Modified Jordan-Brans-Dicke theory with scalar current and the Eddington-Robertson γ-parameter

J. W. Moffat[⋆,†] and V. T. Toth[⋆]

[⋆]Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
[†]Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

The Jordan-Brans-Dicke theory of gravitation, which promotes the gravitational constant to a dynamical scalar field, predicts a value for the Eddington-Robertson post-Newtonian parameter γ that is significantly different from the general relativistic value of unity. This contradicts precision solar system measurements that tightly constrain γ around 1. We consider a modification of the theory, in which the scalar field is sourced explicitly by matter. We find that this leads to a modified expression for the γ-parameter. In particular, a specific choice of the scalar current yields $\gamma = 1$, just as in general relativity, while the weak equivalence principle is also satisfied. This result has important implications for theories that mimic Jordan-Brans-Dicke theory in the post-Newtonian limit in the solar system, including our scalar-tensor-vector modified gravity theory (MOG).

PACS numbers: 04.20.Cv,04.50.Kd,04.80.Cc,98.80.-k

Jordan-Brans-Dicke [1, 2] theory is a theory of gravitation in which the gravitational constant G is replaced with the inverse of a dynamical scalar field ϕ. It can be demonstrated by straightforward derivation that this scalar field is effectively sourced by the curvature of space-time (see, e.g., [3]). There is, however, no scalar current: in the Lagrangian formulation, the variation of matter fields with respect to the scalar field is assumed to be zero.

Jordan-Brans-Dicke theory runs into severe observational constraints within the solar system. Notably, the theory predicts that the value of the post-Newtonian γ-parameter, first introduced by Eddington [4] and Robertson [5] and also Schiff [6], and effectively measuring the amount of spatial curvature produced by unit rest mass, will deviate from the standard general relativistic value of 1[1]. Instead, its value will be $\gamma = (\omega + 1)/(\omega + 2)$ [3], where ω is the dimensionless coupling constant of the dynamical field. Constraints established by precision measurements of the Cassini spacecraft [7] require the uncomfortably large value of $|\omega| > 4 \times 10^4$.

Nonetheless, there is no *a priori* reason to exclude the possibility of a scalar current. A phenomenological matter Lagrangian could be constructed such that it depends explicitly on $G = \phi^{-1}$. The variation of such a Lagrangian with respect to ϕ would be non-zero, introducing a scalar current into the field equations. To demonstrate this, we write the scalar theory Lagrangian as follows:

$$\mathcal{L} = \frac{1}{16\pi}\left[(R - 2\Lambda)\phi + f(\phi, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)\right]\sqrt{-g} + \mathcal{L}_{\rm O.F.}, \tag{1}$$

where R is the Ricci-scalar constructed from the metric $g_{\mu\nu}$, g is the metric determinant, Λ is the cosmological constant, ϕ is a scalar field, f is an arbitrary function, and O.F. stands for terms that represent other fields, which, we assume, depend only on ϕ, not on its derivatives. We set $c = 1$, use the $(+,-,-,-)$ metric signature, and define the Ricci tensor as $R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$, where the Γ are the usual Christoffel-symbols.

The field equations of the theory are the Euler-Lagrange equations corresponding to (1):

$$\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \partial_\kappa\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}_{,\kappa}} + \partial_\kappa\partial_\lambda\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}_{,\kappa\lambda}} = 0, \tag{2}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0, \tag{3}$$

where ∇_μ is the covariant derivative with respect to x^μ. These equations can be recast in the form,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda + \frac{1}{\sqrt{-g}}\frac{1}{\phi}\frac{\partial f\sqrt{-g}}{\partial g^{\mu\nu}} - \frac{\partial R}{\partial g^{\mu\nu}_{,\kappa}}\frac{\partial_\kappa\phi}{\phi} + \frac{2}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\frac{\partial R}{g^{\mu\nu}_{,\kappa\lambda}}\right)\frac{\partial_\kappa\phi}{\phi} + \frac{\partial R}{\partial g^{\mu\nu}_{,\kappa\lambda}}\frac{\partial_\kappa\partial_\lambda\phi}{\phi} = \frac{8\pi}{\phi}T_{\mu\nu}, \tag{4}$$

$$R - 2\Lambda + \frac{\partial f}{\partial\phi} - \nabla_\kappa\frac{\partial f}{\partial(\partial_\kappa\phi)} = 16\pi J, \tag{5}$$

[1] The other Eddington-Robertson parameter, β, is identically 1 in Jordan-Brans-Dicke theory, just as in general relativity.

where $T_{\mu\nu} = -(2/\sqrt{-g})\partial\mathcal{L}_{\rm O.F.}/\partial g^{\mu\nu}$ and $J = -(1/\sqrt{-g})\partial\mathcal{L}_{\rm O.F.}/\partial\phi$. The existence of a non-zero variation of matter fields with respect to ϕ represents a significant generalization of the archetypal scalar field theory of Jordan, Brans and Dicke.

Equation (4) can be rewritten using covariant derivatives, yielding

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda + \frac{1}{\sqrt{-g}}\frac{1}{\phi}\frac{\partial f\sqrt{-g}}{\partial g^{\mu\nu}} - \frac{\partial R}{\partial g^{\mu\nu}_{\kappa\lambda}}\frac{\nabla_\kappa\nabla_\lambda\phi}{\phi} = \frac{8\pi}{\phi}T_{\mu\nu}, \tag{6}$$

Spelled out, the field equations now take the following form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda + w\frac{\partial_\mu\phi\partial_\nu\phi}{\phi} - \frac{1}{2}g_{\mu\nu}\frac{f}{\phi} + (g^{\kappa\lambda}g_{\mu\nu} - \delta^\kappa_\mu\delta^\lambda_\nu)\frac{\nabla_\kappa\nabla_\lambda\phi}{\phi} = \frac{8\pi}{\phi}T_{\mu\nu}, \tag{7}$$

$$R - 2\Lambda + v - 2\partial_\mu w\partial^\mu\phi - 2w\nabla_\mu\nabla^\mu\phi = 16\pi J. \tag{8}$$

where $v = \partial f/\partial\phi$ and $w = \partial f/\partial(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)$. Taking the trace of (7), we obtain

$$-R + 4\Lambda + w\frac{\partial_\mu\phi\partial^\mu\phi}{\phi} - 2\frac{f}{\phi} + 3\frac{\nabla_\mu\nabla^\mu\phi}{\phi} = \frac{8\pi}{\phi}T, \tag{9}$$

allowing us to rewrite (7) and (8) as

$$\begin{aligned} R_{\mu\nu} =& \frac{8\pi}{\phi}\left\{T_{\mu\nu} + \frac{1}{3-2w\phi}\left[\phi J - (1-w\phi)\left(T - \frac{1}{4\pi}\phi\Lambda\right)\right]g_{\mu\nu}\right\} + \frac{1-w\phi}{3-2w\phi}\left(w\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}f\right)g_{\mu\nu} \\ & - \frac{1}{3-2w\phi}\left(\frac{1}{2}v - \partial_\mu w\partial^\mu\phi\right)g_{\mu\nu} + \frac{\nabla_\mu\nabla_\nu\phi}{\phi} - \frac{w\partial_\mu\phi\partial_\nu\phi}{\phi}, \end{aligned} \tag{10}$$

$$\nabla_\mu\nabla^\mu\phi = \frac{2}{3-2w\phi}\left(16\pi T - 4\phi\Lambda + 32\pi\phi J + 4f - 2v\phi - 2w\partial_\mu\phi\partial^\mu\phi + 4\partial_\mu w\partial^\mu\phi\right). \tag{11}$$

For Jordan-Brans-Dicke theory, $f(\phi, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = -\omega\partial_\mu\phi\partial^\mu\phi/\phi$, hence $v = -f/\phi$ and $w = -\omega/\phi$. Therefore, the equations read

$$R_{\mu\nu} = \frac{8\pi}{\phi}\left\{T_{\mu\nu} + \frac{1}{2\omega+3}\left[\phi J - (\omega+1)\left(T - \frac{1}{4\pi}\phi\Lambda\right)\right]g_{\mu\nu}\right\} + \omega\frac{\partial_\mu\phi\partial_\nu\phi}{\phi^2} + \frac{\nabla_\mu\nabla_\nu\phi}{\phi}, \tag{12}$$

$$\nabla_\mu\nabla^\mu\phi = \frac{8\pi}{2\omega+3}\left(T + 2\phi J - \frac{1}{4\pi}\phi\Lambda\right), \tag{13}$$

which, apart from the presence of J, are the equations of Jordan-Brans-Dicke theory in the standard form. To the first post-Newtonian order, terms quadratic in derivatives vanish; the second derivative in (12) can, in turn, be eliminated by a suitable gauge choice (for a thorough derivation, see Appendix A of [8]). In the post-Newtonian metric [9], $T \simeq T_{00}$ and the γ-parameter can be read off as the ratio of the ii and 00 components of (12). In the absence of a cosmological term, $\Lambda = 0$, we get

$$\gamma = \frac{(\omega+1)T - \phi J}{(\omega+2)T + \phi J} \tag{14}$$

If the scalar current vanishes ($J = 0$), we get back the usual post-Newtonian result for Jordan-Brans-Dicke theory:

$$\gamma = \frac{\omega+1}{\omega+2}. \tag{15}$$

This result is frequently cited as a reason for rejecting Jordan-Brans-Dicke theory within the solar system, as precision measurements by the Cassini spacecraft yielding $\gamma - 1 = (2.1 \pm 2.3)\times 10^{-5}$, for instance, are consistent with the theory only if $|\omega| \gtrsim 4\times 10^4$ [7].

However, if a scalar current is present, the situation changes. Specifically, we can choose a scalar current in the form

$$\phi J = -\frac{1}{2}T, \tag{16}$$

which is equivalent to

$$-\phi\frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}_{\rm O.F.}}{\partial\phi} = \frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}_{\rm O.F.}}{\partial g^{\mu\nu}}g^{\mu\nu}. \tag{17}$$

This choice can be made, in part, because J is not a conserved quantity, just as T is not conserved. In this case, equations (12) and (13) read

$$R_{\mu\nu} = \frac{8\pi}{\phi}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} + \frac{1}{4\pi}\frac{\omega+1}{2\omega+3}\phi\Lambda g_{\mu\nu}\right) + \omega\frac{\partial_\mu\phi\partial_\nu\phi}{\phi^2} + \frac{\nabla_\mu\nabla_\nu\phi}{\phi}, \tag{18}$$

$$\nabla_\mu\nabla^\mu\phi = -\frac{2\phi\Lambda}{2\omega+3}. \tag{19}$$

Considering the trace of the bracketed term in Eq. (18), if

$$|\Lambda| \ll \pi\left|\frac{2\omega+3}{\omega+1}\phi^{-1}T\right|, \tag{20}$$

the general relativistic result that is also consistent with solar system data,

$$\gamma \simeq 1, \tag{21}$$

is easily satisfied.

The result (15) has been used as an argument against theories that, within the solar system, yield the same solution as Jordan-Brans-Dicke theory to the first post-Newtonian order. We mention in particular our scalar-tensor-vector (STVG) modified gravity theory (MOG) [10, 11], which, according to an extensive analysis by Deng, et al. [8], shows the same behavior in the solar system as Jordan-Brans-Dicke theory. This problem is avoided by a suitable choice of J yielding (21), as demonstrated above.

Nonetheless, we note that in the case of $J \neq 0$, the theory is no longer a metric theory: material particles carry a scalar charge and no longer move along geodesics. To determine the equations of motion for a test particle, we use a test particle Lagrangian in the form

$$L_{\rm TP} = -m\sqrt{g_{\mu\nu}u^\mu u^\nu} - q\phi, \tag{22}$$

where q is the scalar charge associated with a particle of mass m, moving with four-velocity $u^\mu = dx^\mu/d\tau$ and τ is the proper time along the particle's world line. Integration of (16) over a three-volume encompassing a test particle gives

$$q = -\frac{1}{2}\phi^{-1}m, \tag{23}$$

and $\frac{1}{2}\phi^{-1}m \simeq \frac{1}{2}G_N m$ at the present epoch (G_N is Newton's constant of gravitation.) The equation of motion obtained by varying (22) contains an extra term when compared to the standard geodesic equation of motion:

$$m\left(\frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu}u^\mu u^\nu\right) - qg^{\kappa\lambda}\frac{\partial\phi}{\partial x^\lambda} = 0. \tag{24}$$

Given (23), we obtain

$$m\left(\frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu}u^\mu u^\nu\right) + mg^{\kappa\lambda}\frac{1}{2\phi}\frac{\partial\phi}{\partial x^\lambda} = 0. \tag{25}$$

We observe that m cancels out in the equation of motion, hence the theory satisfies the weak equivalence principle.

Finally, we note that equation (19) can be rewritten in the familiar form

$$(\Box + \mu^2)\phi = 0, \tag{26}$$

with $\Box = \nabla_\nu\nabla^\nu$ and μ given by

$$\mu^2 = \frac{2\Lambda}{2\omega+3}. \tag{27}$$

This last term can be interpreted as the mass μ of the scalar field ϕ. Using $\Lambda \simeq 1.2\times10^{-52}$ m^{-2}, we obtain the mass of an ultralight scalar field, $\mu \simeq 3.9\sqrt{2/(2\omega+3)}\times10^{-69}$ kg.

[1] C. Brans and R. H. Dicke. Mach's Principle and a Relativistic Theory of Gravitation. *Phys. Rev.*, 124(3):925–935, 1962.

[2] P. Jordan. *Schwerkraft und Weltall, Grundlagen der Theoretische Kosmologie.* Vieweg und Sohn, Braunschweig, 1955.
[3] S. Weinberg. *Gravitation and Cosmology.* John Wiley & Sons, 1972.
[4] A. S. Eddington. *The Mathematical Theory of Relativity.* Cambridge University Press, 1957.
[5] H. P. Robertson. Note on the preceding paper: The two body problem in general relativity. *Ann. Math.*, 39(1):101–104, January 1938.
[6] L. I. Schiff. On experimental tests of the general theory of relativity. *Amer. J. Phys.*, 28(4):340–343, 1960.
[7] B. Bertotti, L. Iess, and P. Tortora. A test of general relativity using radio links with the Cassini spacecraft. *Nature (London)* , 425:374–376, September 2003.
[8] X.-M. Deng, Y. Xie, and T.-Y. Huang. Modified scalar-tensor-vector gravity theory and the constraint on its parameters. *Phys. Rev. D* , 79(4):044014–+, February 2009.
[9] C. M. Will. *Theory and Experiment in Gravitational Physics.* Cambridge University Press, 1993.
[10] J. W. Moffat. Scalar-tensor-vector gravity theory. *Journal of Cosmology and Astroparticle Physics*, 2006(03):004, 2006.
[11] J. W. Moffat and V. T. Toth. Fundamental parameter-free solutions in Modified Gravity. *Class. Quant. Grav.*, 26:085002, 2009.