frac{i\hbar \epsilon a}{2}(\frac{\epsilon \triangle l_I}{a} + \frac{\epsilon \triangle l_{I+1}}{a})}$ as

$$e^{\mp \frac{i\hbar \epsilon a}{2}(\frac{\epsilon \triangle l_I}{a} + \frac{\epsilon \triangle l_{I+1}}{a})} = 1 \mp \frac{i\hbar \epsilon a}{2}[(\frac{\epsilon \triangle l_I}{a} + \frac{\epsilon \triangle l_{I+1}}{a}) + O(\epsilon^2 a^2)] \quad (111)$$

Substituting (111) in (107), we get

$\hat{a}_n^{poly}|\Psi^+\rangle =$
$\frac{1}{2i\epsilon(2\pi L)} \sum_{I=0}^{A-1} e^{ink_I} \sum_{\vec{\triangle l}}$

$$c(\vec{\triangle l}) \exp[-\frac{\hbar 2\pi L^2 \epsilon^2}{a^2} \sum_{m>0} \frac{|\tilde{\triangle \epsilon}^{(I)}(m)|^2}{f(m)}|]$$

$$\left[ \left(1 - i\frac{\hbar \epsilon a}{2}(\frac{\epsilon(\triangle l_I + \triangle l_{I+1})}{a}) + O(\epsilon^2 a^2)\right) e^{\frac{\hbar \epsilon^2 2\pi L^2}{a^2} \sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle \tilde{\epsilon}^{(I)}(-m)+c.c.]} \right.$$

$$\left. - \left(1 + i\frac{\hbar \epsilon a}{2}(\frac{\epsilon(\triangle l_I + \triangle l_{I+1})}{a}) + O(\epsilon^2 a^2)\right) e^{-\frac{\hbar \epsilon^2 2\pi L^2}{a^2} \sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle \tilde{\epsilon}^{(I)}(-m)+c.c.]} \right]$$

$$|\vec{\triangle l}\rangle$$

(112)

We now show that the exponent $\frac{\hbar \epsilon^2 2\pi L^2}{a^2} \sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle \tilde{\epsilon}^{(I)}(-m)+c.c.]$ is bounded.
Using Cauchy-Schwarz,

$$(\frac{\hbar \epsilon^2 2\pi L^2}{a^2})^2 |\sum_n \frac{1}{f(n)}[\tilde{\triangle l}(n)\triangle \tilde{\epsilon}^{(I)}(-n) + c.c.]|^2 \leq [\frac{\hbar \epsilon^2 2\pi L^2}{a^2} \sum_n \frac{1}{f(n)}|\tilde{\triangle l}(n)|^2]$$
$$[\frac{\hbar \epsilon^2 2\pi L^2}{a^2} \sum_n \frac{1}{f(n)}|\tilde{\triangle \epsilon}^{(I)}(n)|^2]$$
$$= [\frac{\hbar \epsilon^4 2\pi L^2}{a^4} \sum_n \frac{1}{f(n)}|\tilde{\triangle l}(n)|^2]$$
$$[\hbar 2\pi L^2 \sum_n \frac{1}{f(n)}|\tilde{\triangle \epsilon}^{(I)}(n)|^2]$$

(113)

Using, $f(n) \geq \frac{n}{2} \geq \frac{1}{2} \forall n$, $\sum_n |\tilde{\triangle l}(n)|^2 = \frac{1}{A} \sum_I \triangle l_I^2$ and $\frac{\epsilon^2}{a^2} \triangle l_I^2 < \frac{C}{a^{1+\delta}}$ we have,

$$\frac{\hbar \epsilon^4 2\pi L^2}{a^4} \sum_n \frac{1}{f(n)}|\tilde{\triangle l}(n)|^2 <$$
$$\frac{\hbar \epsilon^4 2\pi L^2}{a^4} \left(\frac{2}{A}\frac{C}{a^{1+\delta}}\right) =$$
$$(C\hbar L)\frac{\epsilon^4}{a^{4+\delta}}$$

(114)

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And the sum

$$\sum_m \frac{1}{f(m)}|\tilde{\triangle\epsilon}^{(I)}(m)|^2 =$$

$$\sum_{m=1}^{A-1} \frac{1}{A^2} 4 \frac{\sin^2(\frac{m\pi}{A})}{f(m)} = \sum_{m=1}^{A-1} \frac{\sin^2(\frac{m\pi}{A})}{\frac{m\pi}{A}}$$

$$\leq \sum_m \frac{2}{A\pi}(\frac{2m\pi}{A}) \qquad (115)$$

$$= \frac{4}{A^2} \sum m$$

$$\sim O(1)$$

Together (114) and (115) imply that,

$$(\frac{\hbar\epsilon^2 2\pi L^2}{a^2})^2 |\sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle\tilde{\epsilon}^{(I)}(-m) + c.c.]|^2 \leq [(C\hbar L)\frac{\epsilon^4}{a^{4+\delta}}]$$
$$[\hbar\pi L^2 O(1)] \qquad (116)$$

As $1 < \delta < 2, \epsilon \sim a^{\triangle}$ with $\triangle > 4$ means that $e^{\pm \frac{\hbar\epsilon^2 2\pi L^2}{a^2}\sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle\tilde{\epsilon}^{(I)}(-m)+c.c.]}$ can be expanded as a power-series in $\frac{\epsilon}{a}$. This observation simplifies (112) to,

$$\hat{a}_n^{poly}|\Psi^+\rangle =$$
$$\frac{1}{2i\epsilon(2\pi L)} \sum_{I=0}^{A-1} e^{ink_I} \sum_{\vec{\triangle l}} c(\vec{\triangle l}) \exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2}\sum_{m>0} \frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}|]$$

$$\left[\left(1 - i\frac{\hbar\epsilon a}{2}(\frac{\epsilon(\triangle l_I + \triangle l_{I+1})}{a} + O(\epsilon^2 a^2)\right)\right.$$

$$\left(1 + \frac{\hbar\epsilon^2 2\pi L^2}{a^2}\sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle\tilde{\epsilon}^{(I)}(-m) + c.c.] + O(\frac{\epsilon^4}{a^4})\right)\right]$$

$$- \left[\left(1 + i\frac{\hbar\epsilon a}{2}(\frac{\epsilon(\triangle l_I + \triangle l_{I+1})}{a}) + O(\epsilon^2 a^2)\right)\right.$$

$$\left(1 - \frac{\hbar\epsilon^2 2\pi L^2}{a^2}\sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle\tilde{\epsilon}^{(I)}(-m) + c.c.] + O(\frac{\epsilon^4}{a^4})\right)\right]$$

$$|\vec{\triangle l}\rangle$$
$$(117)$$

If we assume for a moment that all the sub-leading terms (that is terms which on using $\epsilon \sim a^{\triangle}$ vanish in the limit $a \to 0$) are bounded[13], then the

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[13]Notice that this is a highly non-trivial assumption as although each term inside the sum over $\vec{\triangle l}$ is finite and vanishes in $a \to 0$ limit (with $\epsilon \sim a^{\triangle}$), it is not at all clear if after summing over all possible matter charges, such terms remain bounded. That this is the case is shown in Appendix-B.

leading order terms in the above equation are given by,

$$\hat{a}_n^{poly}|\Psi^+\rangle \approx$$
$$\frac{1}{2i\epsilon(2\pi L)}\sum_{I=0}^{A-1} e^{ink_I} \sum_{\vec{\triangle l}\in S_1(\epsilon,a)} c(\vec{\triangle l})\exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2}\sum_{m>0}\frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}|]$$
$$\left(-i\hbar\epsilon a(\frac{\epsilon(\triangle l_I+\triangle l_{I+1})}{a}) + \frac{2\hbar\epsilon^2 2\pi L^2}{a^2}\sum_m \frac{1}{f(m)}[\tilde{\triangle l}(m)\triangle\tilde{\epsilon}^{(I)}(-m)+c.c.]\right)$$
$$|\vec{\triangle l}\rangle$$
$$(118)$$

where $\tilde{\triangle\epsilon}^{(I)}(\pm m) = \mp\frac{2i}{A}e^{\pm imk_I^+}e^{\frac{\pm im\pi}{A}}\sin(\frac{m\pi}{A})$.
We use the following formulae in (118)

$$\frac{1}{2\pi L}\sum_I e^{ink_I}(\triangle l_I+\triangle l_{I+1}) = \frac{1}{a}e^{-\frac{i\pi n}{A}}2\cos(\frac{2\pi n}{A})\tilde{\triangle l}(n)$$
$$\sum_I (a\,e^{ink_I})\tilde{\triangle\epsilon}^{(I)}(\pm m) = \sum_I \frac{\mp 2i}{A}a\,e^{i(n\pm m)k_I}e^{\frac{\pm im\pi}{A}}\sin(\frac{m\pi}{A})$$
$$= (\mp 2i)(a)\delta_{n,\mp m}e^{\frac{\pm im\pi}{A}}\sin(\frac{m\pi}{A}) \qquad (119)$$

and

$$\sum_I e^{ink_I}\sum_{m>0}\tilde{\triangle l}(-m)\tilde{\triangle\epsilon}^{(I)}(m)\ \alpha\ \sum_{m>0}\delta_{n,-m} = 0$$

and obtain,

$$\hat{a}_n^{poly}|\Psi^+\rangle \approx$$
$$\frac{1}{2i\epsilon aL}\sum_{\vec{\triangle l}\in S_1(\epsilon,a)} c(\vec{\triangle l})\exp[-\frac{\hbar 2\pi L^2\epsilon^2}{a^2}\sum_{m>0}\frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}|]$$
$$\left((-i\hbar\epsilon a)(A)e^{-\frac{i\pi n}{A}}(2\cos(\frac{2\pi n}{A})(\frac{\epsilon\tilde{\triangle l}(n)}{a})\right.$$
$$\left.+(4\pi iL)(\frac{\hbar\epsilon^2}{a})\tilde{\triangle l}(n)e^{-\frac{i\pi n}{A}}\cos(\frac{2\pi n}{A})\right)|\vec{\triangle l}\rangle$$
$$(120)$$

Using $A = \frac{2\pi L}{a}$, it is easy to see that L.H.S. of (120) is exactly zero![14] Notice that the form of $f(m) = \frac{\tan(\frac{m\pi}{A})}{\frac{\pi}{A}}$ is crucial for the above result. Whence for long-wavelength modes (our assumption on $N_0$ is that it is much smaller then A) of the scalar field, $|\Psi^+\rangle$ is a suitable approximant to Fock vacuum (for $\hat{a}_n^{poly}$ with $n > 0$) provided that all the terms that we have neglected above are really error-terms. That is, they are bounded and tend to zero in the continuum limit.
One can do exactly analogous analysis for the right-moving modes ($\hat{a}_{(-)n}^{poly}$) and obtain a corresponding approximant to the Fock vacuum $|\Psi^-\rangle$. The error-terms which arise in the (+)-sector computation, are exactly the same error-terms that arise in the (-)-sector computation. This implies that

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[14] From appendix-A $\Psi^+\rangle$ is normalizable. It is easy to see that $||\Psi^+|| > 1$. Hence the vanishing of R.H.S. of (120) establishes vanishing of (105) to leading order.

$|\Psi^+\rangle \otimes |\Psi^-\rangle$ is the discrete approximant to the Fock vacuum for all the long wavelength modes of the scalar field.

## 10.5 The commutation relation condition

In this section we show that:

$$
\begin{aligned}
\frac{\langle\Psi^+| \ [\hat{a}_n^{\dagger\ poly},\hat{a}_m^{poly}] \ |\Psi^+\rangle}{||\Psi^+||^2} &= -i\hbar\{a_n^*, a_m\} \ + \ O(a) \\
&= \frac{\hbar}{\pi L^2}n\delta_{n,m} + O(a)
\end{aligned}
\tag{121}
$$

where we indicate the error-terms by O(a).
Recall that,

$$
e^{iO_{F_I^+}}|\mathbf{s}_0\rangle \ = \ e^{-i\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})}|\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)}\rangle
\tag{122}
$$

where $|\mathbf{s}_0\rangle = |\gamma, \vec{k}, \vec{\triangle l}) =: |\vec{\triangle l}\rangle$ (Recall that we are supressing all the (+)-labels on the charge-networks), and $(\triangle\epsilon^{(I)})_J = (\delta_{I,J} - \delta_{I,J-1})$.
A straightforward computation reveals,

$$
[e^{iO_{F_I^+}}, e^{iO_{F_J^+}}]|\vec{\triangle l}\rangle \ = \ (-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])e^{iO_{F_I^+ + F_J^+}}|\vec{\triangle l}\rangle
\tag{123}
$$

Using (84), (123) it is easy to see that,

$$
[\hat{a}_n^{\dagger\ poly}, \hat{a}_m^{poly}]|\mathbf{s}_0) \ =
$$
$$
-\frac{1}{\epsilon^2 4\pi L^2}\sum_{I,J}e^{-ink_I + imk_J}\left(\left[\ e^{iO_{F_I^+}} - e^{-iO_{F_I^+}},\ e^{iO_{F_J^+}} - e^{-iO_{F_J^+}}\ \right]\right)|\mathbf{s}_0)
$$
$$
= \ -\frac{1}{\epsilon^2 4\pi L^2}\sum_{I,J}e^{-ink_I + imk_J}\left((-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])\right.
$$
$$
\left.\left[e^{iO_{F_I^+ + F_J^+}} + e^{iO_{F_I^+ + F_J^+}} + e^{iO_{F_I^+ + F_J^+}} + e^{iO_{F_I^+ + F_J^+}}\right]\right)|\mathbf{s}_0)
$$

$$\tag{124}$$

Let $\left[e^{iO_{F_I^+ + F_J^+}} + e^{iO_{F_I^+ + F_J^+}} + e^{iO_{F_I^+ + F_J^+}} + e^{iO_{F_I^+ + F_J^+}}\right] =: \ \hat{\mathbf{F}}(I, J)$.
Then (124) can be written as,

$$
[\hat{a}_n^{\dagger\ poly}, \hat{a}_m^{poly}]|\mathbf{s}_0) \ =
$$
$$
\frac{1}{\epsilon^2 4\pi L^2}\sum_{I,J}e^{-ink_I + imk_J}\left((-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])\mathbf{1}\right)|\mathbf{s}_0)
$$
$$
+ \ \frac{1}{\epsilon^2 4\pi L^2}\sum_{I,J}e^{-ink_I + imk_J}\left((-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])(\hat{\mathbf{F}}(I, J) - 1)\right)|\mathbf{s}_0)
$$
$$
= \left((-2i)\frac{1}{\epsilon^2 4\pi L^2}\frac{\hbar\epsilon^2}{2}\sum_I e^{i(n-m)k_I}\left[\exp(\frac{2\pi im}{A}) - \exp(-\frac{2\pi im}{A})\right]\mathbf{1} + O(\epsilon^2)\mathbf{1}\right)|\mathbf{s}_0)
$$
$$
+ \ \frac{1}{\epsilon^2 4\pi L^2}\sum_{I,J}e^{-ink_I + imk_J}\left((-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])(\hat{\mathbf{F}}(I, J) - 1)\right)|\mathbf{s}_0)
$$
$$
= \left(4\frac{\hbar}{8\pi L^2}\sin(\frac{2\pi m}{A})A\delta_{n,m}\mathbf{1} + O(\epsilon^2)\mathbf{1}\right)|\mathbf{s}_0)
$$
$$
+ \frac{1}{\epsilon^2 4\pi L^2}\sum_{I,J}e^{-ink_I + imk_J}\left((-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])(\hat{\mathbf{F}}(I, J) - 1)\right)|\mathbf{s}_0)
$$

$$\tag{125}$$

where $\mathbf{1}$ is the identity operator.

As $m << A$,

$$[\hat{a}_n^{\dagger\ poly}, \hat{a}_m^{poly}]|\mathbf{s}_0) =$$
$$\frac{\hbar}{\pi L^2} n \delta_{n,m} \mathbf{1}|\mathbf{s}_0) + O(a)\mathbf{1}|\mathbf{s}_0)$$
$$+ \frac{1}{\epsilon^2 4\pi L^2} \sum_{I,J} e^{-ink_I + imk_J} \left( (-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])(\hat{\mathbf{F}}(I,J) - 1) \right) |\mathbf{s}_0)$$
$$(126)$$

Whence,

$$\frac{\langle \Psi^+\ [\hat{a}_n^{\dagger\ poly}, \hat{a}_m^{poly}]\ \Psi^+\rangle}{||\Psi^+||^2} =$$
$$\frac{\hbar}{\pi L^2} n \delta_{n,m} + O(a)$$
$$+ \frac{1}{\epsilon^2 4\pi L^2} \sum_{I,J} e^{-ink_I + imk_J}(-2i)\left( \sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])\frac{\langle\Psi^+(\hat{\mathbf{F}}(I,J)-1)\Psi^+\rangle}{||\Psi^+||^2} \right)$$
$$(127)$$

Thus we need to show that
$\frac{1}{\epsilon^2 4\pi L^2} \sum_{I,J} e^{-ink_I + imk_J} \left( (-2i)\sin(\frac{\hbar\epsilon^2}{2}[\delta_{J,I+1} - \delta_{J,I-1}])\frac{\langle\Psi^+(\hat{\mathbf{F}}(I,J)-1)\Psi^+\rangle}{||\Psi^+||^2} \right)$ are
finite and vanish in $a \to 0$ limit. Once again the various terms in the above
expression can be extimated using exactly the same techniques that are used
in appendix-B. The upshot is that, two point functions of low frequency (left-
moving) modes in the state $|\Psi^+\rangle$ approximate the corresponding Fock-space
functions upto arbitrary accuracy.

Exactly analogous computation shows that two point functions of low fre-
quency (right-moving) modes in the state $|\Psi^-\rangle$ are equal to the Fock-space
functions.

The existence of state $|\Psi^+\rangle \otimes \Psi^-\rangle$ and its properties under the action of
long-wavelengh mode operators shows us there is a continuum limit of the
polymer quantized Parametrized field theory, and this is the classical theory
plus fluctuations given by the Fock vacuum.

It is also easy to lift the above constructions to the physical Hilbert space.
Define

$$|\Psi^+\rangle_{phy} := \sum_{\vec{\triangle l}} c(\vec{\triangle l})\eta^+(|\vec{\triangle l}\rangle) \tag{128}$$

Note that $|\Psi^+\rangle_{phy} \in \mathcal{H}_{(0)phy}^{+ss}$.

As $\hat{a}_n^{poly}$ commute with the group averaging map we have,

$$\hat{a}_n^{poly'}|\Psi^+\rangle_{phy} = \sum_{\vec{\triangle l}} c(\vec{\triangle l})\eta^+(\hat{a}_n^{poly}|\vec{\triangle l}\rangle) \tag{129}$$

Using (129) it is straight forward to show that all our results continue to
hold in the physical Hilbert space $\mathcal{H}_{(0)phy}^{+ss}$.

# 11 Discussion and Open Issues

A detailed summary of our results is available in Section 1; the interested reader is invited to peruse that section again for an informed global perspective of our work in this paper. We proceed directly to a discussion of our results in section 11.1 and of avenues for further research in section 11.2

## 11.1 Discussion of Results.

**(i) Role of the quantum constraints**: In the classical theory the constraints ensure *foliation independence* of flat spacetime free scalar field dynamics [16]. This, together with the interpretation of the embedding variables as spacetime coordinates, is responsible for the emergence of a classical *spacetime* picture from the Hamiltonian theory in which explicit spacetime covariance is absent. Our treatment of the quantum constraints via Group Averaging together with the fact that the embedding coordinate operators are well defined and admit a complete set of eigen states, leads us to ascribe exactly the same role to these structures in quantum theory. From section 8.1 it follows that any charge network state $\mathbf{s}_0$ defines a 'quantum slice' (consisting of the set of points $(X^+, X^-) = (k_e^+, k_e^-)$, one point for every edge $e$ of the charge network), that the 1 parameter set of gauge related charge networks generated by the action of a 1 parameter set of gauge transformations on $\mathbf{s}_0$ defines a 1 parameter set of discrete slices (a 'quantum foliation') and that the Group Averaging of the charge network defines a 'quantum spacetime' (consisting of the set of embedding charges for all the gauge related charge networks).

The fact that the set of gauge related discrete slices correspond to a single discrete *spacetime* is extremely non- trivial. Note that our treatment of the quantum constraints ensures that the group of finite gauge tranformations is represented correctly. This is how the classical constraint algebra is encoded in quantum theory. We believe that this faithful encoding of the group of finite gauge transformations is a key component in the emergence of a spacetime picture at the quantum level.

**(ii) Scalar field dynamics on discrete spacetime**: While spacetime discreteness can be seen at the level of physical states in **(i)** above, the arguments of sections 4.2, 8.2 show that the polymer scalar field evolves on this discrete spacetime.

This may be seen as follows. From section 4.2 it follows that only discrete Poincare translations are represented as well defined unitary operators on the superselected physical states of interest. From section 8.2, it follows that classical symplectic structure which emerges from the polymer quantization of the true degrees of freedom of PFT (which are coordinatized by

the Dirac Observables of section 4.1) corresponds to that of a lattice scalar field theory [15]. Using the action of a 1 (integer- valued) parameter family discrete Poincare transformations of section 4.1 on any Dirac observable of section 8.2 yields a 1 (integer- valued) parameter family of "evolving Dirac observables. It is then straightforward to see that these evolving Dirac observables (which encode the dynamics of the true degrees of freedom) evolve on the underlying discrete spacetime. A particularly simple set of Dirac observables for which this can be checked, are those corresponding to equation (83).

**(iii) Fock like behaviour and the Continuum Limit**: As discussed in section 10.1, the polymer state (73) approximates the Fock vacuum at the level of coarse grained 2 point functions. The nature of the approximation is endowed with precision due to the availability of the notion of a continuum limit. We think that two properties of the continuum limit are noteworthy in the context of semiclassical issues in LQG:
(a)The limit has the crucial property of being *independent of* $\hbar$ thus allowing a seperation of the notions of *continuum* and *quantum.*
(b) The limit is *not* a property of a 1 parameter set of ad- hoc triangulations in a *single* quantum theory; rather, it is a property of a 1 parameter family of unitarily inequivalent quantizations where the parameter is anologous to the Barbero- Immirizi parameter in LQG.

**(iv) The absence of ad- hoc triangulations**: Our construction of the physical state space and the action of Dirac observables thereon is free of the "triangulation" type ambiguities which plague LQG. The only ambiguity in our work is in the detailed choice of lattice approximants (83) to the true (scalar field) degrees of freedom. Since this arises only at the level of semiclassical analysis, it is different from potential ambiguities in the very definition of the polymer quantum dynamics. Indeed, the freedom of choice in lattice approximants is akin that encountered in elementary point particle quantum mechanics wherein semiclassical considerations are based on the "$\hat{q}, \hat{p}$" operators rather than more complicated functions of $\hat{q}, \hat{p}$.

## 11.2   Directions for Further Research

**(i) Thiemann Quantization and Spacetime Covariance**: This, to us, is the most interesting issue in that we believe the issue in polymer PFT closely mirrors that in LQG. The issue is as follows. As indicated in **(i)** of section 11.1 above, our treatment of the quantum constraints via Group Av-

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[15]Note that while the original continuum symplectic structure is faithfully represented by the basic *kinematic* operators of the polymer representation on $\mathcal{H}_{kin}$, this need not (and is not) true of the Dirac observables of section 4.1 which are non-trivially constructed from the quantum kinematics.

eraging implemented the correct constraint algebra and obtained spacetime covariance at the quantum level. The constraints (see section 2) are density weight two constraints. An equivalent set of constraints exist at the classical level which are closer to those of classical Hamiltonian gravity. Specifically, the density weight 2 constraint, $C_{diff}$, obtained as the linear combination $H^+ + H^-$ is given by,

$$C_{diff}(x) = \left[ \Pi_+(x)X^{+'}(x) + \Pi_-(x)X^{-'}(x) + \pi_f(x)f'(x) \right]. \quad (130)$$

It generates spatial diffeomorphisms of the Cauchy slice. The density weight one constraint, $C_{ham}$ obtained by rescaling a linear combination,$H^+ - H^-$, of the density weight 2 ones by the square root of the determinant of the induced spatial metric is given by

$$C_{ham}(x) = \frac{1}{\sqrt{X^{+'}(x)X^{-'}(x)}} \left[ \Pi_+(x)X^{+'}(x) - \Pi_-(x)X^{-'}(x) + \frac{1}{2}(\pi_f^2 + f'^2) \right].$$
$$(131)$$

It generates evolution normal to the Cauchy slice. Further, the Poisson algebra generated by $C_{diff}$ and $C_{ham}$ is the Dirac algebra,

$$\begin{aligned}
\{C_{diff}[\vec{N}], \ C_{diff}[\vec{M}]\} &= C_{diff}[\vec{N}, \vec{M}] \\
\{C_{diff}[\vec{N}], \ C_{ham}[M]\} &= C_{ham}[L_{\vec{N}}M] \\
\{C_{ham}[N], \ C_{ham}[M]\} &= C_{diff}[\beta(\vec{N}, M)]
\end{aligned} \qquad (132)$$

wherein the structure function $\beta^a(N, M) := q^{ab}(N\nabla_b M - M\nabla_b N)$ in (132) is defined by the induced spatial metric $q_{ab}$. This is the exact analog of the constraint algebra of gravity. It is easy to see that spatial diffeomorphisms are unitarily implemented in the polymer kinematics developed here, that they can be solved by Group Averaging to yield spatial difffeomorphism invariant distributions and that the physical states constructed by Group Averaging of the density weight 2 constraints are invariant under the (dual) action of finite spatial diffeomorphisms. A key question is: *Can we implement $\hat{C}_{ham}$ as an operator on the space of spatially diffeomorphism invariant distributions in a manner similar to that developed by Thiemmann [10] in LQG ?*. Preliminary work seems to indicate that the answer is in the affirmative, that the definition of $\hat{C}_{ham}$ suffers from regularization ambiguities similar to the LQG ones, that the most straightforward regularizations do not admit the physical states constructed here as solutions and that there is almost certainly a *non- trivial* regularization which does. Clearly there are extremely interesting open issues related to the relation between regularization ambiguities in the definition of $\hat{C}_{ham}$, the implementation of the correct constraint algebra and spacetime covariance at the quantum level. We hope to confirm our preliminary findings and analyse such issues in a

future publication.

**(ii) Semiclassical States**: Within the context of our work in section 10, we would like to see if the polymer state (73) yields coarse grained $n$- point functions for $n > 2$. Distinct from this, our quantization offers a test bed for proposals to define semiclassical states (see for example [24]) and, thus, avenues for more work.

**(iii) Non- compact spatial topology**: Generalization of our work here to the case of planar spacetime topology is an open problem of particular interest because of its relation to the Callen-Giddings-Harvey-Strominger model of black holes [25, 26, 27, 28]. Preliminary work shows that the non-trivial asymptotic boundary conditions can be implemented on a suitably generalised space of states; it would be of interest to compare our preliminary results with the Infinite Tensor Product arena proposed by Thiemann in [29]. Our (preliminary) implementation of the asymptotic boundary conditions ensures that we admit asymptotically boosted families of Cauchy slices; this is distinct from the issue of unitary implementation of Global Lorentz transformations because the former are implemented in the sense of *gauge* transformations of PFT, whereas the latter are symmetries of the classical theory of true degrees of freedom.

**(iv) Local Lorentz Invariance (LLI)**: This is an open issue and our comments here are of a slightly imprecise nature. Recall that, as mentioned in Section 1, the true degrees of freedom of classical PFT are exactly those of flat spacetime free scalar field theory. Note that the compact spatial topology ensures, already at the classical level, that there is no global Lorentz symmetry; there are only Poincare translations. However the classical theory of a free scalar field on the Minkowskian cylinder has local Lorentz invariance. In contrast our quantization yields a field theory on (the light cone) lattice so that LLI is broken. Note however that our choice of quantization splits the degrees of freedom in left and right movers, and, as result there is no anomalous dispersion in the polymer dynamics. Our intuition is that 'long time but low energy' phenomena (such as time of delay) do not magnify the effects of this broken LLI by virtue of the unitary implementation of discrete Poincare translations. Thus we believe that only phenomena localised in spacetime at the lattice scale (i.e. effects of very high energy processes as opposed to accumulated effects of low energy ones over large spacetime regions) would capture this propery of broken LLI in a noticeable way. However an exhaustive effective field theory analysis of our polymer quantization still needs to be performed and constitutes an open problem. Note also that for the case of compact spatial toplogy studied here, the absence of Lorentz boosts implies that there is a unique global inertial rest frame (Indeed, $X^{\pm}$ are coordinates in this rest frame). However for the non-

compact case (see **(iii)** above), our preliminary work indicates that global Lorentz invariance would be broken; again, we do not yet know in which physical processes such a broken invariance would manifest.

# Appendix

## A. Normalizability of the candidate Fock vacuum

In this appendix, we show that our candidate Fock vacuum state has a finite norm. Note that this statement is not at all obvious as one is summing over all possible matter charge configurations and each term in the sum is positive definite. As always, we only look at $|\Psi^+\rangle$, as $||\Psi^+||^2 = ||\Psi^-||^2$.

$$|\Psi^+\rangle = [\sum_{\vec{\triangle l}\in \mathbf{S_1}(\epsilon,a)} + \sum_{\vec{\triangle l}\in \mathbf{S_2}(\epsilon,a)} ]c(\vec{\triangle l})|\vec{\triangle l}\rangle \tag{133}$$

Where we have denoted the charge-network state $|\gamma, \vec{k}, \vec{\triangle l}\rangle$ by $|\vec{\triangle l}\rangle$

$$||\Psi^+||^2 = \sum_{\vec{\triangle l}\in \mathbf{S_1}(\epsilon,a)} |c(\vec{\triangle l})|^2 + \sum_{\vec{\triangle l}\in \mathbf{S_2}(\epsilon,a)} |c(\vec{\triangle l})|^2$$
$$=: ||\Psi_1||^2 + ||\Psi_2||^2 \tag{134}$$

Recall that $\mathbf{S_2}(\epsilon,a) = \{\vec{\triangle l}|\frac{\epsilon^2}{a^2}\sum_I |\triangle l_I|^2 a > \frac{C}{a^\delta}\}$ and $\mathbf{S_1} = S - \mathbf{S_2}$ where S is the set of all possible matter-charge configurations. We choose $\delta > 1$ and C is a positive constant which is independent of $\epsilon$ and $a$.
(None of the results, here or elsewhere in the appendix depend on the choice of C).
So to show that $|\Psi^+\rangle$ has a finite norm, it suffices to show that $||\Psi_2||^2 < \infty$. We will first show that

$$c(\vec{\triangle l}) \leq \exp[-\frac{1}{A}\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\sum_n |\tilde{\triangle}l(n)|^2] \tag{135}$$

and then prove that

$$\sum_{\vec{\triangle l}\in \mathbf{S_2}(\epsilon,a)} \exp[-\frac{1}{A}\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\sum_n |\tilde{\triangle}l(n)|^2] < \infty \tag{136}$$

To prove (135), notice that (for sufficiently large A)

$$f(n) = \frac{\tan(\frac{n\pi}{A})}{\frac{n\pi}{A}} \leq n_{max} = A \tag{137}$$

Whence (135) follows.
Using, $\sqrt{\sum_n a_n^2} \leq \sum_n a_n$ if all $a_n$'s are positive,

$$||\Psi_2|| \leq \sum_{\vec{\triangle l}} c(\vec{\triangle l})$$
$$< \sum_{\vec{\triangle l}\in \mathbf{S_2}} \exp[-\frac{\hbar}{A}\frac{\epsilon^2}{a^2}(4\pi^2 L)\sum_I a(\triangle l_I)^2] \tag{138}$$

Now set $\sum_I (\triangle l_I)^2 = r^2$ and $\frac{\epsilon^2}{a^2} r^2 = R^2$.

As $\frac{\epsilon^2}{a^2} \sum_I (\triangle l_I)^2 > \frac{C}{a^{1+\delta}}$, $R^2$ varies (almost) continuously as $\vec{\triangle l} \rightarrow \vec{\triangle l} + \vec{\triangle l}_{min}$. Whence the number of $vec\triangle l$ which lie between R and R + dR is,

$$D^A \frac{R^{A-1} dR}{(\frac{\epsilon}{a})^A}$$

Here D is a constant of O(1).

Thus

$$\begin{aligned} \sum_{\vec{\triangle l}} c(\vec{\triangle l}) &< \sum_{\vec{\triangle l} \in \mathbf{S}_2} \exp[-\hbar 2\pi a^2 \sum_I \frac{\epsilon^2}{a^2} \triangle l_I^2] \\ &\approx \int_{R_>}^\infty D^A \frac{R^{A-1} dR}{(\frac{\epsilon}{a})^A} \exp[-\hbar(2\pi a^2)R^2] \end{aligned} \tag{139}$$

where $R_> = \frac{C}{a^{1+\delta}}$.

Let $aR = X$. It is easy to see that,

$$\begin{aligned} \sum_{\vec{\triangle l} \in \mathbf{S}_2} c(\vec{\triangle l}) &< (\frac{a}{\epsilon})^A \frac{D^A}{a^A} \int_{X_>}^\infty X^{A-1} dX \exp[-\hbar(2\pi X^2)] \\ &< (\frac{D}{\epsilon})^A \int_{X_>}^\infty \exp[-\hbar(2\pi X^2)] < (\frac{D}{\epsilon})^A \int_{X_>}^\infty 2\,X\,dX\,\exp[-\hbar(2\pi X^2)] \\ &= (\frac{D}{\epsilon})^A (\frac{1}{2\pi\hbar}) \exp[-2\pi\hbar X_>^2] \end{aligned} \tag{140}$$

where $X_>^2 = a^2 R_>^2 = \frac{1}{a^{\delta-1}}$. So finally we get the following bound on $\|\Psi_{II}\|$.

$$\|\Psi_{II}\| \le \sum_{\vec{\triangle l} \in \mathbf{S}_2} c(\vec{\triangle l}) < (\frac{1}{2\pi\hbar}) \exp[-\frac{2\pi\hbar}{a^{\delta-1}} + A\ln(\frac{D}{\epsilon})] \tag{141}$$

As $\delta > 1$ and $\epsilon \sim a^{\triangle}$ with $\triangle > 4$ implies that the R.H.S of above equation is finite and bounded from above $\forall \epsilon$ and $a \ne 0$. This proves normalizability. Infact it is easy to see that R.H.S tends to zero rapidly as $a \rightarrow 0$.

## B. Estimation of sub-leading terms

In this section we derive bounds on the sub-leading terms in (107), and show that these terms vanish in the limit $\epsilon$, a $\to 0$.

Once again recall that (as shown in appendix-C) it suffices to restrict attention to $|\Psi_1\rangle = \sum_{\vec{\triangle l} \in \mathbf{S}_1(\epsilon,a)} c(\vec{\triangle l}) \hat{a}_n^{poly} |\vec{\triangle l}\rangle$.

Our strategy to show that the sub-leading terms vanish in the continuum limit will be as follows.

Notice that, $|\Psi_1\rangle$ can be succintly written in the following form.

$$|\Psi_1\rangle = \exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}]\sum_{j=1}^{N}\sum_{\vec{\triangle l}\in S_1(\epsilon a)} c(\triangle l)F_j(\triangle l)|\vec{\triangle l}\rangle$$

(142)

where $\frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)} = \frac{\sin^2(\frac{m\pi}{A})}{\frac{m\pi}{A}}$ and is independent of I.

In the above equation N is the number of sub-leading O(a) terms in (107). (we showed in the main text that the leading order term in $\Psi_1$ is zero). Whence

$$|\delta\Psi^+\rangle = \frac{|\Psi_1\rangle}{||\Psi^+||}$$

(143)

Thus it is easy to see that,

$$||\delta\Psi^+||^2 \leq \frac{1}{||\Psi^+||^2}\exp[-2\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}]\sum_{\vec{\triangle l}\in S_1(\epsilon a)} c(\triangle l)^2[\sum_j |F_j(\triangle l)|]^2$$

(144)

Thus if we can show that

**(i)** $\exp[-2\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}]$ is bounded [16] and

**(ii)** $|F_j(\vec{\triangle l})|_{\vec{\triangle l}\in S_1} \leq F_{max}| \ \forall \ j$,

Then we get the following upper bound on $||\delta\Psi^+||$.

$$||\delta\Psi^+||^2$$
$$\leq \frac{N}{||\Psi^+||^2}\exp\left[-2\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}\right]|F_{max}|^2\sum_{\vec{\triangle l}\in S_1(\epsilon a)} c(\vec{\triangle l})^2$$
$$< N\exp\left[-2\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{|\tilde{\triangle\epsilon}^{(I)}(m)|^2}{f(m)}\right]|F_{max}|^2$$

(145)

where $\frac{\sum_{\vec{\triangle l}\in S_1(\epsilon a)} c(\vec{\triangle l})^2}{||\Psi||^2} < 1$ has been used, and N is defined in (142).

If we can also show that $F_{max} \to 0$ in the continuum, then (145) would

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[16]This has already been shown in the main text in the equations (109) and (115).

imply that $lim_{a\to 0}||\delta\Psi^+|| = 0$.

Rest of this appendix is devoted to establishing $F_{max}$ and showing that $lim_{a\to 0}F_{max} = 0$.

Let us recap (106).

$$
|\Psi_1\rangle =
\frac{1}{2i\epsilon L}\sum_I \left( e^{ink_I}\sum_{\vec{\triangle l}\in S_1} \right.
$$
$$
\left. \left[ e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})}c(\vec{\triangle l} - \vec{\triangle}\epsilon^{(I)}) \; - \; e^{\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})}c(\vec{\triangle l} + \vec{\triangle}\epsilon^{(I)}) \right] \right)
$$
$$
|\vec{\triangle l}\rangle
$$
$$
(146)
$$

We extract out the sub-leading terms by re-expressing the above equation as,

$$
|\Psi_1\rangle =
$$
$$
\frac{1}{2i\epsilon L}\sum_I \left( e^{ink_I}\sum_{\vec{\triangle l}\in S_1} c(\vec{\triangle l}) \exp[-\frac{\hbar\epsilon^2}{a^2}2\pi L^2 \sum_m \frac{|\tilde{\triangle}\epsilon^{(I)}(m)|^2}{f(m)}] \right.
$$
$$
\left[ (1 - \frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})) + \left( e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})} - (1 - \frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})) \right) \right]
$$
$$
\cdot \left[ (1 - \frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}) + \right.
$$
$$
\left. \left( \exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}] - (1 - \frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}) \right) \right]
$$
$$
- \left[ (1 + \frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})) + \left( e^{+\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})} - (1 + \frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})) \right) \right]
$$
$$
\cdot \left[ (1 + \frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}) + \right.
$$
$$
\left. \left. \left( \exp[+\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}] - (1 + \frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}) \right) \right] \right)
$$
$$
|\vec{\triangle l}\rangle
$$
$$
(147)
$$

It is easy to show that if we write all the sub-leading terms in the form

$$
\exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{|\tilde{\triangle}\epsilon^{(I)}(m)|^2}{f(m)}]\sum_{j=1}^N \sum_{\triangle l} c(\vec{\triangle l})F_j(\vec{\triangle l})|\vec{\triangle l}\rangle \qquad (148)
$$

then, the $F_j$'s are given by,

$$F_1 =$$
$$\frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left[\left(e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})}-(1-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1}))\right)\right]$$

$$F_2 =$$
$$\frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left(-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}\right)$$
$$\cdot\left[e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})}-(1-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1}))\right]$$

$$F_3 =$$
$$\frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left[\exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}]-\right.$$
$$\left.\left(1-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}\right)\right]$$

$$F_4 =$$
$$\frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left[\left(e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})}-(1-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1}))\right)\right.$$
$$\left.\cdot\left(\exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}]-(1-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)})\right)\right]$$

$$F_5 =$$
$$\frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left[(-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1}))\right.$$
$$\left.\cdot\left(\exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}]-(1-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)})\right)\right]$$

$$F_6 =$$
$$\frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left[(-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1}))\right.$$
$$\left.\left(-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}\right)\right]$$

$$(149)$$

There are 6 more terms obtained by replacing $\frac{-i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})$ by $\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})$ and $-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)[...]$ by $\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)[...]$ in the above terms.
We now show that $|F_j| < F_{max} \forall j$.

**Bound on $|F_1|$**

$$F_1(\vec{\triangle l}) = \frac{1}{2i\epsilon L} \sum_I e^{ink_I}[(e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})} - (1 - \frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})))]$$

$$(150)$$

$$\begin{aligned}
|F_1| &\leq \frac{1}{2\epsilon L} \sum_I \sum_{n=2}^{\infty} |\frac{[\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})]^n}{n!}| \\
&< \frac{1}{2\epsilon L} \sum_I (\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1}))^2 e^{\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})} \\
&< \frac{2}{2\epsilon L} \sum_I (\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1}))^2
\end{aligned}$$

$$(151)$$

Where we have used the a simple inequality, $\alpha^2 e^\alpha < 2\alpha^2$ if $\alpha << 1$.
The exponent in our case is $\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})$. That it is arbitrarily small can be understood by estimating $\epsilon^2(\triangle l_I)$ for all $\vec{\triangle l} \in \mathbf{S}_1$.

$$\begin{aligned}
\frac{\epsilon}{a}(\triangle l_I) &< \frac{C}{a^{\frac{1+\delta}{2}}} \\
\Rightarrow \epsilon(\triangle l_I) &< \frac{C}{a^{\frac{\delta-1}{2}}} \\
\Rightarrow \epsilon^2(\triangle l_I) &< \frac{C\epsilon}{a^{\frac{\delta-1}{2}}}
\end{aligned}$$

$$(152)$$

As $\epsilon = C_0 a^{\triangle}$ with $\triangle > 4$ (and $C_0$ being a finite dimensionful constant that we fix once and for all) implies that $\epsilon^2(\triangle l_I) \to 0$ Let us go back to (151). Using a simple inequality $\sum_I (\triangle l_I + \triangle l_{I+1})^2 < 4\sum_I (\triangle l_I)^2$ we get,

$$\begin{aligned}
|F_1| &< \frac{8}{2\epsilon L}(\frac{\hbar\epsilon^2}{2})^2 \sum_I (\triangle l_I)^2 \\
&= \hbar^2 \frac{4\epsilon a^2}{L} \sum_I (\frac{\epsilon^2}{a^2}(\triangle l_I)^2) \\
&< \hbar^2 \frac{4\epsilon a^2}{L} \frac{1}{a^{1+\delta}}
\end{aligned}$$

$$(153)$$

which tends to zero as $\frac{\epsilon}{a^{\delta-1}} = C_0 a^{\triangle+1-\delta}$.

**Bound on $|F_2|$**

$$F_2 = \frac{1}{2i\epsilon L} \sum_I e^{ink_I} \left(-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2) \sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}\right.$$
$$\left.[(e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I+\triangle l_{I+1})} - (1 - \frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})))]\right) \tag{154}$$

Following estimate will be essential for putting bounds on $|F_2|$

$$\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\sum_m \left[\frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}\right] \leq \hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\sqrt{\sum_m |\tilde{\triangle}\epsilon^{(I)}(m)|^2}\sqrt{\sum_m \frac{|\tilde{\triangle}l(m)|^2}{f(m)}} \tag{155}$$

It is easy to show that **(i)** $\sum_m |\tilde{\triangle}\epsilon^{(I)}(m)|^2 = 2a$.
And using $f(m) > \frac{1}{2} \ \forall \ m$, we also have
**(ii)**

$$\frac{\epsilon^2}{a^2}\sum_m \frac{|\tilde{\triangle}l(m)|^2}{f(m)} < 2\frac{\epsilon^2}{a^2}\sum_m |\tilde{\triangle}l(m)|^2$$
$$= 2\frac{\epsilon^2}{a^2}a\sum_J (\triangle l)_J^2 \tag{156}$$
$$< 2\frac{1}{a^\delta}$$

Using (**(i)** and **(ii)** in (155),

$$\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\sum_m \left[\frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}\right] < \hbar\frac{\epsilon}{a}(2\pi L^2)\sqrt{2a}\sqrt{2}\frac{1}{a^{\frac{\delta}{2}}}$$
$$= \hbar(2\pi L^2)\sqrt{2}\frac{\epsilon}{a^{\frac{\delta+1}{2}}} \tag{157}$$

Now it is straightforward to put an upper bound on $|F_2|$,

$$|F_2| < \frac{1}{2\epsilon L}\sum_I |\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}|$$
$$2|\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})|^2$$
$$< \hbar(2\pi L^2)\sqrt{2}\frac{\epsilon}{a^{\frac{\delta+1}{2}}}\frac{1}{2\epsilon L}\sum_I 2|\frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})|^2 \tag{158}$$
$$< \hbar(2\pi L^2)\sqrt{2}\frac{\epsilon}{a^{\frac{\delta+1}{2}}}\frac{1}{2\epsilon L}\hbar^2\epsilon^2 a^2\frac{1}{a^{1+\delta}}$$
$$= \hbar^3(\pi L)\sqrt{2}\epsilon^2\frac{1}{a^{\frac{-3+\delta}{2}}}$$

which clearly converges to zero at the rate $a^{\triangle+3-\delta}$.

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**Bound on $|F_3|$**

$$F_3 = \frac{1}{2i\epsilon L} \sum_I e^{ink_I}$$

$$\left[ \exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2) \sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle \epsilon}^{(I)}(-m)+c.c.]}{f(m)}] - \left(1 - \frac{\hbar\epsilon^2}{a^2}(2\pi L^2) \sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle \epsilon}^{(I)}(-m)+c.c.]}{f(m)}\right) \right]$$

$$(159)$$

Let

$$\alpha_I(\vec{\triangle l}) := \frac{\hbar\epsilon^2}{a^2}(2\pi L^2) \sum_m \frac{[\tilde{\triangle l}(m)\tilde{\triangle \epsilon}^{(I)}(-m) + c.c.]}{f(m)} \tag{160}$$

Applying Cauchy Schwarz,

$$\begin{aligned}
|\alpha_I(\vec{\triangle l})| &< 2\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sqrt{\sum_m |\tilde{\triangle \epsilon}^{(I)}(m)|^2}\sqrt{\sum_m \frac{|\tilde{\triangle l}(m)|^2}{f(m)}} \\
&= 2\frac{\hbar\epsilon}{a}(2\pi L^2)\sqrt{\sum_m |\tilde{\triangle \epsilon}^{(I)}(m)|^2}\sqrt{\frac{\epsilon^2}{a^2}\sum_m \frac{|\tilde{\triangle l}(m)|^2}{f(m)}}
\end{aligned} \tag{161}$$

Notice that

$$|\tilde{\triangle \epsilon}^{(I)}(m)|^2 = 2a \tag{162}$$

and,

$$\begin{aligned}
\frac{\epsilon^2}{a^2}\sum_m \frac{|\tilde{\triangle l}(m)|^2}{f(m)} &< 2\frac{\epsilon^2}{a^2}\sum_m |\tilde{\triangle l}(m)|^2 \\
&= 2a\sum_I \frac{\epsilon^2}{a^2}(\triangle l_I)^2 \\
&< 2a\frac{C}{a^{1+\delta}}
\end{aligned} \tag{163}$$

where in the first line we have used $f(m) > \frac{1}{2} \ \forall \ m$, in the second line, the forumla for Fourier transform and in the third line, we have used the fact that $\vec{\triangle l} \in \mathbf{S}_1(\epsilon, a)$.
Using (162), (163) in (161) we get,

$$\begin{aligned}
|\alpha_I(\vec{\triangle l})| &< 2\frac{\hbar\epsilon}{a}(2\pi L^2)(2a)\frac{\sqrt{C}}{a^{\frac{1+\delta}{2}}} \\
&= (8\pi L^2)\frac{\sqrt{C}\hbar\epsilon}{a^{\frac{1+\delta}{2}}}
\end{aligned} \tag{164}$$

Once again we use the same trick we used in (151) to conclude that for sufficiently small $|\alpha_I(\vec{\triangle l})|$,

$$|e^{-\alpha_I} - (1 - \alpha_I)| < 2|\alpha_I|^2$$

Whence,

$$\begin{aligned}
|F_3| &< \frac{\pi}{\epsilon a}2\ (8\pi L^2)^2\frac{C\hbar^2\epsilon^2}{a^{1+\delta}} \\
&= C\hbar^2\pi(8\pi L^2)^2\frac{\epsilon}{a^{2+\delta}}
\end{aligned} \tag{165}$$

as $2 > \delta > 1$ and $\epsilon \sim a^{\triangle}$ with $\triangle > 4$ implies that $|F_3|$ is bounded from above and tends to zero as $a$ tends to zero.

**Bound on $|F_4|$**

$$F_4 = \frac{1}{2i\epsilon L} \sum_I e^{ink_I} [(e^{-\frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})} - (1 - \frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})))$$
$$(\exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}] - (1 - \frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}))]$$

$$(166)$$

Let $\beta_I = \frac{\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})$, $\omega_I = \frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}]$, then using the identity that we have used many times so far,

$$|e^{if} - (1 - if)| < 2f^2 \ \forall f << 1 \tag{167}$$

we get,

$$|F_4| < \frac{4}{2\epsilon L}\sum_I \alpha_I^2 \beta_I^2 \tag{168}$$

Whence

$$|F_4| < \frac{4}{2\epsilon L}(\alpha_I^2)_{max}(\beta_I^2)_{max} A \tag{169}$$

Once again using $\sum_I(\triangle l_I + \triangle l_{I+1})^2 < 4\sum_I(\triangle l_I)^2$ and the fact that $\frac{\epsilon^2}{a^2}(\sum_I \triangle l_I^2)_{max} = \frac{C}{a^{1+\delta}}$ we have,

$$(\beta_I^2)_{max} < \frac{\hbar^2\epsilon^4}{4}4\frac{a^2}{\epsilon^2}\frac{C}{a^{1+\delta}}$$
$$= C\hbar^2\frac{\epsilon^2}{a^{\delta-1}} \tag{170}$$

and from (164),

$$(\alpha_I^2)_{max} < C\hbar^2(8\pi L^2)^2\frac{\epsilon^2}{a^{1+\delta}} \tag{171}$$

Thus

$$|F_4| < C^2\hbar^4 4\pi(8\pi L^2)^2\frac{\epsilon^3}{a^{2\delta+1}} \tag{172}$$

Once again with $\triangle > 4$, $|F_4|$ tends to zero as $a \to 0$.

**Bound on $|F_5|$**

$$F_5 = \frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left[\left(-\frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})\right)\right.$$
$$\left.\cdot\left(\exp[-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}] - (1 - \frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)})\right)\right)$$

(173)

Using same techniques that we have used for the terms above, it is easy to see that

$$|F_5| < $$
$$2\frac{\pi}{\epsilon a}(\hbar\epsilon^2)(\triangle l_I)_{max}(\alpha_I^2)_{max}$$
$$< \frac{\pi}{\epsilon a}(\hbar\epsilon^2)\frac{1}{\epsilon}\frac{1}{a^{\frac{\delta-1}{2}}}C\hbar^2(8\pi L^2)^2\frac{\epsilon^2}{a^{1+\delta}}$$
$$= C\hbar^2(8\pi L^2)^2\pi\frac{\epsilon^2}{a^{\frac{3}{2}(\delta+1)}}$$
$$= C\hbar^2(8\pi L^2)^2\pi\ C_0\ a^{\left(2\triangle - \frac{3}{2}(\delta+1)\right)}$$

(174)

as $\triangle > 4$, $|F_5| \to 0$.

**Bound on $|F_6|$**

$|F_6|$ is straightforward to estimate in light of the previous computations.

$$F_6 = \frac{1}{2i\epsilon L}\sum_I e^{ink_I}\left[\left(-\frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})\right)\right.$$
$$\left.\left(-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{[\tilde{\triangle}l(m)\tilde{\triangle}\epsilon^{(I)}(-m)+c.c.]}{f(m)}\right)\right]$$

(175)

We supress all the details. Interested reader can easily re-trace the missing steps.

$$|F_6| < \frac{\pi}{\epsilon a}\hbar\epsilon^2(\triangle l_I)_{max}\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)|\alpha_I|_{max}$$
$$< \frac{\pi}{\epsilon a}(\hbar\epsilon^2)\frac{1}{\epsilon}\frac{1}{a^{\frac{\delta-1}{2}}}\sqrt{C}\hbar\ (8\pi L^2)\frac{\epsilon}{a^{\frac{1+\delta}{2}}}$$
$$= \pi\sqrt{C}\hbar^2\ (8\pi L^2)\frac{\epsilon}{a^{1+\delta}}$$

(176)

Once again it is easy to see that with $\epsilon = C_0 a^{\triangle}$, $|F_6| \to 0$.

The remaining 6 terms, that are obtained by replacing $\frac{-i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})$ by $\frac{i\hbar\epsilon^2}{2}(\triangle l_I + \triangle l_{I+1})$ and $-\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)[...]$ by $\frac{\hbar\epsilon^2}{a^2}(2\pi L^2)[...]$ in the above terms can be estimated in exactly analogous ways.
All $|F_j|$'s are dimensionless quantities, and in the limit $a \to 0$ they all tend

to zero. Whence we can assume an existence of $|F_{max}|$ which tends to zero in the continuum. This proves the assertion we stated in the beginning of this appendix.

# C. Proof of $\lim_{a\to 0} ||\Psi_2|| = 0$

Consider a disjoint union of the set of all matter charges into two mutually exclusive subsets, $S_1(\epsilon, a) := \{\vec{\triangle l}|\frac{\epsilon^2}{a^2}\sum_I a(\triangle l_I)^2 < \frac{C}{a^\delta}\}$, $S_2(\epsilon, a) = S(\epsilon, a) - S_1(\epsilon, a)$. (107) can be written in a schematic form as,

$$
\begin{aligned}
\hat{a}_n^{poly}|\Psi^+\rangle &= \frac{1}{2i\epsilon L}\sum_{\vec{\triangle l}\in S(\epsilon,a)}\sum_I e^{ink_I}\left[e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)})\right.\\
&\qquad\qquad \left. - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})\right]|\vec{\triangle l}\rangle\\
&= \left[\sum_{\vec{\triangle l}\in S_1} + \sum_{\vec{\triangle l}\in S_2}\right]\\
&\qquad \sum_I e^{ink_I}\left[e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)})\right.\\
&\qquad\qquad \left. - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon},\vec{\triangle l})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})\right]|\vec{\triangle l}\rangle\\
&= |\Psi_1\rangle + |\Psi_2\rangle
\end{aligned}
\tag{177}
$$

**Lemma:** Show that, $||\Psi_2|| \to 0$ rapidly as $a \to 0$
**Proof:**
Note the following inequality

$$
\begin{aligned}
||\Psi_2||^2 &= \frac{1}{4\epsilon^2 L^2}\sum_{\vec{\triangle l}\in S_2}\sum_{I,J}e^{in(k_I-k_J)}\\
&\quad [e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(I)},\vec{\triangle l})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(I)},\vec{\triangle l})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})]\\
&\quad [e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(J)},\vec{\triangle l})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(J)}) - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(J)},\vec{\triangle l})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})]\\
&\leq \frac{1}{4\epsilon^2 L^2}\sum_{\vec{\triangle l}\in S_2}[\sum_I[e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(I)},\vec{\triangle l})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(I)},\vec{\triangle l})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})]\\
&\qquad [\sum_J[e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(J)},\vec{\triangle l})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(J)}) - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(J)},\vec{\triangle l})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(J)})]]\\
&\leq \frac{1}{4\epsilon^2 L^2}\sum_{\vec{\triangle l}\in S_2}[\sum_I|e^{-\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(I)},\vec{\triangle l})}c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) - e^{+\frac{i\hbar}{2}\alpha(\vec{\triangle\epsilon}^{(I)},\vec{\triangle l})}c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})|]^2\\
&= \frac{1}{4\epsilon^2 L^2}\sum_{\vec{\triangle l}\in S_2}[\sum_I\sqrt{c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)})^2 + c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})^2}]^2
\end{aligned}
\tag{178}
$$

Thus

$$
\begin{aligned}
||\Psi_2|| &\leq \frac{1}{4\epsilon L}\sum_{\vec{\triangle l}\in S_2}\left[\sum_I\sqrt{c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)})^2 + c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})^2}\right]\\
&\leq \frac{1}{4\epsilon L}\sum_{\vec{\triangle l}\in S_2}\sum_I\left[c(\vec{\triangle l} - \vec{\triangle\epsilon}^{(I)}) + c(\vec{\triangle l} + \vec{\triangle\epsilon}^{(I)})\right]
\end{aligned}
\tag{179}
$$

Note however that,

$$
\begin{aligned}
c(\vec{\triangle l} \pm \vec{\triangle \epsilon}^{(I)}) \ &= \ \exp[-\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\sum_m \frac{|\tilde{\triangle l}(m)|^2 + 2\pm(\tilde{\triangle l}(m)\tilde{\triangle \epsilon}^{(I)}(-m)+c.c.)}{f(m)}] \\
&< \ \exp[-\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\frac{1}{A}\sum_m(|\tilde{\triangle l}(m)|^2 \pm (\tilde{\triangle l}(m)\tilde{\triangle \epsilon}^{(I)}(-m)+c.c.)) \\
&= \ \exp[-\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\frac{a}{A}\sum_J((\triangle l_J)^2 \pm (\triangle l_J)(\triangle \epsilon^{(I)})_J) \\
&< \ \exp[-\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\frac{4\pi L}{A^2}\sum_J(\triangle l_J)^2) - \sqrt{\sum_J(\triangle l_J)^2})
\end{aligned}
$$

$$(180)$$

where in the first step we used $f(n) < n_{max} = A \ \forall \ n$. In the second step $\sum_m \tilde{G}(m)\tilde{G}(-m) \ = \ \sum_J a \ G_J^2$ and in the third step $\pm\sum_J(\triangle l_J)(\triangle \epsilon^{(I)})_J \geq -\sqrt{2}\sqrt{\sum_J(\triangle l)_J^2}$ via Cauchy-Schwarz. ($\triangle \epsilon^{(I)}$ is a vector of length $\sqrt{2}$).

Also note that, as for small enough $\epsilon$, a and for all $\vec{\triangle l} \in S_2$ it is easy to see that,

$$
\frac{1}{2}\sum_J(\triangle l)_J^2 > \sqrt{\sum_J(\triangle l)_J^2} \tag{181}
$$

Whence

$$
c(\vec{\triangle l} \pm \vec{\triangle \epsilon}^{(I)}) \ < \ \exp[-\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\frac{4\pi L}{A^2}\sum_J\frac{(\triangle l_J)^2}{2}] \tag{182}
$$

Using this bound in (179) we see that,

$$
||\Psi_2|| \ < \ \frac{1}{4\epsilon L}2A\sum_{\vec{\triangle l}\in S_2}\exp[-\hbar\frac{\epsilon^2}{a^2}(2\pi L^2)\frac{4\pi L}{A^2}\sum_J\frac{(\triangle l_J)^2}{2}] \tag{183}
$$

However as we have shown in appendix-A, the R.H.S vanishes in the limit $\epsilon, a \to \ 0$.

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