Propagation of extended objects across singularity of time dependent orbifold

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In this paper we argue that the compactified Milne space is a promising model of the cosmological singularity. It is shown that extended objects like strings propagate in a well-defined manner across the singularity of the embedding space. Then a proposal for quantization of extended objects in the case of a membrane is given.

1 Introduction

One of the simplest models of the neighborhood of the cosmological singularity, inspired by string/M theory [1], is the compactified Milne space, $\mathcal{M}_C$. It has been used in the cyclic universe scenario [1, 2, 3]. This model seems to be attractive, because it consists of pre-singularity and post-singularity epochs and can be described in terms of *quantum* elementary objects propagating in *classical* spacetime.

Let us consider a two-dimensional spacetime with the line element:

$$ds^2 = -dt^2 + t^2 d\theta^2. \tag{1}$$

We identify the points $\theta \sim \theta + \beta$ for some fixed value of β, so that $\theta \in [0, \beta[$. Generalization of (1) to the $d+1$ dimensional spacetime, which will be denoted by $\mathcal{M}_C$, has the form:

$$ds^2 = -dt^2 + t^2 d\theta^2 + \delta_{kl}\, dx^k dx^l, \tag{2}$$

where $t, x^k \in R,\ \theta \in S^1\ (k = 2, \ldots, d)$.

One term in the metric (2) disappears/appears at $t = 0$, thus the space $\mathcal{M}_C$ may be used to model the big-crunch/big-bang type singularity. Orbifolding S^1 to the segment gives a model of spacetime in the form of two orbifold planes which collide and re-emerge at $t = 0$. Our results apply to both choices of topology of the compact dimension.

The Polyakov action integral for a test p-brane (i.e. 0-brane = particle, 1-brane = string, 2-brane = membrane, . . .) embedded in a fixed background spacetime with metric $g_{\mu\nu}$ reads:

$$S_p = -\frac{1}{2}\mu_p \int d^{p+1}\sigma \sqrt{-\gamma}\, \left[\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - (p-1)\right] \tag{3}$$

where μ_p is mass per unit of p-volume, σ^a are p-brane worldvolume coordinates, γ_{ab} is p-brane worldvolume metric, $\gamma := det[\gamma_{ab}]$, $(X^\mu) \equiv (T, X^k, \Theta) \equiv (T, X^1, \ldots, X^{d-1}, \Theta)$ are embedding functions of p-brane, i.e. $X^\mu = X^\mu(\sigma^a)$, corresponding to $(t, x^1, \ldots, x^{d-1}, \theta)$ directions of $d+1$ dimensional background spacetime. The case of a particle propagating in $\mathcal{M}_C$ is not clear and was studied in [4, 5].

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2 Dynamics of classical string

Propagation of classical string across the singularity of $\mathcal{M}_C$ is the best example of how extended objects can successfully 'cure' spacetime singularities. In what follows we use the *local flatness* of $\mathcal{M}_C$ to solve the dynamics of a string. The well-known string's propagation in Minkowski space is given by:

$$x^\mu(\tau,\sigma) = x^\mu_+(\tau+\sigma) + x^\mu_-(\tau-\sigma), \tag{4}$$

$$\partial_\tau x^\mu \partial_\tau x_\mu + \partial_\sigma x^\mu \partial_\sigma x_\mu = 0,\ \partial_\tau x^\mu \partial_\sigma x_\mu = 0, \tag{5}$$

where x^μ $(\mu = 0, 1, \dots)$ are Minkowski coordinates and $x^\mu_\pm$ are any functions. The equations (5) are just gauge constraints. For *winding* modes $\overline{x}(t,\theta)$ in $\mathcal{M}_C$, where $\overline{x} := (x^2, x^3, \dots, x^d)$ one shows in [6] that the extra conditions hold:

$$x^0 = f(\tau+\sigma) - f(-\tau+\sigma),\ x^1 = g(\tau+\sigma) - g(-\tau+\sigma), \tag{6}$$

and

$$x^k_+(\sigma_+) + x^k_-(\sigma_-) = \sum_n a^k_n(t) \exp\big(\imath \frac{2\pi n}{\beta}\theta\big),\ k > 1. \tag{7}$$

Satisfying the last condition is not straightforward and rests upon the fact that the dynamics is governed by a second order differential equation. Thus it is sufficient to satisfy the condition (7) by specifying x^k, $\partial_t x^k$ on a single Cauchy's line. In this way one rules out one of the variables in (7) and compares functions dependent on just a single variable. This strategy works [6] and leads to the solutions:

$$x^0 = q\sinh(\sigma_+) + q\sinh(\sigma_-),\ x^1 = q\cosh(\sigma_+) - q\cosh(\sigma_-), \tag{8}$$

$$x^k = \sum_n a^k_{n+} \exp\big(\imath \frac{2\pi n}{\beta}\sigma_+\big) + \sum_n a^k_{n-} \exp\big(\imath \frac{2\pi n}{\beta}\sigma_-\big) + c^k_0(\sigma_+ + \sigma_-), \tag{9}$$

where $k > 1$ and $a^k_{n+},\ a^k_{n-},\ c^k_0$ are constants. These solutions should satisfy the gauge conditions (5), which in the case of $\mathcal{M}_C$ read

$$\partial_+ x_k \partial_+ x^k = q^2 = \partial_- x_k \partial_- x^k. \tag{10}$$

Alternatively, the solutions (9) in terms of t and θ have the form

$$\begin{aligned} x^k(t,\theta) &= \sum_n \Big(a^k_{n+} e^{\imath \frac{2\pi n}{\beta} \mathrm{arcsinh}\left(\frac{t}{2q}\right)} + a^k_{n-} e^{-\imath \frac{2\pi n}{\beta} \mathrm{arcsinh}\left(\frac{t}{2q}\right)}\Big) \exp\big(\imath \frac{2\pi n}{\beta}\theta\big) \\ &+ 2c^k_0 \mathrm{arcsinh}\Big(\frac{t}{2q}\Big), \end{aligned} \tag{11}$$

where n denotes n-th excitation. The number of arbitrary constants in (11) can be reduced by the imposition of the gauge condition (10).

One observes that the above solutions are well-defined everywhere and it is reasonable to expect that the same holds for higher dimensional objects like classical membrane. The quantization should not spoil this as it was proven in the case of a string in [7].

3 Canonical quantization of membrane

Total Hamiltonian, H_T, corresponding to the Polyakov action reads (see e.g. [8]):

$$H_T = \int d^p\sigma \mathcal{H}_T\ ,\ \ \mathcal{H}_T := AC + A^i C_i, \tag{12}$$

where $A = A(\sigma^a)$ and $A^i = A^i(\sigma^a)$ are any 'regular' functions, and C and C_i are first-class constraints:

$$C = \Pi_\mu \, \Pi_\nu \, g^{\mu\nu} + \mu_p^2 \, \partial_\sigma X^\mu \partial_\sigma X^\nu g_{\mu\nu} \approx 0 \, , \quad C_i = \Pi_\mu \partial_i X^\mu \approx 0, \; (i = 1, \ldots, p) \tag{13}$$

with Poisson bracket $\{\cdot, \cdot\} := \int d^p\sigma \Big(\frac{\partial \cdot}{\partial X^\mu} \frac{\partial \cdot}{\partial \Pi_\mu} - \frac{\partial \cdot}{\partial \Pi_\mu} \frac{\partial \cdot}{\partial X^\mu} \Big)$. We will consider *uniformly winding* modes of p-branes in $\mathcal{M}_C$, i.e. $\sigma^p = \theta = \Theta$ and $\partial_\theta X^\mu = 0 = \partial_\theta \Pi_\mu$. This reduces number of world-volume coordinates and subsequently number of constraints by one so that it is now equivalent to the dynamics of a $(p-1)$-brane in the d-dimensional 'flat' FRW universe with the metric $ds^2_{red} = T \eta_{\mu\nu} \, dX^\mu dX^\nu$. In the case of membrane it leads to two constraints of the form:

$$C := \frac{1}{2\mu_2\theta_0 T} \Pi_\alpha \Pi_\beta \eta^{\alpha\beta} + \frac{\mu_2\theta_0}{2} \, T \, \partial_a X^\alpha \partial_b X^\beta \eta_{\alpha\beta} \approx 0 \, , \quad C_1 := \partial_\sigma X^\alpha \Pi_\alpha \approx 0, \tag{14}$$

which effectively are constraints of a string in the spacetime with the line element ds^2_{red}. We redefine the constraints (14) and smear them with test functions:

$$L_n^\pm := \int C_\pm(\sigma) \cdot \exp{(in\sigma)} \, d\sigma, \; n \in Z \, , \quad C_\pm := \frac{C \pm C_1}{2} \tag{15}$$

and check that the new constraints satisfy the following Lie algebra:

$$\{L_n^+, L_m^+\} = i(m-n)L^+_{m+n} \, , \quad \{L_n^-, L_m^-\} = i(m-n)L^-_{m+n} \, , \quad \{L_n^+, L_m^-\} = 0, \tag{16}$$

where $(L_n^\pm)^* = L^\pm_{-n}$. Now we define Hilbert space encoding many-field degrees of freedom as in [9]:

$$\mathcal{H} \ni \Psi[\overrightarrow{Y}] := \int \psi(\overrightarrow{Y}, \acute{\overrightarrow{Y}}, \sigma) d\sigma, \; \overrightarrow{Y} := \overrightarrow{Y}(\sigma) \tag{17}$$

such that $\|\Psi\| < \infty$ and $\langle \Psi | \Phi \rangle := \int \overline{\Psi}[\overrightarrow{Y}] \Phi[\overrightarrow{Y}][d\overrightarrow{Y}]$. We define the operators $\hat{L}_n$ as follows:

$$\begin{aligned} \hat{L}_n \Psi[\overrightarrow{Y}] \quad &:= \quad i \int \Big(\frac{\partial \psi}{\partial Y^\mu} e^{in\sigma} \frac{d}{d\sigma} Y^\mu + \frac{\partial \psi}{\partial \acute{Y}^\mu} \frac{d}{d\sigma} [e^{in\sigma} \frac{d}{d\sigma} Y^\mu] \Big) d\sigma \\ &= \quad \int e^{in\sigma} \Big(- i \frac{\partial \psi}{\partial \sigma} + n \frac{\partial \psi}{\partial \acute{Y}^\mu} \acute{Y}^\mu - n\psi \Big) d\sigma \; \in \mathcal{H} \end{aligned} \tag{18}$$

One may check that:

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m}, \quad \langle \hat{L}_n \Psi | \Phi \rangle = \langle \Psi | \hat{L}_n^\dagger \Phi \rangle = \langle \Psi | \hat{L}_{-n} \Phi \rangle, \tag{19}$$

which is a quantum counterpart for each subalgebra contained in the full algebra (16). To construct the representation of the full algebra (16), which consists of two commuting subalgebras, one may use standard techniques, i.e. direct sum or tensor product of the representations of both subalgebras. Now, following the Dirac prescription one solves the quantum constraints, i.e. one looks for such Ψ that:

$$\hat{L}_n \Psi[\overrightarrow{Y}] = 0, \; n \in Z. \tag{20}$$

For $\psi = \psi(\overrightarrow{Y}, \acute{\overrightarrow{Y}})$ the condition (20) reads:

$$\int (\acute{e^{in\sigma}})[-\psi + \frac{\partial \psi}{\partial \acute{Y}^\mu} \acute{Y}^\mu] \, d\sigma = 0, \; n \in Z, \tag{21}$$

which has the solution [9]:

$$\psi = \Big(\sum_i \alpha_i(\overrightarrow{Y}) \prod_\mu |\acute{Y}^\mu|^{\rho_i^\mu} \Big)^{\frac{1}{\rho}} - c, \tag{22}$$

where $\sum_\mu \rho_i^\mu = \rho$. This is an expected result since the measure $\sqrt[\rho]{\prod_\mu |\acute{Y}^\mu|^{\rho^\mu}} d\sigma$ is invariant with respect to σ-diffeomorphisms.

All operators acting on the solutions (22) are observables since they act on gauge invariant states. The whole variety of states includes many subspaces, which we can use to construct representations of observables. An example of such subspace is spanned by:

$$\psi := \alpha_\mu(\overrightarrow{Y})\acute{Y}^\mu \implies \Psi[Y] = \int \alpha_\mu(\overrightarrow{Y}) dY^\mu. \tag{23}$$

We may introduce quantum observables $\widehat{O}_s,\ s = s^\lambda(Y^\mu)\partial_{Y^\lambda}$ such that:

$$\widehat{O}_s\left(\int \alpha_\mu dY^\mu\right) = \int \left(s^\lambda \alpha_{\mu,\lambda} + s^\lambda_{,\mu}\alpha_\lambda\right) dY^\mu \implies [\widehat{O}_s, \widehat{O}_t] = \widehat{O}_{[s,t]}. \tag{24}$$

But what are the fields Y^μ? It seems that one needs to postulate (find?) a relation between Y^μ and $\{X^\mu \times \acute{X}^\mu \times \Pi_\nu\}$. Such a relation was proposed in [9]. Study of this relation would enable to interpret the observables in (24) in physical terms and thus complete the proposed quantization scheme for membrane.

4 Conclusions

It seems that the compactified Milne space, $\mathcal{M}_C$, is suitable for modelling higher dimensional cosmological singularity. We showed that classical propagation of excited string is well-defined and unambiguous. The natural expectation would be: quantization should not spoil it!

We have proposed a quantization procedure for uniformly winding membrane, within which we made some progress, particularly we found non-trivial quantum states. It would be interesting to find some relation of our quantization of membrane with M-theory (in our procedure there is no critical dimensionality). However, our work is a first step toward the full resolution of the cosmological singularity that would require quantization of both spacetime and physical p-branes.

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