Killing Vector Fields in Three Dimensions: A Method to Solve Massive Gravity Field Equations

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Abstract

Killing vector fields in three dimensions play important role in the construction of the related spacetime geometry. In this work we show that when a three dimensional geometry admits a Killing vector field then the Ricci tensor of the geometry is determined in terms of the Killing vector field and its scalars. In this way we can generate all products and covariant derivatives at any order of the ricci tensor. Using this property we give ways of solving the field equations of Topologically Massive Gravity (TMG) and New Massive Gravity (NMG) introduced recently. In particular when the scalars of the Killing vector field (timelike, spacelike and null cases) are constants then all three dimensional symmetric tensors of the geometry, the ricci and einstein tensors, their covariant derivatives at all orders, their products of all orders are completely determined by the Killing vector field and the metric. Hence the corresponding three dimensional metrics are strong candidates of solving all higher derivative gravitational field equations in three dimensions.

1 Introduction

Einstein gravity in three dimensions is trivial because it contains no dynamics. A theory in three dimensions which dynamics is known as topologically massive gravity (TMG) [1],[2]. The action of TMG is the sum of the standard Einstein-Hilbert and Chern-Simons terms in three dimensions. TMG with cosmological constant is considered recently [3], [4]. This theory violates parity, it is renormalizable but not unitary. Recently a new massive gravity theory (NMG) is introduced. In the framework of perturbation theory NMG is both renormalizable and unitary [5]-[7] (see also [8], [9]). Although higher derivative theories play important role to have a consistent quantum gravity it is however very difficult to obtain classical solutions of these theories. For this purpose there have been several attempts to solve these theories, in particular to find black hole solutions. In these attempts different methods were used. In [10] we used perfect fluid solutions of Einstein theory to solve the TMG field equations. In [11] - [16] by assuming commuting two Killing vector fields Clement gave some solutions of the TMG and NMG field equations. Another method is the principle of symmetric criticality [17], [18], [19]. In [20] algebraic classification of the ricci tensor is used and reviewed all known solutions of the TMG. In this work it has been also emphasized the importance of the Killing vector fields in obtaining the solutions of the TMG field equations. It was shown that (see also [21]) all solutions are locally belong to one the three solutions: Timelike-squashed AdS_3, Spacelike-squashed AdS_3 and AdS pp-waves [22]. In [23] Chow et al used the Kundt metrics to solve TMG field equations and they obtained new solutions of the TMG. Another method is to use the Gödel type metrics [24]-[26] in three dimensions [27]. There are also some other attempts [28]-[30] to solve the field equations of TMG. The importance of the Killing vectors is observed in some of these works but , as far as we see, the crucial point is missed. Killing vectors in three dimensions are not only important in derivation of the exact solutions but they control the whole geometry of spacetime.

In this work we shall introduce a new method for three dimensional gravity theories. The starting point is to assume a one parameter family of isomorphism. This implies existence of a Killing vector field in the spacetime geometry. In general this assumption may simplify and reduce the number the field equations by choosing a suitable coordinate patch so that one coordinate in the line element becomes cyclic. In the case of four and higher dimensional geometries this may give information about some components of the curvature and hence the ricci tensors but it does not provide all components of these tensors. We shall show that this is possible in three dimensions.

If a three dimensional spacetime admits a non-null Killing vector field, writing it in terms of two scalars, its norm and rotation scalar, we show that the metric and the ricci tensors are completely determined in terms of the Killing vector, its scalars and vectors obtained through these scalars (Theorem 1). Physical interpretation of the corresponding spacetime metric is that it solves the Einstein-Maxwell- Dilaton field equations in three dimensions. When the scalars of the Killing vector field become constant then the corresponding ricci tensor becomes relatively simple (Theorem 2). We show that the spacetime metric, in this case, solve the field equations of TMG and NMG. Indeed these metrics (for timelike and spacelike cases) solve not only these field equations, but they solve all higher (curvature) derivative theories at any order. When the Killing vector field is null we have similar result (Theorem 3). Without referring to the metric tensor we construct the Einstein tensor when the spacetime geometry admits a null vector. The corresponding metric solves Einstein field equations with a null fluid distribution and an exotic scalar field. When the scalar of the null Killing vector field become constant the Einstein tensor becomes more simpler (Theorem 4) and the metric solves the field equations of TMG and NMG. Both for non-null and null cases when the scalars of the Killing vector fields become constant then the metric solves not only the field equations of TMG and NMG bu solves also all higher derivative gravitational filed equations in three dimensions .

In Section 2 we give the ricci and metric tensors in terms of the non-null Killing vector field and its scalars. In particular if the scalars are constants the ricci tensor is expressed in terms of the Killing vector field and he Gaussian curvature of the two dimensional background space. In Section 3 we show that when the Killing vector field is null then the ricci tensor is calculated in terms of the null Killing vector filed and its scalar. In Section 4 we investigate the the case when the scalar is a constant. In Sections 5 and 6, by using the ricci tensors found in the previous Sections, we solve the field equations of the TMG and NMG respectively.

2 Non-Null Killing Vectors in Three Dimensions

In this section we shall construct the metric and Einstein tensors in terms of scalar functions of non-null Killing vector. Our conventions in this paper are similar to the convention of Hawking-Ellis [31].

Let u^μ be a non-null Killing vector field satisfying the Killing equation

$$u_{\mu\,;\,\nu} + u_{\nu\,;\,\mu} = 0 \tag{1}$$

with $u_\alpha\, u^\alpha = \alpha \neq 0$. One can show that

$$u^\alpha\, u_{\alpha\,;\,\mu} = \frac{1}{2}\,\alpha_{,\mu}, \quad u^\alpha\, u_{\mu\,;\,\alpha} = -\frac{1}{2}\,\alpha_{,\mu} \tag{2}$$

The norm function α is in general not a constant. It is possible to write the Killing equation as

$$u_{\mu\,;\,\nu} = \frac{1}{2}\,(u_{\mu\,;\,\nu} - u_{\nu\,;\,\mu}) = \eta_{\mu\nu\alpha}\, v^\alpha \tag{3}$$

where v^α is any vector field and $\eta_{\mu\nu\alpha}$ is the Levi-Civita alternating symbol. Using (2) we find that

$$v^\rho = w u^\rho + \eta^{\rho\mu\sigma}\,\phi_{,\mu}\, u_\sigma \tag{4}$$

where $w = \frac{u_\mu v^\mu}{\alpha}$. Using (4) in (3) we get

$$u_{\mu\,;\,\nu} = w\,\eta_{\mu\nu\alpha}\, u^\alpha + u_\mu\,\phi_{,\nu} - u_\nu\,\phi_{,\mu} \tag{5}$$

where $\phi = \frac{1}{2}\ln\alpha$. The functions w and ϕ specify the Killing vector field and they are assumed to be linearly independent. Differentiating (5) with respect to x^σ and anti-symmetrizing both sides with respect ν and σ we get

$$\begin{aligned} u_{\mu\,;\,\nu\,\sigma} - u_{\mu\,;\,\sigma\,\nu} \;=\;& w_{,\sigma}\,\eta_{\mu\nu\alpha}\, u^\alpha - w_{,\nu}\,\eta_{\mu\sigma\alpha}\, u^\alpha - w\,\eta_{\mu\nu\alpha}\,\phi^\alpha\, u_{,\sigma} + w\,\eta_{\mu\sigma\alpha}\,\phi^\alpha\, u_\nu \\ & -2\,\eta_{\nu\sigma\alpha}\, u^\alpha\,\phi_{,\mu} - u_\nu\,\phi_{,\sigma}\,\phi_{,\mu} + u_\sigma\,\phi_{,\mu}\,\phi_{,\nu} - u_\nu\,\phi_{;\mu\sigma} + u_\sigma\,\phi_{;\mu\nu} \\ & +w^2\,(g_{\mu\nu}\, u_\sigma - g_{\mu\sigma}\, u_\nu) \end{aligned} \tag{6}$$

Using the identities

$$w_{,\sigma}\,\eta_{\mu\nu\alpha}\,u^\alpha + w_{,\mu}\,\eta_{\nu\sigma\alpha}\,u^\alpha + w_{,\nu}\,\eta_{\sigma\mu\alpha}\,u^\alpha = \eta_{\sigma\mu\nu}\,u^\alpha\,\phi_{,\alpha} = 0, \tag{7}$$
$$\eta_{\mu\nu\alpha}\,\phi^\alpha\,u_{,\sigma} + \eta_{\nu\sigma\alpha}\,\phi^\alpha\,u_{,\mu} + \eta_{\sigma\mu\alpha}\,\phi^\alpha\,u_{,\nu} = \eta_{\sigma\mu\nu}\,\phi^\alpha\,u_{,\alpha} = 0 \tag{8}$$

then Eq. (6) reduces to

$$\begin{aligned} u_{\mu\,;\,\nu\,\sigma} - u_{\mu\,;\,\sigma\,\nu} &= w\,\eta_{\nu\sigma\alpha}\,\phi^\alpha\,u_\mu - \eta_{\nu\sigma\alpha}\,u^\alpha\,y_\mu + \phi_{,\mu}\,(u_\sigma\,\phi_{,\nu} - u_\nu\,\phi_\sigma) \\ &\quad + w^2\,(g_{\mu\nu}\,u_\sigma - g_{\mu\sigma}\,u_\nu) \end{aligned} \tag{9}$$

where

$$y_\mu = w_\mu + 2w\,\phi_\mu \tag{10}$$

Using the ricci identity $u_{\mu\,;\,\nu\,\sigma} - u_{\mu\,;\,\sigma\,\nu} = R^\rho{}_{\mu\nu\sigma}\,u_\rho$ in (9) we get

$$\begin{aligned} R^\rho{}_{\mu\nu\sigma}\,u_\rho &= w\,\eta_{\nu\sigma\alpha}\,\phi^\alpha\,u_\mu - \eta_{\nu\sigma\alpha}\,u^\alpha\,y_\mu + \phi_{,\mu}\,(u_\sigma\,\phi_{,\nu} - u_\nu\,\phi_\sigma) \\ &\quad + w^2\,(g_{\mu\nu}\,u_\sigma - g_{\mu\sigma}\,u_\nu) \end{aligned} \tag{11}$$

and hence we obtain

$$R^\rho{}_\nu\,u_\rho = -\zeta_\nu - (\nabla^2\,\phi + 2w^2)\,u_\nu \tag{12}$$

where

$$\zeta_\mu = \eta_{\mu\alpha\beta}\,y^\alpha\,u^\beta \tag{13}$$

In three dimensions the Weyl tensor vanishes and hence curvature tensor is expressed solely in terms of the ricci tensor

$$R_{\rho\mu\nu\alpha} = -R_{\mu\nu}\,g_{\rho\alpha} + R_{\rho\nu}\,g_{\mu\alpha} + R_{\mu\alpha}\,g_{\rho\nu} - R_{\rho\alpha}\,g_{\mu\nu} + \frac{R}{2}\,(g_{\rho\alpha}\,g_{\mu\nu} - g_{\rho\nu}\,g_{\mu\alpha}) \tag{14}$$

Combining equations (11) and (14) we obtain

$$\bar{R}_{\mu\nu}\,u_\sigma - \bar{R}_{\mu\sigma}\,u_\nu + \zeta_\sigma\,g_{\mu\nu} - \zeta_\nu\,g_{\mu\sigma} = -\eta_{\nu\sigma\alpha}\,u^\alpha\,y_\mu + w\,\eta_{\nu\sigma\alpha}\,\phi^\alpha\,u_\mu, \tag{15}$$

where

$$\bar{R}_{\mu\nu} = -R_{\mu\nu} + (P + \tfrac{R}{2} - w^2)\, g_{\mu\nu} - \phi_{;\mu\nu} - \phi_{,\mu}\,\phi_{,\nu}, \tag{16}$$

$$P = \nabla^2\,\phi + 2\phi_{,\alpha}\,\phi^{\alpha} + 2w^2 \tag{17}$$

In addition to the identities (7) we have also the following identities

$$\begin{aligned} z_{\mu}\,u_{\sigma} - z_{\sigma}\,u_{\nu} &= \alpha\,\eta_{\mu\sigma\rho}\,\phi^{\rho}, & (18)\\ \zeta_{\mu}\,y_{\sigma} - \zeta_{\sigma}\,y_{\nu} &= -(y_{\alpha}\,y^{\alpha})\,\eta_{\mu\sigma\rho}\,u^{\rho} & (19)\end{aligned}$$

Using these identities in (20) we arrive at the following theorem:

Theorem 1. *If a three dimensional spacetime admits a non-null Killing vector field u^{μ} then its metric and the Einstein tensors are respectively given by*

$$\begin{aligned} g_{\mu\nu} &= \frac{1}{\alpha}\,u_{\mu}\,u_{\nu} + \frac{1}{\zeta^{\alpha}\,\zeta_{\alpha}}\,\zeta_{\mu}\,\zeta_{\nu} + \frac{1}{y_{\alpha}y^{\alpha}}\,y_{\mu}\,y_{\nu}, & (20)\\ G_{\mu\nu} &= -\frac{H}{\alpha}\,u_{\mu}\,u_{\nu} + \frac{1}{\alpha}\,[(w\,z_{\mu} - \zeta_{\mu})\,u_{\nu} + (w\,z_{\nu} - \zeta_{\mu})\,u_{\mu}] & (21)\\ &\quad +(\nabla^2\,\phi + w^2)\,g_{\mu\nu} - \phi_{\mu}\,\phi_{,\nu} - \phi_{;\mu\nu}, & (22)\\ H &= 2\nabla^2\phi + 3w^2 + \frac{1}{2}\,R - \phi^{\alpha}\,\phi_{,\alpha}, & (23)\\ z_{\mu} &= \eta_{\mu\alpha\beta}\,\phi^{\alpha}\,u^{\beta} & (24)\end{aligned}$$

where R is the ricci scalar provided that the functions w and ϕ are not constant at the same time.

Remark 1. When u^{μ} is a timelike Killing vector field, $\alpha = -e^{2\phi}$, then it can be shown that the source of the Einstein equations is composed of a charged fluid distribution and a dilaton field ϕ (in Einstein frame). The electromagnetic field tensor $f_{\mu\nu}$ is given by

$$f_{\mu\nu} = w\,\eta_{\mu\nu\alpha}\,u^{\alpha} - u_{\mu}\,\phi_{,\nu} + u_{\nu}\,\phi_{,\mu} \tag{25}$$

so that

$$f^{\mu\nu}{}_{;\mu} = -\zeta^{\nu} + (\nabla^2 \phi - 2w^2)\, u^{\nu} \tag{26}$$

The Maxwell energy momentum tensor is given as

$$\begin{aligned} M_{\mu\nu} &= -\frac{1}{2}\alpha\,(w^2 + \phi_{,\alpha}\,\phi^{\alpha})\, g_{\mu\nu} + (w^2 + \phi_{,\alpha}\,\phi^{\alpha})\, u_{\mu}\, u_{\nu} \\ &\quad +\alpha\,\phi_{,\mu}\,\phi_{,\nu} - w\,(z_{\mu}\, u_{\nu} + z_{\nu}\, u_{\mu}) \end{aligned} \tag{27}$$

Then field equations are

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{4}\, e^{-2\phi}\, M_{\mu\nu} + e^{-2\phi}\,(-2\nabla^2\phi + 4\,\phi_{,\alpha}\,\phi^{\alpha} + e^{-2\phi}\, f^2 - \frac{R}{2})\, u_{\mu}\, u_{\nu} \\ &\quad +(\nabla^2\phi + \frac{1}{4}\, e^{-2\phi}\, f^2) g_{\mu\nu} - e^{-2\phi}\,(J_{\mu}\, u_{\nu} + J_{\nu}\, u_{\mu}) - \phi_{;\mu\nu} \end{aligned} \tag{28}$$

where $J^{\mu} = f^{\alpha\mu}{}_{;\alpha}$ and $f^2 = f_{\alpha\beta}\, f^{\alpha\beta}$.

If w and ϕ are both constants then we get

$$u_{\mu\,;\,\nu} = w\,\eta_{\mu\nu\alpha}\, u^{\alpha} \tag{29}$$

Then following the above procedure we have the following result.

Theorem 2. *If a three dimensional spacetime admits a non-null Killing vector field u^{μ} with w and α are both constants then the Einstein tensor is given as*

$$G_{\mu\nu} = -\frac{1}{\alpha}\,(3w^2 + \frac{1}{2}\, R)\, u_{\mu}\, u_{\nu} + w^2\, g_{\mu\nu} \tag{30}$$

In this case, since the vector ζ_{μ} vanishes in (15), the metric tensor is not obtained.

Remark 2. If u^{μ} is a timelike Killing vector field the above Einstein tensor represents a dust distribution in a spacetime with a positive cosmological constant.

For completeness let us give the spacetime metrics of each case: We shall consider the case α and w are constants.

1. Let $u^{\mu} = -\frac{1}{u_0}\,\delta^{\mu}_{0}$ be the timelike vector field then $u^{\mu}\,u_{\mu} = -1$. Hence $\alpha = -1$ and $u_{\mu} = g_{\mu\alpha}\,u^{\alpha}$, then $g_{00} = -u_0^2, g_{01} = -u_0\,u_1, g_{02} = -u_0\,u_2$. Then the spacetime metric can be taken as ($x^{\mu} = (t, r, z)$)

$$ds^2 = -(u_0\,dt + u_1\,dr + u_2\,dz)^2 + M^2\,dr^2 + 2Ldrdz + N^2\,dz^2 \tag{31}$$

where $M^2 = g_{11} + u_1^2, N^2 = g_{22} + u_2^2, L = g_{12} + u_1\,u_2$. Here u_0 is a constant (due to the Killing equation) and the metric functions M,N, L , u_1 and u_2 depend on the variables r and z. The metric in (31) is of Gödel type which was used in [27]. The Einstein tensor is given in (30) with $\alpha = -1$. The only field equation is

$$u_{2,r} = u_{1,z} + 2w\,\Delta/u_0, \tag{32}$$

where $\Delta = |u_0|\sqrt{M^2\,N^2 - L^2}$. For simplicity consider the case $L = 0$. Then $R = 2K + 2w^2$ where K is the Gaussian curvature of the locally Euclidian two dimensional background space with the line element $ds_2^2 = M^2\,dr^2 + +N^2\,dz^2$. Metric (31) gives the Einstein tensor (30) where the only field equation is given in equation (32) with $\alpha = -1$ and w constant.

2. Let $u^{\mu} = \frac{1}{u_2}\,\delta^{\mu}_{2}$ be the spacelike vector field then $u^{\mu}\,u_{\mu} = 1$. Hence $\alpha = 1$ and $u_{\mu} = g_{\mu\alpha}\,u^{\alpha} = \frac{1}{u_2}\,g_{\mu 2}$, then $g_{22} = u_2^2, g_{12} = u_1\,u_2, g_{02} = u_0\,u_2$. Then spacetime metric can be taken as

$$ds^2 = (u_2\,dz + u_1\,dr + u_0\,dt)^2 + M^2\,dr^2 + 2L\,drdt - N^2\,dt^2, \tag{33}$$

where $M^2 = g_{11} - u_1^2$, $N^2 = u_0^2 - g_{00}, L = g_{01} - u_0\,u_1$, u_2 is a constant (due to the Killing equation) and the metric functions M,N, L, u_1 and u_0 depend on the variables t and r. The Einstein tensor is given in (30) with $\alpha = 1$. The only field equation is

$$u_{0,r} = u_{1,t} + 2\,w\,\Delta/u_2, \tag{34}$$

where $\Delta = |u_2|\,\sqrt{M^2\,N^2 + L^2}$. In both cases the ricci scalar is R is not constant. In this case as well, take $L = 0$ for simplicity we have $R = 2K + 2w^2$ where K is the Gaussian curvature of two dimensional locally minkowskian background space with the line element $ds_2^2 = M^2\,dr^2 - N^2\,dt^2$. Metric (33) gives the Einstein tensor (30) where the only field equation is given in (34) with $\alpha = 1$ and w constant. In Both cases the ricci tensor takes the form

$$
\begin{aligned}
R_{\mu\nu} &= -\frac{1}{\alpha}\,(4w^2 + K)\,u_\mu\,u_\nu + (2w^2 + K)\,g_{\mu\nu}, \qquad (35)\\
G_{\mu\nu} &= -\frac{1}{\alpha}\,(4w^2 + K)\,u_\mu\,u_\nu + w^2\,g_{\mu\nu}. \qquad (36)
\end{aligned}
$$

It is possible to pass from timelike case to spacelike case by complex transformation [13]. When the background two dimensional spaces are de-sitter od anti de-sitter, in each case, these solutions are known as timelike squashed AdS_3 and spacelike squashed AdS_3 [20],[21], [11].

When R is a constant, the Cotton tensor and the Laplacian of the ricci tensor take simple forms (for both cases). They will used later.

$$
\begin{aligned}
C_{\mu\nu} &= -w\,(4w^2 + K)[g_{\mu\nu} - \frac{3}{\alpha}\,u_\mu\,u_\nu], \qquad (37)\\
\nabla^2 R_{\mu\nu} &= -2w^2\,(4w^2 + K)[g_{\mu\nu} - \frac{3}{\alpha}\,u_\mu\,u_\nu] \qquad (38)
\end{aligned}
$$

3 Three Dimensional Spacetime Admitting a Null Killing Vector Field

In this section we shall construct the Einstein tensor with respect to the scalar functions of a null Killing vector field.

Let ξ_μ be Killing Vector field. It satisfies the Killing equation

$$
\xi_{\mu;\,\nu} = \frac{1}{2}(\xi_{\mu;\,\nu} - \xi_{\nu;\,\mu}) = \eta_{\mu\nu\alpha}\,v^\alpha \qquad (39)
$$

where $\eta_{\mu\nu\alpha} = \sqrt{-det(g)}\,\epsilon_{\mu\nu\alpha}$ and v^μ is an arbitrary vector field and a semi-colon defines the covariant derivative with respect to the metric $g_{\mu\nu}$. Here the $\epsilon_{\mu\nu\alpha}$ is the three dimensional Levi-Civita alternating symbol. Since $\xi^\alpha\,\xi_{\alpha;\beta} = 0$ and $\xi^\alpha\,\xi_{\beta;\alpha} = 0$, then $v^\mu = w\xi^\mu$. Hence (39) becomes

$$\xi_{\mu;\,\nu} = w\,\eta_{\mu\nu\alpha}\,\xi^{\alpha} \tag{40}$$

where w is an arbitrary function. Taking one more covariant derivative of (40) we find that

$$\xi_{\mu;\,\nu\alpha} = w_{,\alpha}\,\eta_{\mu\nu\rho}\,\xi^{\rho} - w^2\,(g_{\mu\alpha}\,\xi_{\nu} - g_{\nu\alpha}\,\xi_{\mu}) \tag{41}$$

Using this equation and the Ricci identity we get

$$\begin{aligned}
\xi_{\mu;\,\nu\alpha} - \xi_{\mu;\,\alpha\nu} &= w_{,\alpha}\,\eta_{\mu\nu\rho}\,\xi^{\rho} - w_{,\nu}\,\eta_{\mu\alpha\rho}\,\xi^{\rho} - w^2\,(g_{\mu\alpha}\,\xi_{\nu} - g_{\nu\mu}\,\xi_{\alpha}) \\
&= R^{\rho}{}_{\mu\nu\alpha}\,\xi_{\rho}.
\end{aligned} \tag{42}$$

On the other hand, since the Weyl tensor vanishes in three dimensions the Riemann tensor is expressed totally in terms of the ricci tensor as expressed in (14). Using (42) and (14) we find

$$Q_{\mu\nu}\,\xi_{\alpha} - Q_{\mu\alpha}\,\xi_{\nu} - \zeta_{\nu}\,g_{\mu\alpha} + \zeta_{\alpha}\,g_{\mu\nu} = w_{,\alpha}\,\eta_{\mu\nu\rho}\,\xi^{\rho} - w_{,\nu}\,\eta_{\mu\alpha\rho}\,\xi^{\rho} \tag{43}$$

where

$$\zeta_{\mu} = \eta_{\mu\alpha\beta}\,w^{,\alpha}\,\xi^{\beta}, \tag{44}$$

$$Q_{\mu\nu} = -R_{\mu\nu} + (w^2 + \frac{R}{2})\,g_{\mu\nu}. \tag{45}$$

Here R is the ricci scalar.

Let a vector X_{μ} be defined as

$$X_{\mu} = \eta_{\mu\alpha\beta}\,F^{,\alpha}\,\xi^{\beta} \tag{46}$$

where $F^{,\mu} = g^{\mu\alpha}\,F_{,\alpha}$ and F is any function independent of t. It is easy to show that $\xi^{\mu}\,X_{\mu} = X^{\mu}\,X_{\mu} = 0$. Since the orthogonal null vectors can only be parallel then we have

$$X_{\mu} = \sigma\xi_{\mu}, \tag{47}$$

Hence

$$\zeta_{\mu} = \psi\xi_{\mu}, \tag{48}$$

and

$$\xi_\mu \, w_\nu - \xi_\nu \, w_\mu = \psi \, \eta_{\mu\nu\alpha} \, \xi^\alpha, \quad \psi \neq 0 \tag{49}$$

which leads also

$$w^{,\alpha} \, w_{,\alpha} = \psi^2. \tag{50}$$

Using (49) in (43) we obtain

$$Q_{\mu\nu} + \psi \, g_{\mu\nu} - \frac{1}{\psi} \, w_{,u} \, w_{,\nu} + \rho \xi_\mu \, \xi_\nu = 0. \tag{51}$$

Hence

$$R_{\mu\nu} = \rho \, \xi_\mu \, \xi_\nu - (\psi + 2w^2) \, g_{\mu\nu} - \frac{1}{\psi} \, w_{,u} \, w_{,\nu} \tag{52}$$

where ρ is a function to be determined and the ricci scalar is given by

$$R = -6 \, w^2 - 4\psi. \tag{53}$$

Then we have the following theorem:

Theorem 3. *Let ξ^μ be a null Killing vector field of a three dimensional spacetime, then the corresponding Einstein tensor is given by*

$$G_{\mu\nu} = \rho \, \xi_\mu \, \xi_\nu - \frac{1}{\psi} \, w_{,u} \, w_{,\nu} + (w^2 + \psi) \, g_{\mu\nu} \tag{54}$$

where ρ is a function to be determined. Furthermore the scalar field satisfies the following partial differential equation

$$\nabla_\mu \, (\frac{1}{\psi} \, w^{,\mu}) = 2w. \tag{55}$$

It is quite interesting that by imposing a null Killing vector in a three dimensional spacetime geometry we obtain the form of the Einstein tensor without knowing the metric tensor explicitly, except the function ρ. The source of the Einstein field equations is composed of a null fluid and a scalar field. The scalar field satisfies a nonlinear differential equation (55). From the Einstein equations the energy momentum tensor $T_{\mu\nu}$ of the source satisfies $T_{\mu\nu}\,\xi^{\mu}\,\xi^{\nu}=0$. For any timelike unit vector field u^{μ} we have $T_{\mu\nu}\,u^{\mu}\,u^{\nu}=\rho\,(\xi^{\mu}\,u_{\mu})^2-\frac{1}{\psi}\,(w_{,\mu}\,u^{\mu})^2+(\psi+w^2)$. Choosing $\psi=-\sqrt{w_{\mu}\,w^{\mu}}$ then the sign of the energy depends on the sign of the function ρ.

As a summary we can conclude in this section as: If ξ_{μ} is a null Killing vector field of a three dimensional spacetime geometry then the corresponding metric solves the Einstein-null fluid-scalar field equations.The only field equations are (40) and the scalar field equation (55).

4 Null Case Without Scalar Field

In the previous section we considered the case $\psi \neq 0$. Vanishing of ψ is a distinct case. Due to this reason we consider it in a separate section. We have the following result:

Theorem 4.

When $\psi=0$ in Theorem 3 then w becomes a constant and the ricci, Einstein tensors take simple forms

$$\begin{aligned} R_{\mu\nu} &= \rho\,\xi_{\mu}\,\xi_{\nu}-2w^2\,g_{\mu\nu}, && (56)\\ G_{\mu\nu} &= \rho\,\xi_{\mu}\,\xi_{\nu}+w^2\,g_{\mu\nu}, && (57)\end{aligned}$$

with $R=-6w^2$.

For explicit solutions we shall adopt some simplifying assumptions.

(4.A). As an example let us assume that $\xi^{\mu}=\delta^{\mu}_{0}$. Then $\xi_{\mu}=g_{\mu\alpha}\,\xi^{\alpha}=g_{\mu 0}$. Let $\xi_0=0$, hence we get $g_{00}=0, g_{01}=\xi_1, g_{02}=\xi_2$. The corresponding spacetime metric can be given as $(x^{\mu}=(t,r,z))$

$$ds^2=2(\xi_{\mu}\,dx^{\mu})\,dt+m^2\,dr^2+n^2dz^2+2\,\ell\,dr dz \qquad (58)$$

where ξ_1,ξ_2, m, n and l are functions of r and z. Let $\ell=0$, then $\det(g)\equiv -\Delta^2=-(\xi_1^2\,n^2+\xi_2^2\,m^2)$, $\psi=\frac{\xi_1\,w_{,z}-\xi_2\,w_{,r}}{\Delta}$, and

$$g_{\mu\nu}=\begin{pmatrix} 0 & \xi_1 & \xi_2\\ \xi_1 & m^2 & 0\\ \xi_2 & 0 & n^2\end{pmatrix}, \quad g^{\mu\nu}=\frac{1}{\Delta^2}\begin{pmatrix} -m^2\,n^2 & \xi_1\,n^2 & \xi_2\,m^2\\ \xi_1\,n^2 & \xi_2^2 & -\xi_1\,\xi_2\\ \xi_2\,m^2 & -\xi_1\,\xi_2 & \xi_1^2\end{pmatrix} \qquad (59)$$

The equation (40) reduces to

$$\xi_{1,z} - \xi_{2,r} = 2w\Delta \tag{60}$$

The function ρ can be calculated in terms of the metric functions ξ_1, ξ_2, m and n. By using a transformation $r = f(R, Z)$ and $z = g(R, Z)$ where f and g are any differentiable functions, without loosing any generality, we can let one of the functions ξ_1 or ξ_2 zero . Here in this work we will take $\xi_1 = 0$ and $\xi_2 = q$. The with these simplifications $\Delta = q\,m_{,}$, $q_{,r} = -2w\,q\,m$, w is a nonzero constant and

$$ds^2 = 2\,q\,dz\,dt + m^2\,dr^2 + n^2 dz^2. \tag{61}$$

With such a choice we have

$$\rho = \frac{1}{m^3\,q^3}\,[2wqm^2\,n\,n_{,r} + nqm_{,r}n_{,r} - m^2 qm_{,zz} + m^2 m_{,z}q_{,z} - mnqn_{,rr} - mq(n_{,r})^2] \tag{62}$$

If $w = 0$ the Killing vector becomes hypersurface orthogonal and q becomes a constant. Then

$$\rho = \frac{1}{m^3\,q^2}\,[nqm_{,r}n_{,r} - m^2 m_{,zz} + m^2 m_{,z}q_{,z} - mnn_{,rr} - m(n_{,r})^2]. \tag{63}$$

(4.B). Letting $q = e^y$ and $n = m$ we get $m = -\frac{1}{2w}\,y_{,r}$, $w \neq 0$. Here y is a function of r and z. Then

$$\rho = \frac{1}{e^{2y}\,y_{,r}}\,[-(y_{,zz} + y_{,rr}) + \frac{1}{2}\,((y_{,r})^2 + (y_{,z})^2)]_{,r} \tag{64}$$

We get a nonlinear partial differential equation for y when $\rho = \rho_0$ is a constant.

$$-(y_{,zz} + y_{,rr}) + \frac{1}{2}\,((y_{,r})^2 + (y_{,z})^2) = \frac{1}{2}\,\rho_0\,e^{2y} + \rho_1 \tag{65}$$

where ρ_1 is an integration constant. An exact solution of the above equation is given as $y = -\ln[A + B(r - r_0)^2]$ where $\rho_1 = 0$ and $\rho_0 = 4AB$. After performing some scale transformations the metric becomes

$$ds^2 = \frac{dt\,dz}{\rho_0 + w\,r^2} + \frac{r^2}{[\rho_0 + w\,r^2]^2}\,(dr^2 + dz^2) \tag{66}$$

This metric solves Einstein field equations (57) with null fluid whose energy density $\rho = \rho_0$ is a constant. All curvature invariants are constants. For instance $R = -6w^2$, $R^{\alpha\beta}\, R_{\alpha\beta} = 12 w^4$. Hence the spacetime is not asymptotically flat. When $\rho_0 = 0$ the metric becomes anti-de Sitter in three dimensions.

(4.C). When $w = 0$, the Killing vector becomes covariantly constant. By taking $m = n$ the function ρ becomes

$$\rho = -\frac{1}{mq}\,(m_{zz} + m_{rr}) \tag{67}$$

and the metric takes the form

$$ds^2 = 2\,q\,dz\,dt + m^2\,(dr^2 + dz^2). \tag{68}$$

Einstein tensor becomes

$$G_{\mu\nu} = \rho \xi_\mu\,\xi_\nu \tag{69}$$

All scalars constructed from the ricci tensor vanish. We will not consider this case in the sequel.

5 Topologically Massive Gravity Theory

Topologically Massive Gravity (TMG) equations found by Deser, Jackiw and Templeton (DJT) [1]. Recently [3], [4] this theory was extended to the case with a cosmological constant. They are given as follows.

$$G^\mu_\nu + \frac{1}{\mu}\,C^\mu{}_\nu = \lambda\,\delta^\mu_\nu. \tag{70}$$

Here $G_{\mu\nu}$ and $R_{\mu\nu}$ are the Einstein and Ricci tensors respectively and C^μ_ν is the Cotton tensor which is given by

$$C^\mu{}_\nu = \eta^{\mu\beta\alpha}\,(R_{\nu\beta} - \frac{1}{4}\,R\,g_{\nu\beta})_{;\alpha}. \tag{71}$$

The constants μ and λ are respectively the DJT parameter and the cosmological constant. Solutions of this theory were studied by several authors [10]-[27].

1. When the spacetime admits a non-null Killing vector field u^{μ}, assuming that the ricci scalar R is a constant (or the Gaussian curvature, $R = 2K + 2w^2$, of the two spaces orthogonal to the Killing directions is a constant) and using the Eq. (30) we get

$$C_{\mu\nu} = -w\,(3w^2 + \frac{R}{2})[g_{\mu\nu} - \frac{3}{\alpha}\,u_{\mu}\,u_{\nu}]. \tag{72}$$

Using the TMG field equations we obtain

$$\mu = 3w, \quad \lambda = -\frac{R}{6}. \tag{73}$$

which is valid for both spacelike and timelike cases. When $w = 0$ then Killing vector becomes hypersurface orthogonal and the Cotton tensor vanishes. In this case TMG reduces to the Einstein theory, i.e., vacuum spacetime with a cosmological constant [32]

2. If the spacetime admits a null Killing vector the Einstein tensor takes the form (57). Using the Killing equation (40) and the Einstein tensor (57) we get

$$C^{\mu}{}_{\nu} = (w\rho - \sigma)\,\xi_{\mu}\,\xi_{\nu}, \quad \sigma = \frac{\xi_1\,\rho_{,z} - \xi_2\,\rho_{,r}}{\Delta} \tag{74}$$

which leads to the following equations

$$\lambda = w^2, \tag{75}$$

$$(\mu + w)\rho = \sigma = \tfrac{\xi_1\,\rho_{,z} - \xi_2\,\rho_{,r}}{\Delta}. \tag{76}$$

In addition to the Killing equation (60) these equations constitute the field equations to be solved for the DJT Theory. With the simplification done in **4.A** of the last section we get

$$\rho = \rho_0\,q^{\frac{\mu+w}{2w}}, \quad m = -\frac{q_{,r}}{2wq} \tag{77}$$

where ρ_0 is an arbitrary constant. As far as the solutions are concerned we have the following three classes:

a). We obtain a simple solution of (76) when $\mu = -w$ which leads to $\rho = \rho_0$ constant. Then using the simplifications done in **4.B** of last section ($\xi_1 = 0$

and $\xi_2 = q = e^y$, $n = m = -\frac{1}{2w}\, y_{,r})$ we get all metric functions related to the function y which satisfies the differential equation (65). An exact solution and metric of the spacetime is given in (66) of **4.B** part of the last section.

b). A more general solution is obtained when $\mu + 3w = 0$ where the function y satisfies the equation

$$-(y_{,zz} + y_{,rr}) + \frac{1}{2}\,((y_{,r})^2 + (y_{,z})^2) = \rho_0\, y + \rho_2 \qquad (78)$$

where ρ_2 is an integration constant. A solution of this equation is $y = \frac{\rho_0}{2}\, r^2,\ \rho_2 = -\rho_0$. Then the metric becomes

$$ds^2 = e^{\frac{\rho_0}{2}\, r^2}\, dt\, dz + \frac{\rho_0^2\, r^2}{4\, w^2}\, [dr^2 + dz^2] \qquad (79)$$

Here $\rho_0 \neq 0$.

c). The case when $\mu + 5w \neq 0$. Function y satisfies the equation

$$-(y_{,zz} + y_{,rr}) + \frac{1}{2}\,((y_{,r})^2 + (y_{,z})^2) = \frac{\rho_0}{\epsilon}\, e^{\epsilon\, y} + \rho_3 \qquad (80)$$

where ρ_3 is an integration constant and $\epsilon = \frac{\mu+5w}{2w}$. The circularly symmetric metric has the form

$$ds^2 = e^y\, dt\, dz + \frac{1}{4\, w^2}\, e^{\epsilon y}\, [-\frac{2\rho_0}{\epsilon\,(\epsilon-1)} + 2\rho_3\, e^{-\epsilon\, y} + \rho_4\, e^{(1-\epsilon)\, y}]\, (dr^2 + dz^2) \qquad (81)$$

where the function $y = y(r)$ satisfies the equation

$$y_r = \pm\, e^{\frac{\epsilon}{2}\, y}\, \sqrt{-\frac{2\rho_0}{\epsilon\,(\epsilon-1)} + 2\rho_3\, e^{-\epsilon\, y} + \rho_4\, e^{(1-\epsilon)\, y}} \qquad (82)$$

Here ρ_4 is also an integration constant. All other cases will be considered later.

6 A New Massive Gravity in Three Dimensions

Recently a new, parity-preserving theory introduced by Bergshoeff-Hohm-Townsend (BHT) [5] in three dimensions which is equivalent to Pauli-Fierz massive field theory at the linearized level. Originally this theory, known as NMG does not contain a cosmological term in the action. Adding a cosmological constant λ the new massive gravity field equations (CNMG) are given as

$$2m_0^2\, G_{\mu\nu} + K_{\mu\nu} + \lambda\, g_{\mu\nu} = 0 \tag{83}$$

where

$$\begin{aligned} K_{\mu\nu} &= 2\nabla^2\, R_{\mu\nu} - \frac{1}{2}(R_{;\mu\nu} + g_{\mu\nu}\, \nabla^2\, R) - 8R^{\rho}{}_{\mu}\, R_{\nu\rho} \\ &\quad + \frac{9}{2} R\, R_{\mu\nu} + [3R^{\alpha\beta}\, R_{\alpha\beta} - \frac{13}{8} R^2]\, g_{\mu\nu} \end{aligned} \tag{84}$$

where m_0 is relative mass parameter and ∇^2 is the Laplace-Beltrami operator. Here we used the mass parameter as m_0 not to confuse with the metric function m. Solutions of these equations have been recently studied in [15], [6]

1. When the spacetime admits a non-null Killing vector field, assuming that the ricci scalar R is a constant and using the equation (30) we get $R = 2K + 2w^2$ and

$$\nabla^2 R_{\mu\nu} = -2w^2\, (3w^2 + \frac{R}{2})[g_{\mu\nu} - \frac{3}{\alpha}\, u_\mu\, u_\nu], \tag{85}$$

$$R^{\mu\alpha}\, R_{\nu\alpha} = \frac{1}{\alpha}\, (3w^2 + \frac{R}{2})(w^2 - \frac{R}{2})\, u^\mu\, u_\nu + (w^2 + \frac{R}{2})^2\, \delta^\mu{}_\nu, \tag{86}$$

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{\alpha}\, (3w^2 + \frac{R}{2})\, (4w^2 - \frac{R}{2})\, u_\mu\, u_\nu + (-2w^4 - 4w^2\, R \\ &\quad + \frac{R^2}{8})\, g_{\mu\nu}. \end{aligned} \tag{87}$$

Then using the field equations of NTM gravity we obtain $(3w^2 + \frac{R}{2} \neq 0)$

$$2m_0^2 = 4w^2 - \frac{R}{2}, \quad \lambda = -2w^4 - \frac{R^2}{8}. \tag{88}$$

Observe that the solution does not exist when the cosmological constant vanishes, $\lambda = 0$.

2. When the spacetime admits a null Killing vector field, using Eq.(40) and the Einstein tensor (57) we obtain $R = -6w^2$ and

$$\begin{aligned} \nabla^2 R_{\mu\nu} &= (\nabla^2 \rho + 4w\sigma + 4w^2\rho)\, \xi_\mu \, \xi_\nu, & (89) \\ R^\rho{}_\mu \, R_{\rho\nu} &= -4w^2\rho \, \xi_\mu \, \xi_\nu + 4w^4 \, g_{\mu\nu}, & (90) \\ K_{\mu\nu} &= [2\,\nabla^2 \rho + 8w\sigma + 13w^2\rho]\, \xi_\mu \, \xi_\nu - \frac{1}{2}\, w^4 \, g_{\mu\nu}. & (91) \end{aligned}$$

These equations lead to the following results

$$\begin{aligned} 2\,\nabla^2 \rho + (2\,m_0^2 + 13w^2)\,\rho + 8w\sigma = 0, & \quad (92) \\ 4m_0^2 w^2 - w^4 + 2\lambda = 0. & \quad (93) \end{aligned}$$

The full massive gravity field equations reduce to (92), (93) and the equation (40). A circularly symmetric solution of this equation is given by $\rho = \rho_0 \, e^{k\,y}$ where ρ_0 is a constant and k satisfies the quadratic equation $8k^2 + 24k + 27/2 - \frac{\lambda}{w^4} = 0$ with roots $k_{1,2} = \left(-6 \pm \sqrt{9 + \frac{2\lambda}{w^4}}\right)/4$. If $\lambda = 0$ then $k_1 = \frac{-3}{4}$ and $k_2 = \frac{-9}{4}$. The circularly symmetric metric becomes

$$ds^2 = e^{2\,y}\, dt\, dz + \frac{1}{4\,w^2}\,[\frac{2\rho_0}{k-1}\, e^{ky} + \rho_1 e^y]\,(dr^2 + dz^2) \tag{94}$$

Here y satisfies a similar equation like in (82)

$$y_{,r} = \pm \sqrt{\frac{2\rho_0}{k-1}\, e^{ky} + \rho_1 \, e^y} \tag{95}$$

where ρ_1 is an integration constant. For each value of k (k_1 and k_2) we have two different metrics.

BHT introduced also a more general model [5] which also includes the topologically massive gravity as a special case.

$$\lambda\, m_0^2 g_{\mu\nu} + \alpha\, G_{\mu\nu} + \frac{1}{\mu}\, C_{\mu\nu} + \frac{\beta}{2m_0^2}\, K_{\mu\nu} = 0, \tag{96}$$

where λ, α and β are dimensionless parameters. For $m_0 \to \infty$ and for fixed μ generalized BHT equations reduce to the DJT equations (70). We solve these equations with

$$\tfrac{\beta}{m_0^2}\, \nabla^2 \rho + (\alpha + \tfrac{w}{\mu} + \tfrac{13 w^2 \beta}{2 m_0 2})\, \rho + (-\tfrac{1}{\mu} + \tfrac{4\beta w}{m_0^2})\sigma = 0, \tag{97}$$

$$\lambda\, m_0^2 + \alpha w^2 - \tfrac{\beta}{4 m_0^2}\, w^4 = 0 \tag{98}$$

In a similar fashion letting $\rho = \rho_0\, e^{k\, y}$ we get a quadratic equation for k

$$4\, w^2\, \frac{\beta}{m_0^2}\, k(k+1) + 2(-\frac{1}{\mu} + \frac{4\beta w}{m_0^2})\, w\, k + (\alpha + \frac{w}{\mu} + \frac{13 w^2 \beta}{2 m_0 2}) = 0 \tag{99}$$

In order that the roots to exists the following condition should be satisfied

$$(-\frac{1}{\mu} + \frac{4\beta w}{m_0^2} + 2\, \frac{\beta^2}{m_0^2})^2 - 4\, \frac{\beta}{m_0^2}\, (\alpha + \frac{w}{\mu} + \frac{13 w^2 \beta}{2 m_0 2})\, w^2 \geq 0 \tag{100}$$

The metric function y satisfies exactly the same equation (95) but k solves the quadratic equation (99) in this case. The circularly symmetric metric is of the form given (94) where $y(r)$ satisfies (95).

Both for null and non-null cases when the scalars of the Killing vector fields are constants the we have a more general result. The following theorem implies that the corresponding metrics may solve all higher derivative gravitational field equations in three dimensions.

Conjecture. *Let a three dimensional spacetime admit a Killing vector field u_μ (non-null or null) with constant scalars. Let the Gaussian curvature K of the two dimensional spaces be constant for the case of non-null vector fields where corresponding einstein tensors are respectively given in (35) and (57). Then any symmetric second rank covariant tensor constructed from the ricci tensor by covariant differentiation and by contraction is the linear sum of $u_\mu\, u_\nu$ and the metric tensor $g_{\mu\nu}$*

7 Conclusion

In this work we first studied the Killing vector fields in a three dimensional spacetime geometry. We showed that,independent of the type of the Killing vector fields, the ricci tensor can be determined in terms of the Killing vector fields and their scalars. Usually components of this tensor are calculated in terms of the components of metric tensor in a given coordinate system. In three dimensions, when the geometry admits at least a Killing vector field we don't have to follow such a direction to determine the components of the ricci tensor. Using this property in each case, when the Killing vector field is timelike, spacelike or null we first presented solutions of Einstein field equations with sources. Then by using special cases, when the scalars of the Killing vector fields are constants, we gave solutions of the field equations of the Topologically Massive Gravity and New Massive Gravity field equations. Some of the solutions of the Topologically Massive Gravity field equations we obtained in this work may already be known. Our basic purpose in this work is to present a new method to solve higher derivative gravity theories rather finding specific solutions. We conjecture at the end that, for the three type of Killing vector fields with constant scalars our method solves all higher derivative theories in three dimensions.

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