# Zonal polynomials and hypergeometric functions of quaternion matrix argument *

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**Abstract**

We define zonal polynomials of quaternion matrix argument and deduce some important formulae of zonal polynomials and hypergeometric functions of quaternion matrix argument. As an application, we give the distributions of the largest and smallest eigenvalues of a quaternion central Wishart matrix $W \sim \mathbb{Q}W(n, \Sigma)$, respectively.

## 1 INTRODUCTION

Zonal polynomials and hypergeometric functions of real (or complex) symmetric matrices early introduced in [4] and [5, 6, 7] were used to study the density functions and the distributions of eigenvalues of Wishart matrices. Now they are very useful tools in the study of Multivariate Statistical Analysis. There are many ways of defining zonal polynomials. Some of them have appeared in [4], [5] and [10]. Muirhead's definition of zonal polynomials of a real matrix argument is an axiomatic definition which appeared in [9], involving partial differential operators. This definition is easier and more convenient for practical use. Gross and Richards defined zonal polynomials of a matrix argument over the division algebra **F**, including the real and complex fields, and quaternion division by means of the representation of groups. Maybe the authors thought there were some problems in their results, since they do not compute the numerical presentations of $C_\kappa(A)$ ($A$ is a Hermitian quaternion matrix).

In this paper, we modify the definition of zonal polynomials of a real matrix argument given in [9] and define zonal polynomials of a quaternion matrix argument.

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Then we compute the presentations of $C_\kappa(A)$. We also define quaternion hypergeometric functions in terms of zonal polynomials of a quaternion matrix argument and derive some useful formulas for quaternion hypergeometric functions. Using these results, we give the distributions of the largest and smallest eigenvalues of a quaternion Wishart matrix $W = A^H A$ (i.e., $W \sim \mathbb{Q}W_m(n, \Sigma), n \geqslant m,\ A \sim \mathbb{Q}N_{n\times m}(0, I_n \bigotimes \Sigma)$).

The paper is organized as follows. §2 provides the preliminary tools for deriving our results. The zonal polynomials and hypergeometric functions of a quaternion matrix argument will be studied in §3 and §4, respectively. In the last section, we will give the distributions of the largest and smallest eigenvalues.

# 2 PRELIMINARY

Following [13], let $\mathbb{C}$ and $\mathbb{R}$ denote the fields of complex and real numbers, respectively, and let $\mathbb{Q}$ denote the quaternion division algebra over $\mathbb{R}$, i.e., every $a \in \mathbb{Q}$ can be expressed as $a = a_1 + a_2 i + a_3 j + a_4 k$, where $i, j, k$ satisfy the following relations

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Put $a^H = a_1 - a_2 i - a_3 j - a_4 k$ and $\|a\| = (a^H a)^{1/2} = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^{1/2}$. Let $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{Q}^{m\times n}$ denote the set of all $m \times n$ matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$, respectively. Any $A \in \mathbb{Q}^{m\times n}$ can be written as $A = (a_{ij})_{m\times n} = A_1 + A_2 i + A_3 j + A_4 k$, where $a_{ij} \in \mathbb{Q}$, and $A_1,\ A_2,\ A_3,\ A_4 \in \mathbb{R}^{m\times n}$. $A_1$ is the real part of $A$, denoted by $\mathrm{Re}\, A$. We also set $\mathrm{Im}\,(A) = A_2,\ \mathrm{Jm}\,(A) = A_3$ and $\mathrm{Km}\,(A) = A_4$. Put $A^H = (a_{ji}^H)_{n\times m} = A_1' - A_2' i - A_3' j - A_4' k$. We say $A$ is Hermitian if $A^H = A$. The eigenvalues of a Hermitian matrix are all real. If the eigenvalues are all positive, then we say it is a positive definite quaternion matrix.

Let $\mathrm{tr}\,(\cdot)$ be the trace on $\mathbb{Q}^{n\times n}$ and put $\mathrm{Retr}\,(A) = \mathrm{tr}\,(\mathrm{Re}\, A)$ for $A \in \mathbb{Q}^{n\times n}$. We have

$$\mathrm{Retr}\,(A) = \frac{1}{2}\mathrm{tr}\,(A + A^H),\ \mathrm{Retr}\,(AB) = \mathrm{Retr}\,(BA), \quad \forall\, A,\, B \in \mathbb{Q}^{n\times n}.$$

Moreover, if $A = A^H \in \mathbb{Q}^{n\times n}$, then $\mathrm{Retr}\,(A) = \mathrm{tr}\,(A) = \sum\limits_{s=1}^{n} \lambda_s$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. Set ${}_qS(n) = \{A \in \mathbb{Q}^{n\times n} |\ A^H = A\}$ and

$${}_qO(n) = \{A \in \mathbb{Q}^{n\times n} |\ A^H A = AA^H = I_n\},\ {}_qV_{n,m} = \{A \in \mathbb{Q}^{m\times n} |\ A^H A = I_n\}.$$

Let $A \in \mathbb{Q}^{m\times n}$ be $A = A_1 + A_2 i + A_3 j + A_4 k = (A_1 + A_2 i) + (A_3 + A_4 i)j = B_1 + B_2 j$. $A$ has the complex representation $A^\sigma = \begin{pmatrix} B_1 & -B_2 \\ \overline{B_2} & \overline{B_1} \end{pmatrix}$ $(\overline{B_1} = A_1 - A_2 i,\ \overline{B_2} = A_3 - A_4 i)$ and the real representations

$${}_1A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ -A_3 & A_4 & A_1 & -A_2 \\ -A_4 & -A_3 & A_2 & A_1 \end{pmatrix} \quad \text{and } {}_2A = \begin{pmatrix} A_1 & -A_2 & -A_3 & -A_4 \\ A_2 & A_1 & -A_4 & A_3 \\ A_3 & A_4 & A_1 & -A_2 \\ A_4 & -A_3 & A_2 & A_1 \end{pmatrix}.$$

For $A \in \mathbb{Q}^{m\times n}$, denote by $|A|_q = \det(A^\sigma)$ and $|A|_d = |A^H A|$ the q–determinant and double determinant of $A$, respectively; here $|\cdot|$ is the determinant of a square quaternion matrix given in [13]. We have $|A^H|_d = |A|_d$ and $|A|_d = |A|_q$ (cf. [13]). Moreover, we have

**Lemma 2.1.** *Let $A \in \mathbb{Q}^{n\times n}$.*

(1) $|A|_d^2 = \det({}_1A) = \det({}_2A)$;

(2) *Let $A = T^H T$, where $T = (t_{ij})_{n\times n} \in \mathbb{Q}^{n\times n}$ is an upper-triangular matrix with $t_{ii} > 0$, $i = 1, \ldots, m$. Then $|A| = t_{11}^2 \cdots t_{nn}^2$.*

*Proof.* (1) Set $S_1 = \begin{pmatrix} A_1 & -A_3 \\ A_3 & A_1 \end{pmatrix}$, $S_2 = \begin{pmatrix} A_2 & -A_4 \\ -A_4 & -A_2 \end{pmatrix}$. Then by the proof of [8, Lemma 3.1],

$$\begin{aligned} |A|_d^2 = |A|_q^2 &= \left| \det \begin{pmatrix} A_1 + A_2 i & -A_3 - A_4 i \\ A_3 - A_4 i & A_1 - A_2 i \end{pmatrix} \right|^2 = \det(S_1 + S_2 i)\det(S_1 - S_2 i) \\ &= \det \begin{pmatrix} S_1 + S_2 i & \\ & S_1 - S_2 i \end{pmatrix} = \det \begin{pmatrix} 2S_1 & -iS_2 \\ -iS_2 & \frac{1}{2}S_1 \end{pmatrix}. \end{aligned}$$

Note that

$$\begin{pmatrix} 2S_1 & -iS_2 \\ -iS_2 & \frac{1}{2}S_1 \end{pmatrix} = \begin{pmatrix} 2 & \\ & -i \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \frac{i}{2} \end{pmatrix} = \begin{pmatrix} 2 & \\ & i \end{pmatrix} \begin{pmatrix} S_1 & -S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -\frac{i}{2} \end{pmatrix}.$$

So $|A|_d^2 = \det \begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix} = \det \begin{pmatrix} S_1 & -S_2 \\ S_2 & S_1 \end{pmatrix} = \det({}_1A) = \det({}_2A)$.

(2) We have $|A| = |T|_d = |T|_q = \det(T^\sigma) = \det \begin{pmatrix} T_1 & -T_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix}$, where $T = T_1 + T_2 j$ with $T_1, T_2 \in \mathbb{C}^{n\times n}$ and $T_1$, $T_2$ have the form

$$T_1 = \begin{pmatrix} t_{11} & & \ast \\ & \ddots & \\ \mathbf{0} & & t_{nn} \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & & \ast \\ & \ddots & \\ \mathbf{0} & & 0 \end{pmatrix},$$

respectively. A simple computation shows that $\det \begin{pmatrix} T_1 & -T_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix} = t_{11}^2 \cdots t_{nn}^2$. $\square$

Let $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}^{m\times n}$ and $X_1$, $X_2$, $X_3$, $X_4$ are $m \times n$ matrices of functionally independent real variables. Define the volume of $X$ as $(dX) = (dX_1) \bigwedge (dX_2) \bigwedge (dX_3) \bigwedge (dX_4)$, where $(dX_s)$, $s = 1, 2, 3, 4$, are defined in [8].

**Lemma 2.2.** *$X$, $Y \in \mathbb{Q}^{m\times n}$ and $Y = AXB$ where $A \in \mathbb{Q}^{m\times m}$ and $B \in \mathbb{Q}^{n\times n}$ are constant invertible matrices.*

(1) *We have $(dY) = |A|_q^{2n} |B|_q^{2m} (dX)$;*

(2) *Suppose $X \in {}_qS(m)$ and $B = A^H$. Then $(dY) = |A|_q^{2m-1}(dX)$.*

*Proof.* (1) Let $Y = AW$ and $W = XB$. Then $dY = AdW$, $dW = dX\, B$ and

$$\begin{aligned} (dY_1', dY_2', dY_3', dY_4')' &= ({}_2A)(dW_1', dW_2', dW_3', dW_4')' \\ (dW_1, dW_2, dW_3, dW_4) &= (dX_1, dX_2, dX_3, dX_4)({}_1B). \end{aligned}$$

Using the operator vec$(\cdot)$ (defined in [8, Definition 1.2]) to $dX_s$, $dY_s$ and $dW_s$, $s = 1, \ldots, 4$, we have

$$\begin{pmatrix} \mathrm{vec}(dY_1) \\ \mathrm{vec}(dY_2) \\ \mathrm{vec}(dY_3) \\ \mathrm{vec}(dY_4) \end{pmatrix} = {}_2(I \otimes A) \begin{pmatrix} \mathrm{vec}(dW_1) \\ \mathrm{vec}(dW_2) \\ \mathrm{vec}(dW_3) \\ \mathrm{vec}(dW_4) \end{pmatrix}, \quad \begin{pmatrix} \mathrm{vec}(dW_1) \\ \mathrm{vec}(dW_2) \\ \mathrm{vec}(dW_3) \\ \mathrm{vec}(dW_4) \end{pmatrix} = (({}_1B)' \otimes I) \begin{pmatrix} \mathrm{vec}(dX_1) \\ \mathrm{vec}(dX_2) \\ \mathrm{vec}(dX_3) \\ \mathrm{vec}(dX_4) \end{pmatrix}$$

by [8, Lemma 1.1] so that

$$\begin{pmatrix} \mathrm{vec}(dY_1) \\ \mathrm{vec}(dY_2) \\ \mathrm{vec}(dY_3) \\ \mathrm{vec}(dY_4) \end{pmatrix} = {}_2(I \otimes A)(({}_1B)' \otimes I) \begin{pmatrix} \mathrm{vec}(dX_1) \\ \mathrm{vec}(dX_2) \\ \mathrm{vec}(dX_3) \\ \mathrm{vec}(dX_4) \end{pmatrix}.$$

Thus by [8, Lemma 1.2] and Lemma 2.1,

$$(dY) = |{}_1A|^n |{}_2B|^m (dX) = |A|_q^{2n} |B|_q^{2m} (dX).$$

(2) Since $A$ is invertible, it follows from [13, Theorem 4.3] that $A$ is the product of elementary quaternion matrices. Thus using the same method as in the proof of [8, Theorem 1.20], we can get the assertion. $\square$

The following two lemmas, which come from [1, p37, p38], will be used in this paper:

**Lemma 2.3.** *$X \in {}_qS(m)$ with $X > 0$. Suppose $X = T^H T$, where $T = (t_{ij})_{m\times m} \in \mathbb{Q}^{m\times m}$ is an upper-triangular matrix with real diagonal elements. Then*

$$(dX) = 2^m \prod_{i=1}^m t_{ii}^{4(m-i)+1} (dT),$$

*where* $(dT) = \bigwedge_{s=1}^m d\, t_{ss} \bigwedge_{p=1}^4 \bigwedge_{s<t}^m d\ t_{st}^{(p)},\ t_{st} = t_{st}^{(1)} + t_{st}^{(2)} i + t_{st}^{(3)} j + t_{st}^{(4)} k,\ s < t,\ t = 1, \ldots, m.$

**Lemma 2.4.** *Let $Z = H_1 T \in \mathbb{Q}^{n\times m}$ with $H_1 \in {}_qV_{m,n}$, here $T$ the upper triangular matrix with positive diagonal elements. Then we have*

$$(dZ) = \prod_{i=1}^m t_{ii}^{4(n-i)+3} (dT) \wedge (H_1^H dH_1),$$

*where* $(H_1^H dH_1) = \bigwedge_{s=1}^m \bigwedge_{t=s+1}^n h_t^H d\, h_s$ *for* $H = (H_1|H_2) = (h_1, \ldots, h_m | h_{m+1}, \ldots, h_n)$.

In this paper, we shall use the singularvalue decomposition (SVD) of a matrix in $\mathbb{Q}^{m\times n}$ as follows. Let $A \in \mathbb{Q}^{m\times n}$ with rank $A = r$. Then there are $U = (U_1|U_2) \in {}_qO(m)$, $V = (V_1|V_2) \in {}_qO(n)$, with $U_1 \in {}_qV_{r,m},\ V_1 \in {}_qV_{r,n}$ such that

$$A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^H = U_1 D V_1^H \tag{1}$$

([13, Theorem 7.2]), where $D = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_r)$ and $\lambda_1, \ldots, \lambda_r$ are the singular values of $A$. If $A \in {}_qS(n)$ with rank $A = r$, then $V$ and $V_1$ can be taken as $U$ and $U_1$ in (1) respectively.

**Lemma 2.5.** *Let $X \in \mathbb{Q}^{m\times n}$ with* $\operatorname{rank} X = n \le m$. *Let $X = UDV^H$ with $U \in {}_qV_{n,m}$, $V \in {}_qO(n)$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ (assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$). Then*

(1) $(dX) = (2\pi^2)^{-n} \prod\limits_{j<i}^{n} (\lambda_j^2 - \lambda_i^2)^4 \prod\limits_{i=1}^{n} \lambda_i^{4m-4n+3} (dD) \bigwedge (U^H dU) \bigwedge (V^H dV)$ *for* $U \ne V$;

(2) $(dX) = (2\pi^2)^{-n} \prod\limits_{j<i}^{n} (\lambda_j - \lambda_i)^4 (dD) \bigwedge (U^H dU)$ *for* $m = n$ *and* $U = V$,

*where* $(V^H dV) = \bigwedge\limits_{s<t}^{n} v_t^H d\, v_s$ *for* $V = (v_1 \cdots v_n)$, $(U^H dU) = \bigwedge\limits_{s<t}^{n} u_t^H d\, u_s$ *for* $U = (u_1 \cdots u_n)$.

*Proof.* The assertions can be found in [2, p241, p242]. But we must divide the volume elements by $(2\pi^2)^n$ to normalize the arbitrary phases of elements in the first row of $U$. $\square$

**Corollary 2.6.** *Let $X = UDV^H$ with $X$, $U$, $D$, $V$ given in Lemma 2.5. Put $Z = X^H X$. Then*

$$(dX) = 2^{-n} \prod_{i=1}^{n} \lambda_i^{4m-4n+2} (dZ) \wedge (U^H dU) = 2^{-n} |X|_q^{2m-2n+1} (dZ) \wedge (U^H dU).$$

Recall that a quaternion variable $X = X_1 + X_2 i + X_3 j + X_4 k \sim \mathbb{Q}N(0,1)$ if $X_1, X_2, X_3, X_4$ iid. $N(0, \frac{1}{4})$. Thus $X = (x_{ij})_{n\times m} \in \mathbb{Q}^{n\times m}$ is said to be the quaternion normal matrix $\mathbb{Q}N_{n\times m}(0, I_n \otimes I_m)$ (or $X \sim \mathbb{Q}N_{n\times m}(0, I_n \otimes I_m)$) if $\{x_{ij} |\, i = 1, \ldots, n, j = 1, \ldots, m\}$ iid. to $\mathbb{Q}N(0,1)$. It is easy to deduce that the density function of $X \sim \mathbb{Q}N_{n\times m}(0, I_n \otimes I_m)$ is

$$f(X) = \frac{2^{2mn}}{\pi^{2mn}} \exp(-2\operatorname{tr}(X^H X)). \tag{2}$$

By (2) and Lemma 2.4, we can get

$$\operatorname{vol}(V_{m,n}) = \int_{V_{m,n}} (H_1^H dH_1) = \frac{2^m \pi^{2mn - m^2 + m}}{\prod\limits_{i=1}^{m} \Gamma[2n - 2(i-1)]} = \frac{2^m \pi^{2mn}}{\mathbb{Q}\Gamma_m(2n)} \tag{3}$$

where $\mathbb{Q}\Gamma(a) = \pi^{m^2-m} \prod\limits_{i=1}^{m} \Gamma(a - 2(i-1))$ $(\operatorname{Re}(a) > 2(m-1))$ (cf. (4.1) of [1]).

We call $Y \sim \mathbb{Q}N_{n\times m}(\mu, I_n \otimes \Sigma)$ if $Y = \mu + XB^H$, where $X \sim \mathbb{Q}N_{n\times m}(0, I_n \otimes I_m)$, $\Sigma = BB^H$ is invertible. By Lemma 2.1 and (2), we can write the density function of $Y \sim \mathbb{Q}N_{n\times m}(\mu, I_n \otimes \Sigma)$ as follows:

$$\frac{2^{2mn}}{\pi^{2mn} |\Sigma|^{2n}} \exp(\operatorname{Retr}(-2\Sigma^{-1}(Y-M)^H (Y-M))). \tag{4}$$

Let $W = Y^H Y$, we say $W \sim \mathbb{Q}W_m(n, \Sigma)$ $(n \geqslant m)$, if $Y \sim \mathbb{Q}N_{n\times m}(0, I_n \otimes \Sigma)$. $W$ is called the quaternion central Wishart matrix and the density function of $W$ is

$$\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n) |\Sigma|^{2n}} \exp(\operatorname{Retr}(-2\Sigma^{-1} W)) |W|^{2n-2m+1}. \tag{5}$$

As applications of the theory of zonal polynomials of quaternion matrix argument, we discuss the distributions of the maximum and the minimum eigenvalues of $W$, respectively, in the last section.

# 3 ZONAL POLYNOMIAL FOR QUATERNION MATRIX

The zonal polynomials of a Hermitian matrix are defined in terms of partitions of positive integers. Let $k$ be a positive integer; a partition $\kappa$ of $k$ is written as $\kappa = (k_1, k_2, \cdots)$, where $\sum_i k_i = k$, with the convention, unless otherwise stated, that $k_1 \geqslant k_2 \geqslant \cdots$, where $k_1, k_2, \cdots$ are non-negative integers. And if $\kappa = (k_1, k_2, \cdots)$ and $\lambda = (l_1, l_2, \cdots)$ are two partitions of $k$, we will write $\kappa > \lambda$ if $k_i > l_i$ for the first index $i$ for which the parts are unequal.

**Definition 3.1.** *Let $Y \in {}_qS(m)$ with eigenvalues $y_1, y_2, \ldots, y_m$ and let $\kappa = (k_1, k_2, \cdots)$ be a partition of $k$ into not more than $m$ parts. The zonal polynomial of $Y$ corresponding to $\kappa$, denoted by $C_\kappa(Y)$ (in this paper, we use the symbol $C_\kappa(Y)$ to denote the zonal polynomials of Hermitian quaternion matrices for notational simplicity) is a symmetric homogeneous polynomial of degree $k$ in the latent roots $y_1, \ldots, y_m$ such that:*

(i) *The term of highest weight in $C_\kappa(Y)$ is $y_1^{k_1}, \cdots, y_m^{k_m}$, that is,*

$$C_\kappa(Y) = d_\kappa y_1^{k_1} \cdots y_m^{k_m} + \textit{terms of lower weight} \tag{6}$$

*where $d_\kappa$ is a constant.*

(ii) *$C_\kappa(Y)$ is an eigenfunction of the differential operator $\Delta_Y$ given by*

$$\Delta_Y = \sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} 4 \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i} \tag{7}$$

(iii) *As $\kappa$ varies over all partitions of $k$, the zonal polynomials have unit coefficients in the expansion of $(\operatorname{tr} Y)^k$, that is*

$$(\operatorname{tr} Y)^k = (y_1 + y_2 + \cdots + y_m)^k = \sum_{\kappa}^{m} C_\kappa(Y). \tag{8}$$

By the way, if we replace $(ii)$ by $(ii)'$:
$(ii)'$ $C_\kappa(Y)$ is an eigenfunction of the differential operator $\Delta_Y$ given by

$$\Delta_Y = \sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} 2 \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i} \tag{9}$$

Then the conditions $(i), (ii)'$ and $(iii)$ define zonal polynomials for Hermitian complex matrices. We can verify this definition of zonal polynomials is just coincide with the definition of zonal polynomials for Hermitian complex matrices in [6].

By using the same method as in the proof of [9, Theorem 7.2.2], we can obtain the following:

**Lemma 3.2.** *The zonal polynomial $C_\kappa(Y)$ corresponding to the partition $\kappa = (k_1, k_2, \ldots, k_m)$ of $k$ satisfies the partial differential equation*

$$\Delta_Y C_\kappa(Y) = [\rho_\kappa + k(4m - 1)] C_\kappa(Y) \tag{10}$$

*where $\Delta_Y$ is given by (7) and*

$$\rho_\kappa = \sum_{i=1}^{m} k_i(k_i - 4i) \tag{11}$$

If $\kappa = (k_1, k_2, \ldots, k_m)$, the monomial symmetric function of $y_1, y_2, \ldots, y_m$ corresponding to $\kappa$ is defined as $M_\kappa = y_1^{k_1} \cdots y_m^{k_m} +$ *symmetric terms*. For example,

$$M_1(Y) = y_1 + \cdots + y_m, \quad M_2(Y) = y_1^2 + \cdots + y_m^2, \quad M_{1,1}(Y) = \sum_{i<j}^{m} y_i y_j$$

and so on.

When $k = 1$, $C_{(1)} = \operatorname{tr} Y = y_1 + \cdots + y_m$ by (8).

When $k = 2$, there is two partitions $(1,1)$, $(2,0)$ by definition 3.1 and Lemma 3.2, so we have following equations,

$$\begin{aligned} C_{(2)} =& d_{(2)} M_{(2)}(Y) + \beta M_{(1,1)}(Y) && (12)\\ C_{(1,1)} =& (2 - \beta) M_{(1,1)}(Y) && (13)\\ \Delta_Y C_{(2)}(Y) =& (8m - 6) C_{(2)}(Y) && (14) \end{aligned}$$

We have $d_{(2)} = 1$ from above, since $C_{(2)} + C_{(1,1)} = (\operatorname{tr} Y)^2$. Also we can verify

$$\begin{aligned} \Delta_Y M_{(2)}(Y) =& (8m - 6) M_{(2)}(Y) + 8 M_{(1,1)}(Y) && (15)\\ \Delta_Y M_{(1,1)}(Y) =& (8m - 12) M_{(1,1)}(Y). && (16) \end{aligned}$$

By means of (15), (16) and (14), we have $\beta = \dfrac{4}{3}$ by the following equation,

$$(8m-6)(M_{(2)}(Y) + \beta M_{(1,1)}(Y)) = (8m-6) M_{(2)}(Y) + 8 M_{(1,1)}(Y) + (8m-12)\beta M_{(1,1)}(Y).$$

Then the two zonal polynomials for Hermitian quaternion matrices in the case $k = 2$ are

$$C_{(2)} = M_{(2)}(Y) + \frac{4}{3} M_{(1,1)}(Y), \quad C_{(1,1)} = \frac{2}{3} M_{(1,1)}(Y).$$

Now we consider the case $k = 3$. We have three partitions $(3), (2,1), (1,1,1)$ when $k = 3$. Thus,

$$\begin{aligned} C_{(3)} =& M_{(3)}(Y) + \beta M_{(2,1)}(Y) + \gamma M_{(1,1,1)}(Y)\\ C_{(2,1)} =& (3 - \beta) M_{(2,1)}(Y) + \delta M_{(1,1,1)}(Y)\\ C_{(1,1,1)} =& (6 - \gamma - \delta) M_{(1,1,1)}(Y). \end{aligned}$$

Since

$$\begin{aligned} \Delta_Y M_{(3)}(Y) =& (12m - 6) M_{(3)}(Y) + 12 M_{(2,1)}(Y)\\ \Delta_Y M_{(2,1)}(Y) =& (12m - 14) M_{(2,1)}(Y) + 24 M_{(1,1,1)}(Y)\\ \Delta_Y M_{(1,1,1)}(Y) =& 12(m - 2) M_{(1,1,1)}(Y), \end{aligned}$$

it follows from Lemma 3.2 that

$$\Delta_Y C_{(3)}(Y) = (12m - 6) C_{(3)}(Y), \quad \Delta_Y C_{(2,1)}(Y) = (12m - 14) C_{(2,1)}(Y).$$

From the above equations, we can deduce that $\beta = \frac{3}{2}$, $\gamma = 2$, $\delta = \frac{18}{5}$. Therefore, we have three zonal polynomials for Hermitian quaternion matrices when $k = 3$ as follows:

$$\begin{aligned} C_{(3)}(Y) =& M_{(3)}(Y) + \frac{3}{2}M_{(2,1)}(Y) + 2M_{(1,1,1)}(Y) \\ C_{(2,1)}(Y) =& \frac{3}{2}M_{(2,1)}(Y) + \frac{18}{5}M_{(1,1,1)}(Y) \\ C_{(1,1,1)}(Y) =& \frac{2}{5}M_{(1,1,1)}(Y). \end{aligned}$$

In general, let $\kappa$ be a partition of $k$. Then $C_\kappa(Y)$ can be expressed in terms of monomial symmetric functions as

$$C_\kappa(Y) = \sum_{\lambda \leqslant \kappa} c_{(\kappa,\lambda)} M_{(\lambda)}(Y).$$

By Lemma 3.2, we obtain that the coefficients $c_{(\kappa,\lambda)}$ are determined by the following equation:

$$c_{(\kappa,\lambda)} = \sum_{\lambda < \mu \leqslant \kappa} \frac{4[(l_i + t) - (l_j - t)]}{\rho_\kappa - \rho_\lambda} c_{(\kappa,\mu)}, \tag{17}$$

where $\rho_\kappa = \sum_{i=1}^{m} k_i(k_i - 4i)$, $\lambda = (l_1, \ldots, l_m)$ and $\mu = (l_1, \ldots, l_i + t, \ldots, l_j - t, \ldots, l_m)$ for $t = 1, \cdots, l_j$ such that, when the parts of the partition $\mu$ are arranged in descending order, $\mu$ is above $\lambda$ and below or equal to $\kappa$. The summation in (17) is over all such $\mu$, including possibly, non–descending ones, and any empty sum is taken to be zero.

For example, when $k = 4$, we have five partitions $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$. Then the zonal polynomial $C_{(4)}(Y)$ has the form

$$\begin{aligned} C_{(4)}(Y) =& M_{(4)}(Y) + c_{(4)(3,1)}M_{(3,1)}(Y) + c_{(4),(2,2)}M_{(2,2)}(Y) \\ &+ c_{(4),(2,1,1)}M_{(2,1,1)}(Y) + c_{(4),(1,1,1,1)}M_{(1,1,1,1)}(Y). \end{aligned}$$

By (11), we have

$$\rho_{(4)} = 0, \ \rho_{(3,1)} = -10, \ \rho_{(2,2)} = -16, \ \rho_{(2,1,1)} = -22, \ \rho_{(1,1,1,1)} = -36.$$

Let $\kappa = (4)$, $\lambda = (3,1)$. Then by (17), $c_{(4)(3,1)} = \frac{4 \times 4}{10} \times 1 = \frac{8}{5}$. The coefficient $c_{(4),(2,2)}$ comes from the partitions $(3,1), (4)$, so

$$c_{(4)(2,2)} = \frac{4 \times 2}{16} \times \frac{8}{5} + \frac{4 \times 4}{16} \times 1 = \frac{9}{5}.$$

Since the coefficient $c_{(4),(2,1,1)}$ comes from the partitions $(3,1,0), (3,0,1), (2,2,0)$,

$$c_{(4),(2,1,1)} = 2 \times \frac{4 \times 3}{22} \times \frac{8}{5} + \frac{4 \times 2}{22} \times \frac{9}{5} = \frac{12}{5}.$$

Noting that the coefficient $c_{(4),(1,1,1,1)}$ comes from the partitions $(2,0,1,1)$, $(2,1,0,1)$, $(2,1,1,0)$, $(1,2,1,0)$, $(1,2,0,1)$, $(1,1,2,0)$, we have

$$c_{(4),(1,1,1,1)} = 6 \times \frac{4 \times 2}{36} \times \frac{12}{5} = \frac{16}{5}.$$

We list the coefficients of $M_\lambda(Y)$ in $C_\kappa(Y)$ for quaternion matrix $Y$ in the Table. We see that these coefficients are different from these in the real cases given in [9, p238].

Table: Coefficients of monomial symmetric functions $M_\lambda(Y)$ in $C_\kappa(Y)$

$k = 2$,

| $\kappa$ \ $\lambda$ | (2) | (1,1) |
|---|---|---|
| (2) | 1 | 4/3 |
| (1,1) | 0 | 2/3 |

$k = 3$,

| $\kappa$ \ $\lambda$ | (3,0) | (2,1) | (1,1,1) |
|---|---|---|---|
| (3,0) | 1 | 3/2 | 2 |
| (2,1) | 0 | 3/2 | 18/5 |
| (1,1,1) | 0 | 0 | 2/5 |

$k = 4$,

| $\kappa$ \ $\lambda$ | (4) | (3,1) | (2,2) | (2,1,1) | (1,1,1,1) |
|---|---|---|---|---|---|
| (4) | 1 | 8/5 | 9/5 | 12/5 | 16/5 |
| (3,1) | 0 | 12/5 | 16/5 | 104/15 | 64/5 |
| (2,2) | 0 | 0 | 1 | 4/3 | 16/5 |
| (2,1,1) | 0 | 0 | 0 | 4/3 | 32/7 |
| (1,1,1,1) | 0 | 0 | 0 | 0 | 8/35 |

$k = 5$,

| $\kappa$ \ $\lambda$ | (5) | (4,1) | (3,2) | (3,1,1) | (2,2,1) | (2,1,1,1) | (1,1,1,1,1) |
|---|---|---|---|---|---|---|---|
| (5) | 1 | 5/3 | 2 | 8/3 | 3 | 4 | 16/3 |
| (4,1) | 0 | 10/3 | 5 | 220/21 | 90/7 | 160/7 | 800/21 |
| (3,2) | 0 | 0 | 3 | 4 | 26/3 | 16 | 32 |
| (3,1,1) | 0 | 0 | 0 | 20/7 | 80/21 | 85/7 | 200/7 |
| (2,2,1) | 0 | 0 | 0 | 0 | 5/3 | 4 | 80/7 |
| (2,1,1,1) | 0 | 0 | 0 | 0 | 0 | 1 | 40/9 |
| (1,1,1,1,1) | 0 | 0 | 0 | 0 | 0 | 0 | 8/63 |

Let $X$ be an $m \times m$ positive definite quaternion matrix and put

$$(ds)^2 = \mathrm{Retr}\,(X^{-1}dXX^{-1}dX) \tag{18}$$

where $dX = (dx_{ij})_{m\times m}$. This is a differential form and is invariant under the transformation $X \to LXL^H$, here $L \in \mathbb{Q}^{m\times m}$ is invertible. For then $dX \to LdXL^H$, so that

$$\begin{aligned}\mathrm{Retr}\,(X^{-1}dXX^{-1}dX) &\to \mathrm{Retr}\,((LXL^H)^{-1}LdXL^H(LXL^H)^{-1}LdXL^H)\\ &= \mathrm{Retr}\,(X^{-1}dXX^{-1}dX).\end{aligned}$$

Put $n = 2m^2 - m$, let $x$ be the $n \times 1$ vector

$$x = (x_{11}, \operatorname{Re} x_{12}, \ldots, \operatorname{Re} x_{1m}, x_{22}, \ldots, \operatorname{Re} x_{2m}, \ldots, x_{mm}, \operatorname{Im} x_{12}, \ldots, \operatorname{Im} x_{m,m-1},$$
$$\operatorname{Jm} x_{12}, \ldots, \operatorname{Jm} x_{m,m-1}, \operatorname{Km} x_{12}, \ldots, \operatorname{Km} x_{m,m-1})'.$$

For notational convenience, relabel $x$ as $(x_1, \ldots, x_n)$. Similar to the real case, we have

$$(ds)^2 = \operatorname{Retr}(X^{-1}dXX^{-1}dX) = dx'G(x)dx$$

where $G(x)$ is an $n \times n$ nonsingular symmetric matrix. Define the differential operator $\Delta_X^*$ as

$$\Delta_X^* = \det G(x)^{-1/2} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[ \det G(x)^{1/2} \sum_{i=1}^{n} g(x)^{ij} \frac{\partial}{\partial x_i} \right],$$

where $G(x)^{-1} = (g(x)^{ij})$. Let $\dfrac{\partial}{\partial x} = \left( \dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_n} \right)'$, then we can write $\Delta_X^*$ as

$$\Delta_X^* = \det G(x)^{-1/2} \left( \frac{\partial}{\partial x} \right)' \left[ \det G(x)^{1/2} G(x)^{-1} \frac{\partial}{\partial x} \right] \tag{19}$$

which is invariant under the transformation $X \to LXL^H$ ($L \in \mathbb{Q}^{m\times m}$ is invertible), i.e., $\Delta_X^* = \Delta_{LXL^H}^*$.

The proofs of the above assertions are just the same as in [9, p240] and we do not show them here. Consider the positive definite quaternion matrix $X = HYH^H$, $H \in {}_qO(m)$, $Y = \operatorname{diag}(y_1, \ldots, y_m)$. In terms of $H$ and $Y$, the invariant differential form $(ds)^2$ given by (18) can be written as

$$\begin{aligned}
(ds)^2 &= \operatorname{Retr}(X^{-1}dXX^{-1}dX) \\
&= \operatorname{Retr}(Y^{-1}dYY^{-1}dY) - 2\operatorname{Retr}(d\Theta Y^{-1} d\Theta Y^{-1}) + 2\operatorname{Retr}(d\Theta d\Theta) \\
&= \sum_{i=1}^{m} \frac{(dy_i)^2}{y_i^2} - 2\sum_{i=1}^{m} ((\operatorname{Im} d\theta_{ii})^2 + (\operatorname{Jm} d\theta_{ii})^2 + (\operatorname{Km} d\theta_{ii})^2) \\
&+ 2\sum_{i<j}^{m} \frac{y_i^2 + y_j^2}{y_i y_j} ((\operatorname{Re} d\theta_{ij})^2 + (\operatorname{Im} d\theta_{ij})^2 + (\operatorname{Jm} d\theta_{ij})^2 + (\operatorname{Km} d\theta_{ij})^2) \\
&+ 2\sum_{i=1}^{m} ((\operatorname{Im} d\theta_{ii})^2 + (\operatorname{Jm} d\theta_{ii})^2 + (\operatorname{Km} d\theta_{ii})^2) \\
&- 4\sum_{i<j}^{m} ((\operatorname{Re} d\theta_{ij})^2 + (\operatorname{Im} d\theta_{ij})^2 + (\operatorname{Jm} d\theta_{ij})^2 + (\operatorname{Km} d\theta_{ij})^2) \\
&= \sum_{i=1}^{m} \frac{(dy_i)^2}{y_i^2} + 2\sum_{i<j}^{m} \frac{(y_i - y_j)^2}{y_i y_j} ((\operatorname{Re} d\theta_{ij})^2 + (\operatorname{Im} d\theta_{ij})^2 + (\operatorname{Jm} d\theta_{ij})^2 + (\operatorname{Km} d\theta_{ij})^2) \\
&= ((dy)' \ (\operatorname{Re} d\theta)' \ (\operatorname{Im} d\theta)' \ (\operatorname{Jm} d\theta)' \ (\operatorname{Km} d\theta')G(y) \begin{pmatrix} dy \\ \operatorname{Re} d\theta \\ \operatorname{Im} d\theta \\ \operatorname{Jm} d\theta \\ \operatorname{Km} d\theta \end{pmatrix}
\end{aligned}$$

where $d\Theta = (d\theta_{ij}) = H^H dH = -dH^H H$, $dy = (dy_1, dy_2, \ldots, dy_m)'$, and

$$\begin{aligned}\operatorname{Re} d\theta =& (\operatorname{Re} d\theta_{12}, \operatorname{Re} d\theta_{13}, \ldots, \operatorname{Re} d\theta_{m-1,m})', \quad \operatorname{Im} d\theta = (\operatorname{Im} d\theta_{12}, \operatorname{Im} d\theta_{13}, \ldots, \operatorname{Im} d\theta_{m-1,m})', \\ \operatorname{Jm} d\theta =& (\operatorname{Jm} d\theta_{12}, \operatorname{Jm} d\theta_{13}, \ldots, \operatorname{Jm} d\theta_{m-1,m})', \ \operatorname{Km} d\theta = (\operatorname{Km} d\theta_{12}, \operatorname{Km} d\theta_{13}, \ldots, \operatorname{Km} d\theta_{m-1,m})'.\end{aligned}$$

Therefore $G(y)$ has the form

$$G(y) = \begin{pmatrix} B & & & 0 & & \\ & A_{12} & & & & \\ & & \ddots & & & \\ 0 & & & A_{ij}(i<j) & & \\ & & & & \ddots & \\ & & & & & A_{m-1,m} \end{pmatrix},$$

where

$$B = \begin{pmatrix} y_1^{-2} & & \\ & \ddots & \\ & & y_m^{-2} \end{pmatrix}, \ A_{ij} = \begin{pmatrix} \frac{2(y_i-y_j)^2}{y_i y_j} & & & \\ & \frac{2(y_i-y_j)^2}{y_i y_j} & & \\ & & \frac{2(y_i-y_j)^2}{y_i y_j} & \\ & & & \frac{2(y_i-y_j)^2}{y_i y_j} \end{pmatrix}.$$

In terms of (19) and $\frac{\partial}{\partial y}, \frac{\partial}{\partial R\theta}, \frac{\partial}{\partial I\theta}, \frac{\partial}{\partial J\theta}, \frac{\partial}{\partial K\theta}$, the operator $\Delta_X^*$ can be expressed as

$$\Delta_X^* = \Delta_{HYH^H}^* = |G(y)|^{-1/2} \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial R\theta} \\ \frac{\partial}{\partial I\theta} \\ \frac{\partial}{\partial J\theta} \\ \frac{\partial}{\partial K\theta} \end{pmatrix}' \left[ |G(y)|^{1/2} G(y)^{-1} \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial R\theta} \\ \frac{\partial}{\partial I\theta} \\ \frac{\partial}{\partial J\theta} \\ \frac{\partial}{\partial K\theta} \end{pmatrix} \right],$$

($\frac{\partial}{\partial R\theta}, \frac{\partial}{\partial I\theta}, \frac{\partial}{\partial J\theta}, \frac{\partial}{\partial K\theta}$ are the derivation of $\operatorname{Re}\theta, \operatorname{Im}\theta, \operatorname{Jm}\theta, \operatorname{Km}\theta$ respectively), that is,

$$\begin{aligned}\Delta_X^* =& \Delta_{HYH^H}^* = \sum_{i=1}^m y_i^2 \frac{\partial^2}{\partial y_i^2} + 4 \sum_{i=1}^m \sum_{j=1, j\neq i}^m \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i} \\ &+ (3-2m) \sum_{i=1}^m y_i \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i<j}^m \frac{y_i y_j}{(y_i-y_j)^2} \left( \frac{\partial^2}{\partial R\theta_{ij}^2} + \frac{\partial^2}{\partial I\theta_{ij}^2} + \frac{\partial^2}{\partial J\theta_{ij}^2} + \frac{\partial^2}{\partial K\theta_{ij}^2} \right) \\ =& \Delta_Y + (3-2m) E_Y + \frac{1}{2} \sum_{i<j}^m \frac{y_i y_j}{(y_i-y_j)^2} \left( \frac{\partial^2}{\partial R\theta_{ij}^2} + \frac{\partial^2}{\partial I\theta_{ij}^2} + \frac{\partial^2}{\partial J\theta_{ij}^2} + \frac{\partial^2}{\partial K\theta_{ij}^2} \right)\end{aligned}$$

where $\Delta_Y$ is given in Definition 3.1, $E_Y = \sum_{i=1}^{m} y_i \frac{\partial}{\partial y_i}$, $\frac{\partial^2}{\partial R\theta^2}, \frac{\partial^2}{\partial I\theta^2}, \frac{\partial^2}{\partial J\theta^2}, \frac{\partial^2}{\partial K\theta^2}$ is the second derivation of $\operatorname{Re}\theta,\ \operatorname{Im}\theta,\ \operatorname{Jm}\theta,\ \operatorname{Km}\theta$, respectively. It follows from $E_Y C_\kappa(Y) = kC_\kappa(Y)$ and the above equation that

$$
\begin{aligned}
\Delta_X^* C_\kappa(X) &= \Delta_{HYH^H}^* C_\kappa(Y) \\
&= \left[\Delta_Y + (3-2m)E_Y + \frac{1}{2}\sum_{i<j}^{m} \frac{y_i y_j}{(y_i - y_j)^2}\left(\frac{\partial^2}{\partial R\theta_{ij}^2} + \frac{\partial^2}{\partial I\theta_{ij}^2} + \frac{\partial^2}{\partial J\theta_{ij}^2} + \frac{\partial^2}{\partial K\theta_{ij}^2}\right)\right] C_\kappa(Y) \\
&= [\rho_\kappa + k(4m-1) + (3-2m)k] C_\kappa(Y) \\
&= [\rho_\kappa + 2k(m+1)] C_\kappa(X).
\end{aligned}
$$

In fact, we could have defined the zonal polynomial $C_\kappa(X)$ for $X > 0$ in terms of the operator $\Delta_X^*$ rather than $\Delta_Y$. Here the definition would be that $C_\kappa(X)$ $(= C_\kappa(Y))$ is a symmetric homogeneous polynomial of degree $k$ in the latent roots $y_1, \cdots, y_m$ of $X$ satisfying conditions (i) and (iii) of definition 3.1 and such that $C_\kappa(X)$ is an eigenfunction of the differential operator $\Delta_X^*$. The eigenvalue of $\Delta_X^*$ corresponding to $C_\kappa(X)$ is, from the above equation, equal to $[\rho_\kappa + 2k(m+1)]$. This defines the zonal polynomials for the positive definite quaternion matrix $X$, and since they are polynomials in the latent roots of $X$ their definition can be extended to an arbitrary Hermitian quaternion matrix and then to a non-Hermitian quaternion matrix by using $C_\kappa(XY) = C_\kappa(X^{1/2} Y X^{1/2})$ ($X$ is a positive definite matrix and $Y$ is a Hermitian matrix).

**Theorem 3.3.** *Let $X_1, X_2 \in {}_qS(m)$ with $X_1$ positive definite. Then*

$$
\int_{{}_qO(m)} C_\kappa(X_1 H X_2 H^H)(dH) = \frac{C_\kappa(X_1) C_\kappa(X_2)}{C_\kappa(I_m)},
$$

*where $(dH)$ is the normalized invariant measure on ${}_qO(m)$.*

*Proof.* Let

$$
f_\kappa(X_2) = \int_{{}_qO(m)} C_\kappa(X_1 H X_2 H^H)(dH).
$$

It is easy to verify $f_\kappa(X_2) = f_\kappa(UX_2U^H)$, $U \in {}_qO(m)$ so that $f_\kappa(X_2)$ is a symmetric function of $X_2$; in fact, a symmetric homogeneous polynomial of degree $k$. Set $L = X_1^{1/2} H$ and suppose $X_2 > 0$. Then by use of the invariance of $\Delta_{X_2}^*$, we have

$$
\begin{aligned}
\Delta_{X_2}^* f_\kappa(X_2) &= \int_{{}_qO(m)} \Delta_{X_2}^* C_\kappa(X_1 H X_2 H^H)(dH) \\
&= \int_{{}_qO(m)} \Delta_{X_2}^* C_\kappa(X_1^{1/2} H X_2 H^H X_1^{1/2})(dH) \\
&= \int_{{}_qO(m)} \Delta_{X_2}^* C_\kappa(L X_2 L^H)(dH) = \int_{{}_qO(m)} \Delta_{LX_2L^H}^* C_\kappa(L X_2 L^H)(dH) \\
&= [\rho_\kappa + 2k(m+1)] f_\kappa(X_2)
\end{aligned}
$$

Then $f_\kappa(X_2)$ must be a multiple of the zonal polynomial $C_\kappa(X_2)$, i.e., $f_\kappa(X_2) = \lambda_\kappa C_\kappa(X_2)$. Put $X_2 = I_m$, then $\lambda_\kappa = \frac{C_\kappa(X_1)}{C_\kappa(I_m)}$. Finally, we get the result by analytic continuation. □

Theorem 3.3 plays a vital role in the next evaluation of many integrals involving zonal polynomials.

Let $\mathbb{Q}\Gamma_m(a) = \int_{A>0} \operatorname{etr}(-A)|A|^{a-2m+1}(dA)$ be the quaternion $\Gamma$–function given in [3] and then $\mathbb{Q}\Gamma_n(\alpha) = \pi^{n(n-1)} \prod_{j=1}^{n} \Gamma[\alpha - 2(j-1)]$, $\operatorname{Re}\alpha > 2(n-1)$. Set

$$\mathbb{Q}\Gamma_n(\alpha,\kappa) = \pi^{n(n-1)} \prod_{j=1}^{n} \Gamma[\alpha + k_j - 2(j-1)], \quad \operatorname{Re}\alpha > 2(n-1) - k_n,$$

where $\kappa = (k_1, \ldots, k_n)$ is a partition of the integer $k$: $k = k_1 + k_2 + \cdots + k_n$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n \geqslant 0$. Then we have $(\alpha)_\kappa \triangleq \prod_{j=1}^{n} (\alpha - 2(j-1))_{k_j} = \dfrac{\mathbb{Q}\Gamma_n(\alpha,\kappa)}{\mathbb{Q}\Gamma_n(\alpha)}$, where $(\alpha)_j = \alpha(\alpha+1)\cdots(\alpha+j-1)$.

**Lemma 3.4.** *Let $A = (a_{ij})_{m\times m} \in {}_qS(m)$ with eigenvalues $\lambda_1, \ldots, \lambda_m$ (are all real). Put $r_1 = \sum_{i=1}^{m} \lambda_i$, $r_2 = \sum_{i<j}^{m} \lambda_i\lambda_j$, $\cdots$, $r_m = \lambda_1 \cdots \lambda_m$ and $\operatorname{tr}_k(A) = \sum_{1\le i_1<i_2<\cdots<i_k\le m} \det A_{i_1,i_2\cdots i_k}$, where $A_{i_1,i_2,\cdots,i_k}$ denotes the $k \times k$ matrix formed from $A$ by deleting all but the $i_1, \ldots, i_k$th rows and columns. Then $r_j = \operatorname{tr}_j(A)$.*

*Proof.* We have $P(\lambda) = |A - \lambda I_m| = \sum_{k=0}^{m} (-\lambda)^k r_{m-k}(\lambda_1, \ldots, \lambda_m)$. We also can get $|A - \lambda I_m| = \sum_{k=0}^{m} (-\lambda)^k \operatorname{tr}_{m-k}(A)$ by the definition of the determinant of a quaternion matrix given in [13]. The assertion follows. $\square$

**Lemma 3.5.** *Let $Y = \operatorname{diag}(y_1, y_2, \ldots, y_m)$ be a real diagonal matrix and $X = (x_{ij})_{m\times m}$ be a $m \times m$ positive definite quaternion matrix. Then*

$$C_\kappa(XY) = d_\kappa y_1^{k_1} \cdots y_m^{k_m} x_{11}^{k_1-k_2} \left| \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right|^{k_2-k_3} \cdots |X|^{k_m} + \textit{terms of lower weight}, \tag{20}$$

*where $\kappa = (k_1, \cdots, k_m)$, $d_\kappa$ is the coefficient of the term of highest weight in $C_\kappa(\cdot)$.*

*If $Z = \operatorname{diag}(z_1, z_2, \ldots, z_m)$ is a real diagonal matrix and $Y = (y_{ij})_{m\times m}$ is a $m \times m$ positive definite quaternion matrix, then*

$$\begin{aligned} C_\kappa(Y^{-1}Z) =& d_\kappa z_1^{k_m} \cdots z_m^{k_1} y_{11}^{k_{m-1}-k_m} \left| \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \right|^{k_{m-2}-k_{m-1}} \cdots |Y|^{-k_1} \\ & + \textit{terms of lower weight}, \end{aligned}$$

*where $\kappa = (k_1, \ldots, k_m)$, $d_\kappa$ is the coefficient of the term of highest weight in $C_\kappa(\cdot)$.*

*Proof.* Let $A \in {}_qS(m)$ and $a_1, \ldots, a_m$ be its real eigenvalues. Then

$$\begin{aligned} C_\kappa(A) =& d_\kappa a_1^{k_1} \cdots a_m^{k_m} + \textit{terms of lower weight} \\ =& d_\kappa a_1^{k_1-k_2} (a_1a_2)^{k_2-k_3} \cdots (a_1a_2\cdots a_m)^{k_m} + \textit{terms of lower weight} \\ =& d_\kappa (\sum_{i=1}^{m} a_i)^{k_1-k_2} (\sum_{i<j}^{m} a_ia_j)^{k_2-k_3} \cdots (a_1a_2\cdots a_m)^{k_m} + \textit{symmetric terms} \\ =& d_\kappa r_1^{k_1-k_2} r_2^{k_2-k_3} \cdots r_m^{k_m} + \textit{symmetric terms}. \end{aligned}$$

On the other hand, by Lemma 3.4,

$$\begin{aligned} C_\kappa(A) =& d_\kappa \operatorname{tr}_1(A)^{k_1-k_2} \operatorname{tr}_2(A)^{k_2-k_3}\cdots \operatorname{tr}_m(A) + \textit{symmetric terms} \\ =& d_\kappa a_{11}^{k_1-k_2} \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right|^{k_2-k_3} \cdots \end{aligned}$$

Set $A = XY,\ a_{ij} = x_{ij}y_j$. We have

$$C_\kappa(XY) = d_\kappa y_1^{k_1}\cdots y_m^{k_m} x_{11}^{k_1-k_2} \left| \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right|^{k_2-k_3} \cdots |X|^{k_m} + \textit{terms of lower weight.}$$

Similarly, we can get the second assertion. □

Let $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{Q}^{m\times n}$ and put $\operatorname{re}(A) = A_1 + A_2 i + A_3 j$. Let $\Phi_m = \{T \in {}_qS(m) | \operatorname{re}(T) > 0\}$. $\Phi_m$ is called the generalized right half plane.

**Theorem 3.6.** *Let $Z \in \Phi_m$ and $Y \in {}_qS(m)$. Then*

$$\int_{X>0} \operatorname{etr}(-XZ)|X|^{a-2m+1} C_\kappa(XY)(dX) = (a)_\kappa \mathbb{Q}\Gamma_m(a)|Z|^{-a} C_\kappa(YZ^{-1}),$$

*for* $\operatorname{Re}(a) > 2(m-1)$ *and*

$$\int_{X>0} \operatorname{etr}(-XZ)|X|^{a-2m+1} C_\kappa(X^{-1}Y)(dX) = \frac{(-1)^k \mathbb{Q}\Gamma_m(a)}{(-a+2m-1)_\kappa} |Z|^{-a} C_\kappa(YZ)$$

*for* $\operatorname{Re}(a) > 2(m-1) + k_1$*, where we set* $C_\kappa = 1$ *and* $(a)_\kappa = 1$ *when* $\kappa = (0)$*.*

*Proof.* For $Z = I_m$, we should prove the following equation

$$\int_{X>0} \operatorname{etr}(-X)|X|^{a-2m+1} C_\kappa(XY)(dX) = (a)_\kappa \mathbb{Q}\Gamma_m(a) C_\kappa(Y).$$

Let $f(Y) = \int_{X>0} \operatorname{etr}(-X)|X|^{a-2m+1} C_\kappa(XY)(dX)$ and put $S = H^H XH$, $H \in {}_qO(m)$. Then $(dS) = (dX)$ and

$$\begin{aligned} f(HYH^H) =& \int_{X>0} \operatorname{etr}(-X)|X|^{a-2m+1} C_\kappa(XHYH^H)(dX) \\ =& \int_{S>0} \operatorname{etr}(-S)|S|^{a-2m+1} C_\kappa(SY)(dS) = f(Y) \end{aligned}$$

and hence

$$\begin{aligned} f(Y) =& \int_{{}_qO(m)} f(Y)(dH) = \int_{{}_qO(m)} f(HYH^H)(dH) \\ =& \int_{X>0} \operatorname{etr}(-X)|X|^{a-2m+1} \int_{{}_qO(m)} C_\kappa(XHYH^H)(dH)(dX) \\ =& \int_{X>0} \operatorname{etr}(-X)|X|^{a-2m+1} \frac{C_\kappa(X)C_\kappa(Y)}{C_\kappa(I_m)}(dX) \\ =& \frac{C_\kappa(Y)}{C_\kappa(I_m)} f(I_m). \end{aligned}$$

Since $f(Y)$ is a symmetric homogeneous polynomial in the latent of $Y$, it can be assumed without loss of generality that $Y$ is diagonal, $Y = \operatorname{diag}(y_1, \ldots, y_m)$, using (i) of Definition 3.1, $f(Y) = \dfrac{f(I_m)}{C_\kappa(I_m)} d_\kappa y_1^{k_1} \cdots y_m^{k_m} + \cdots$, since

$$\begin{aligned} f(Y) = \int_{X>0} & \operatorname{etr}(-X)|X|^{a-2m+1} d_\kappa y_1^{k_1} \cdots y_m^{k_m} \times \\ & x_{11}^{k_1-k_2} \left| \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right|^{k_2-k_3} \cdots |X|^{k_m} (dX). \end{aligned}$$

Put $X = T^H T$, where $T$ is a upper triangular with positive diagonal elements. Then

$$\operatorname{tr} X = \sum_{i \le j}^{m} t_{ij}^H t_{ij}, \; x_{11} = t_{11}^2, \; \left| \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right| = t_{11}^2 t_{22}^2, \cdots, \; |X| = \prod_{i=1}^{m} t_{ii}^2$$

(Lemma 2.1 (2)). By Lemma 2.3,

$$\begin{aligned} f(Y) = & \int_{X>0} \exp\Big(-\sum_{i \le j}^{m} t_{ij}^H t_{ij}\Big) \prod_{i=1}^{m} t_{ii}^{2a-4m+2} d_\kappa y_1^{k_1} \cdots y_m^{k_m} \\ & \times \prod_{i=1}^{m} t_{ii}^{2k_i} 2^m \prod_{i=1}^{m} t_{ii}^{4m-4i+1} \bigwedge_{i \le j}^{m} dt_{ij} + \cdots \\ = & d_\kappa y_1^{k_1} \cdots y_m^{k_m} \pi^{m(m-1)} \prod_{i=1}^{m} \Gamma(a + k_i - 2(i-1)) + \cdots \\ = & d_\kappa y_1^{k_1} \cdots y_m^{k_m} (a)_\kappa \mathbb{Q}\Gamma_m(a) + \cdots . \end{aligned}$$

By comparing the coefficients of the two expressions of $f(Y)$, we have $\dfrac{f(I_m)}{C_\kappa(I_m)} = (a)_\kappa \mathbb{Q}\Gamma_m(a)$.

When $Z > 0$, let $V = Z^{1/2} X Z^{1/2}$. Then $(dV) = |Z|_q^{2m-1}(dX)$ and

$$\begin{aligned} & \int_{X>0} \operatorname{etr}(-XZ)(\det X)^{a-2m+1} C_\kappa(XY)(dX) \\ = & |Z|_q^{-a} \int_{X>0} \operatorname{etr}(-Z^{-1/2} V Z^{1/2}) |V|^{a-2m+1} C_\kappa(V Z^{-1/2} Y Z^{-1/2})(dV) \\ = & |Z|_q^{-a} \int_{X>0} \operatorname{etr}(-V) |V|^{a-2m+1} C_\kappa(V Z^{-1/2} Y Z^{-1/2})(dV) \\ = & |Z|_q^{-a} (a)_\kappa \mathbb{Q}\Gamma_m(a) C_\kappa(Y Z^{-1}). \end{aligned}$$

Finally, by analytic continuation, we get the result on $\Phi_m$ since the left–side of the integrations in the theorem is absolutely convergent in $\Phi_m$. □

**Definition 3.7.** *If $f(X)$ is a function of the positive definite $m \times m$ quaternion matrix $X$, the Laplace transform of $f(X)$ is defined to be*

$$g(Z) = \mathcal{L}(f(X)) = \int_{X>0} \operatorname{etr}(-XZ) f(X)(dX)$$

*which is absolutely convergent for $Z \in \Phi_m$. Note that $\mathcal{L}(\cdot)$ is one to one for $Z - P \in \Phi_m$ where $P$ is a complex positive definite matrix* (cf. [3]).

**Theorem 3.8.** *Let $Y \in {}_qS(m)$. Then*

$$\int_{0<X<I_m} |X|^{a-2m+1}|I-X|^{b-2m+1}C_\kappa(XY)(dX) = \frac{\mathbb{Q}\Gamma_m(a,\kappa)\mathbb{Q}\Gamma_m(b)}{\mathbb{Q}\Gamma_m(a+b,\kappa)}C_\kappa(Y)$$

*for* $\mathrm{Re}\,(a) > 2(m-1)$, $\mathrm{Re}\,(b) > 2(m-1)$ *and*

$$\int_{0<X<I_m} |X|^{a-2m+1}|I-X|^{b-2m+1}C_\kappa(X^{-1}Y)(dX) = \frac{\mathbb{Q}\Gamma_m(a,-\kappa)\mathbb{Q}\Gamma_m(b)}{\mathbb{Q}\Gamma_m(a+b,-\kappa)}C_\kappa(Y)$$

*for* $\mathrm{Re}\,(a) > 2(m-1)+k_1$, $\mathrm{Re}\,(b) > 2(m-1)$.

*Proof.* Let $f(Y) = \int_{0<X<I_m} |X|^{a-2m+1}|I-X|^{b-2m+1}C_\kappa(XY)(dX)$. It is easy to check that $f(Y) = f(HYH^H)$, $H \in {}_qO(m)$ and $f(Y)C_\kappa(I_m) = f(I_m)C_\kappa(Y)$ by Theorem 3.3. Take $Z = I_m$ and $Y = I_m$ in Theorem 3.6. Then

$$\begin{aligned}
\int_{W>0} \mathrm{etr}\,(-W)|W|^{a+b-2m+1}f(W)(dW) &= \int_{W>0} \mathrm{etr}\,(-W)|W|^{a+b-2m+1}\frac{f(I_m)C_\kappa(W)}{C_{\kappa(I_m)}}(dW) \\
&= \frac{f(I_m)}{C_{\kappa(I_m)}}\mathbb{Q}\Gamma_m(a+b,\kappa)C_\kappa(I_m) \\
&= f(I_m)\mathbb{Q}\Gamma_m(a+b,\kappa).
\end{aligned}$$

Set $X = W^{-1/2}UW^{-1/2}$. Then

$$\begin{aligned}
&\int_{W>0} \mathrm{etr}\,(-W)|W|^{a+b-2m+1}f(W)(dW) \\
&= \int_{W>0} \mathrm{etr}\,(-W)|W|^{a+b-2m+1}\int_{0<X<I_m} |X|^{a-2m+1}|I-X|^{b-2m+1}C_\kappa(XW)(dX)(dW) \\
&= \int_{W>0} \mathrm{etr}\,(-W)|W|^{a+b-2m+1}\int_{0<U<W} |U|^{a-2m+1}|W|^{-a-b+4m-2}|W-U|^{b-2m+1} \\
&\quad \times C_\kappa(W^{1/2}UW^{-1/2})|W|^{1-2m}(dU)(dW) \\
&= \int_{U>0} \mathrm{etr}\,(-V-U)\int_{V>0} |U|^{a-2m+1}|V|^{b-2m+1}C_\kappa(U)(dV)(dU) \text{ (for } V = W-U) \\
&= \int_{U>0} \mathrm{etr}\,(-U)|U|^{a-2m+1}C_\kappa(U)(dU)\int_{V>0} \mathrm{etr}\,(-V)|V|^{b-2m+1}(dV) \\
&= \mathbb{Q}\Gamma_m(a,\kappa)\mathbb{Q}\Gamma_m(b)C_\kappa(I_m).
\end{aligned}$$

So $f(I_m) = \dfrac{\mathbb{Q}\Gamma_m(a,\kappa)\mathbb{Q}\Gamma_m(b)}{\mathbb{Q}\Gamma_m(a+b,\kappa)}C_\kappa(I_m)$ and hence $f(Y) = \dfrac{\mathbb{Q}\Gamma_m(a,\kappa)\mathbb{Q}\Gamma_m(b)}{\mathbb{Q}\Gamma_m(a+b,\kappa)}C_\kappa(Y)$. □

**Corollary 3.9.** *If $Y \in {}_qS(m)$, then*

$$\int_{0<X<I_m} |X|^{a-2m+1}C_\kappa(XY)(dX) = \frac{(a)_\kappa}{(a+2m-1)_\kappa}\frac{\mathbb{Q}\Gamma_m(a)\mathbb{Q}\Gamma_m(2m-1)}{\mathbb{Q}\Gamma_m(a+2m-1)}C_\kappa(Y)$$

*where* $\mathrm{Re}\,(a) > 2(m-1)$, *and* $\kappa = (k_1, k_2, \ldots, k_m)$.

# 4 HYPERGEOMETRIC FUNCTION FOR QUATERNION MATRIX

**Definition 4.1.** *The hypergeometric functions of a Hermitian quaternion matrix argument are given by*

$$ {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(X)}{k!} \tag{21} $$

*where* $\sum_{\kappa}$ *denotes summation over all partitions* $\kappa = (k_1, \ldots, k_m)$, $k_1 \geqslant \cdots \geqslant k_m \geqslant 0$ *of* $k$ *and* $X \in {}_qS(m)$.

**Remark 4.2.** *We have the special case* ${}_0F_0(A) = \operatorname{etr} A$ *for* $A \in {}_qS(m)$. *From [3], we have*

(1) *If* $p < q$, *then the hypergeometric series (21) converges absolutely for all* $X$;

(2) *If* $p = q + 1$, *then the series (21) converges absolutely for* $\|X\| < 1$ *and diverges for* $\|X\| > 1$;

(3) *If* $p > q$, *then the series (21) diverges unless it terminates.*

**Definition 4.3.** *The hypergeometric functions of Hermitian quaternion matrices* $X$, $Y$ *are given by*

$$ {}_pF_q{}^m(a_1, \cdots, a_p; b_1, \cdots, b_q; X, Y) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(X) C_\kappa(Y)}{C_\kappa(I_m) k!} \tag{22} $$

By Theorem 3.3, we have

**Theorem 4.4.** *If* $X,\ Y \in {}_qS(m)$ *with* $X > 0$, *then*

$$ \int_{{}_qO(m)} {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; XHYH^H)(dH) = {}_pF_q{}^m(a_1, \cdots, a_p; b_1, \cdots, b_q; X, Y). $$

By Theorem 3.6, we also have

**Theorem 4.5.** *Let* $Z \in {}_qS(m)$ *and suppose* $p \leqslant q$, $\operatorname{Re}(a) > 2(m-1)$. *Then*

$$ \begin{aligned} &\int_{X>0} \operatorname{etr}(-XZ)(\det X)^{a-2m+1} {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; X)(dX) \\ &\qquad = \mathbb{Q}\Gamma_m(a)(\det Z)^{-a} {}_{p+1}F_q(a_1, \cdots, a_p, a; b_1, \cdots, b_q; Z^{-1}) \end{aligned} $$

*and*

$$ \begin{aligned} &\int_{X>0} \operatorname{etr}(-XZ)(\det X)^{a-2m+1} {}_pF_q{}^m(a_1, \cdots, a_p; b_1, \cdots, b_q; X, Y)(dX) \\ &\qquad = \mathbb{Q}\Gamma_m(a)(\det Z)^{-a} {}_{p+1}F_q{}^m(a_1, \cdots, a_p, a; b_1, \cdots, b_q; Z^{-1}, Y) \end{aligned} $$

*for all* $Z \in \Phi_m$ *when* $p < q$ *and for* $\|[\operatorname{re}(Z)]^{-1}\| < 1$ *when* $p = q$.

**Corollary 4.6.** *Let $Z \in {}_qS(m)$ with $\|Z\| < 1$ and $\mathrm{Re}\,(a) > 2(m-1)$. Then ${}_1F_0(a; Z) = |I_m - Z|^{-a}$.*

*Proof.* Assume that $0 < Z < I_m$. By Theorem 4.5,

$$\int_{X>0} \mathrm{etr}\,(-XZ^{-1})|X|^{a-2m+1}\mathrm{etr}\,(X)(dX) = \mathbb{Q}\Gamma_m(a)|Z|^a{}_1F_0(a, Z).$$

Let $X = Z^{1/2}UZ^{1/2}$, then $(dX) = |Z|^{2m-1}(dU)$ by Lemma 2.2 (2) and hence

$$\begin{aligned}\int_{X>0} &\mathrm{etr}\,(-XZ^{-1})|X|^{a-2m+1}\mathrm{etr}\,(X)(dX)\\ &=\int_{X>0} \mathrm{etr}\,(X(I - Z^{-1}))|X|^{a-2m+1}(dX)\\ &=|Z|^a\int_{U>0} \mathrm{etr}\,(-U(I - Z))|U|^{a-2m+1}(dU).\end{aligned}$$

Put $P = (I - Z)^{1/2}U(I - Z)^{1/2}$. Then

$$\begin{aligned}\int_{U>0} &\mathrm{etr}\,(-U(I - Z))|U|^{a-2m+1}(dU)\\ &=\int_{P>0} |I - Z|^{-a+2m-1}|P|^{a-2m+1}\mathrm{etr}\,(-P)|I - Z|^{-2m+1}(dP)\\ &=|I - Z|^{-a}\mathbb{Q}\Gamma_m(a).\end{aligned}$$

Finally, we have ${}_1F_0(a; Z) = |I_m - Z|^{-a}$ for $Z \in {}_qS(m)$ with $\|Z\| < 1$, by analytic continuity. □

**Theorem 4.7.** *Let $X \in \mathbb{Q}^{m\times n}$ $(m \le n)$ and $H = (H_1|H_2) \in {}_qO(n)$, $H_1 \in {}_qV_{m,n}$. Then ${}_0F_1(2n, 4XX^H) = \int_{{}_qO(n)} \exp(4\mathrm{Retr}\,(XH_1))(dH)$.*

*Proof.* We use the same method as in the proof of [9, Theorem 7.4.1]. Assume that $\mathrm{rank}\,X = m$. Applying the Laplace transform to $|X|_q^{2n-2m+1}\int_{{}_qO(n)} \exp(4\mathrm{Retr}\,(XH_1))(dH)$ and $|X|_q^{2n-2m+1}{}_0F_1(2n, 4XX^H)$, respectively, we have

$$\begin{aligned}g_l(Z) &= \int_{XX^H>0} \mathrm{etr}\,(-XX^HZ)|X|_q^{2n-2m+1}\int_{{}_qO(n)} \exp(4\mathrm{Retr}\,(XH_1))(dH)(dXX^H)\\ g_r(Z) &= \int_{XX^H>0} \mathrm{etr}\,(-XX^HZ)|X|_q^{2n-2m+1}{}_0F_1(2n, 4XX^H)(dXX^H).\end{aligned}$$

Since $(dX) = 2^{-m}|X|_q^{2n-2m+1}(dXX^H)(U_1^HdU_1)$, it follows that

$$g_l(Z) = \frac{\mathbb{Q}\Gamma_m(2n)}{\pi^{2mn}}\int_{XX^H>0}\int_{{}_qO(n)} \mathrm{etr}\,(-XX^HZ)\exp(4\mathrm{Retr}\,(XH_1))(dH)(dX).$$

Let $Z > 0$ and put $X = Z^{-1/2}Y$. Then $(dX) = |Z|_q^{-n}(dY)$ and hence

$$\begin{aligned}g_l(Z) =&\frac{\mathbb{Q}\Gamma_m(2n)}{|Z|_q^n\pi^{2mn}}\int_{YY^H>0}\int_{{}_qO(n)} \mathrm{etr}\,(2(YH_1Z^{-1/2} + Z^{-1/2}H_1^HY^H) - YY^H)(dH)(dY)\\ =&\frac{\mathbb{Q}\Gamma_m(2n)}{|Z|_q^n\pi^{2mn}}\mathrm{etr}\,(4Z^{-1})\int_{YY^H>0}\int_{{}_qO(n)} \mathrm{etr}\,(-(Y - 2Z^{-1/2}H_1^H)(Y - 2Z^{-1/2}H_1^H)^H)(dH)(dY).\end{aligned}$$

Note $\frac{1}{\pi^{2mn}}\mathrm{etr}\,(-(Y-2Z^{-1/2}H_1^H)(Y-2Z^{-1/2}H_1^H)^H)$ is the density function of $\mathbb{Q}N_{m\times n}(2Z^{-1/2}H_1^H, 2I_m\otimes I_n)$. Thus $g_l(Z)=\mathbb{Q}\Gamma_m(2n)|Z|_q^{-n}\mathrm{etr}\,(4Z^{-1})$.

On the other hand, by Theorem 4.5

$$\begin{aligned}g_r(Z)=&\mathbb{Q}\Gamma_m(2n)\det(Z)^{-2n}{}_1F_1(2n,2n,4Z^{-1})\\=&\mathbb{Q}\Gamma_m(2n)|Z|_q^{-n}{}_0F_0(4Z^{-1})\\=&\mathbb{Q}\Gamma_m(2n)|Z|_q^{-n}\mathrm{etr}\,(4Z^{-1}).\end{aligned}$$

Then $g_l(Z)=g_r(Z)$, $\forall\, Z\in\Phi_m$ by analytic continuation. $\square$

# 5 THE DISTRIBUTION OF EIGENVALUES

The joint density function of the eigenvalues of complex central Wishart matrix is given in [12] and its distribution of the maximum and the minimum eigenvalues is shown in [11]. In this section, we generalize some results in [11, 12] to the quaternion cases.

Let $W=AA^H\sim\mathbb{Q}W_m(n,\Sigma)\quad(n\geqslant m)$, $A\sim\mathbb{Q}N(0,I_n\otimes\Sigma)$. The density function of $W$ is given by (5). Let $W=VDV^H$. Then $(dW)=(2\pi^2)^{-m}\prod\limits_{i<j}^{m}(\lambda_i-\lambda_j)^4(dD)\bigwedge(V^HdV)$ by Lemma 2.5. Then the differential form of the density of $W$ is

$$\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}\exp(\mathrm{Retr}\,(-2\Sigma^{-1}W))|W|^{2n-2m+1}(2\pi^2)^{-m}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^4(dD)\bigwedge(V^HdV).$$

Integrating the above equation on $(V^HdV)$, by Theorem 4.4 we have

$$\begin{aligned}&\int\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}\exp(\mathrm{Retr}\,(-2\Sigma^{-1}W))|W|^{2n-2m+1}(2\pi^2)^{-m}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^4(dD)\bigwedge(V^HdV)\\&=\frac{2^m\pi^{2m^2-2m}}{\mathbb{Q}\Gamma_m(2m)|\Sigma|^{2n}}\int\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)}\exp(\mathrm{Retr}\,(-2\Sigma^{-1}W))|W|^{2n-2m+1}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^4(dD)\bigwedge(dV)\\&=\frac{2^{2mn}\pi^{2m^2-2m}}{\mathbb{Q}\Gamma_m(2m)\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}{}_0F_0(-2\Sigma^{-1},D)|D|^{2n-2m+1}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^4(dD)\end{aligned}$$

which gives the joint density of the eigenvalues. When $\Sigma=\sigma^2I_n$, the joint density of the eigenvalues of $W$ is

$$\frac{2^{2mn}\pi^{2m^2-2m}}{\mathbb{Q}\Gamma_m(2m)\mathbb{Q}\Gamma_m(2n)|\sigma^2|^{2nm}}|D|^{2n-2m+1}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^4\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{m}\lambda_i\right)(dD)\qquad(23)$$

Let $W\sim\mathbb{Q}W_m(n,\Sigma)(n\geqslant m)$ and $\Delta$ be a $m\times m$ positive definite quaternion matrix. We will present the distributions of $P(W>\Delta)$ and $P(W<\Delta)$ as follows

**Theorem 5.1.** *Let $W$ and $\Delta$ be as above. Then*

$$P(W<\Delta) = \frac{2^{2mn}\mathbb{Q}\Gamma_m(2m-1)}{\mathbb{Q}\Gamma_m(2n+2m-1)}\frac{|\Delta|^{2n}}{|\Sigma|^{2n}}\,{}_1F_1(2n,2n+2m-1,-2\Sigma^{-1}\Delta)$$

$$P(W>\Delta) = \sum_{k=0}^{m(2n-2m+1)}\widehat{\sum_{\kappa}}\frac{C_\kappa(2\Sigma^{-1}\Delta)}{k!}\mathrm{etr}\,(-2\Sigma^{-1}\Delta),$$

*where $\widehat{\sum}$ denotes summation over the partitions $\kappa=(k_1,\ldots,k_m)$ of $k$ with $k_1\leqslant 2n-2m+1$.*

*Proof.* By means of the density function of $W$ in (5), we have

$$P(W<\Delta)=\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}\int_{0<W<\Delta}\exp(\mathrm{Retr}\,(-2\Sigma^{-1}W))|W|^{2n-2m+1}(dW)$$

Let $W=\Delta^{1/2}X\Delta^{1/2}$. Then $(dW)=|\Delta|^{2m-1}dX$. By Corollary 3.9, we get that

$$\begin{aligned}
&P(W<\Delta)=P(X<I)\\
&=\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}\int_{0<X<I}\exp(\mathrm{Retr}\,(-2\Sigma^{-1}\Delta^{1/2}X\Delta^{1/2})|\Delta|^{2n-2m+1}|X|^{2n-2m+1}|\Delta|^{2m-1}(dX)\\
&=\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)}\frac{|\Delta|^{2n}}{|\Sigma|^{2n}}\int_{0<X<I}\mathrm{etr}\,(-2\Sigma^{-1}\Delta^{1/2}X\Delta^{1/2}))|X|^{2n-2m+1}(dX)\\
&=\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)}\frac{|\Delta|^{2n}}{|\Sigma|^{2n}}\int_{0<X<I}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\frac{C_\kappa(-2\Delta^{1/2}\Sigma^{-1}\Delta^{1/2}X)}{k!}|X|^{2n-2m+1}(dX)\\
&=\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)}\frac{|\Delta|^{2n}}{|\Sigma|^{2n}}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\frac{\mathbb{Q}\Gamma_m(2n)\mathbb{Q}\Gamma_m(2m-1)}{\mathbb{Q}\Gamma(2n+2m-1)}\frac{C_\kappa(-2\Sigma^{-1}\Delta)}{k!}\frac{(2n)_\kappa}{(2n+2m-1)_\kappa}\\
&=\frac{2^{2mn}\mathbb{Q}\Gamma_m(2m-1)}{\mathbb{Q}\Gamma_m(2n+2m-1)}\frac{|\Delta|^{2n}}{|\Sigma|^{2n}}\,{}_1F_1(2n,2n+2m-1,-2\Sigma^{-1}\Delta).
\end{aligned}$$

Note that

$$P(W>\Delta)=\frac{2^{2mn}}{\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}\int_{W>\Delta}\mathrm{etr}\,(-2\Sigma^{-1}W)|W|^{2n-2m+1}(dW).$$

Put $W=\Delta^{1/2}(I+X)\Delta^{1/2}$. Then $dW=|\Delta|^{2m-1}(dX)$ and so

$$\begin{aligned}
&P(W>\Delta)\\
&=\frac{2^{2mn}|\Delta|^{2n}}{\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}\int_{X>0}\mathrm{etr}\,(-2\Sigma^{-1}\Delta)\mathrm{etr}\,(-2\Sigma^{-1}\Delta^{1/2}X\Delta^{1/2})|I+X|^{2n-2m+1}(dX)\\
&=\frac{2^{2mn}|\Delta|^{2n}}{\mathbb{Q}\Gamma_m(2n)|\Sigma|^{2n}}\int_{X>0}\mathrm{etr}\,(-2\Sigma^{-1}\Delta)\times\mathrm{etr}\,(-2\Sigma^{-1}\Delta^{1/2}X\Delta^{1/2})\\
&\qquad\times|I+X^{-1}|^{2n-2m+1}|X|^{2n-2m+1}(dX).
\end{aligned}$$

Since

$$\begin{aligned}
|I+X^{-1}|^{2n-2m+1}&={}_1F_0(-2n+2m-1,-X^{-1})\\
&=\sum_{k=0}^{m(2n-2m+1)}\widehat{\sum_{\kappa}}\frac{[-(2n-2m+1)]_\kappa C_\kappa(X^{-1})(-1)^k}{k!}
\end{aligned}$$

by Corollary 4.6, it follows from Theorem 3.6 that

$$\begin{aligned}
&\int_{X>0} \operatorname{etr}(-2\Sigma^{-1}\Delta)\operatorname{etr}(-2\Sigma^{-1}\Delta^{1/2}X\Delta^{1/2})|I+X^{-1}|^{2n-2m+1}|X|^{2n-2m+1}(dX)\\
&\qquad = \sum_{k=0}^{m(2n-2m+1)} \widehat{\sum_{\kappa}} \frac{(-1)^k[-2n+2m-1]_\kappa}{k!}\\
&\qquad\quad \times \int_{X>0} \operatorname{etr}(-2\Sigma^{-1}\Delta^{1/2}X\Delta^{1/2})|X|^{2n-2m+1}C_\kappa(X^{-1})(dX)\\
&\qquad = \sum_{k=0}^{m(2n-2m+1)} \widehat{\sum_{\kappa}} \frac{\mathbb{Q}\Gamma_{2m}(2n)}{k!}|2\Delta^{1/2}\Sigma^{-1}\Delta^{1/2}|^{-2n}C_\kappa(2\Sigma^{-1}\Delta).
\end{aligned}$$

Therefore, we obtain the result. $\square$

**Corollary 5.2.** *Let $W \sim \mathbb{Q}W_m(n,\Sigma)$ $(n \geqslant m)$ and let $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and smallest eigenvalue of $W$ respectively. Then distribution of $\lambda_{\max}$ (resp. $\lambda_{\min}$) is given by*

$$P(\lambda_{\max} < x) = \frac{\mathbb{Q}\Gamma_m(2m-1)}{\mathbb{Q}\Gamma_m(2n+2m-1)}\frac{x^{2mn}}{|\Sigma|^{2n}}{}_1F_1(2n, 2n+2m-1, -2x\Sigma^{-1}) \tag{24}$$

$$P(\lambda_{\min} > x) = \sum_{k=0}^{m(2n-2m+1)} \widehat{\sum_{\kappa}} \frac{C_\kappa(2x\Sigma^{-1})}{k!}\operatorname{etr}(-2x\Sigma^{-1}). \tag{25}$$

*The density of $\lambda_{\max}$ (resp. $\lambda_{\min}$) is obtained by differentiating (24) (resp. (25)) with respect to $x$.*

*Proof.* The inequality $\lambda_{\max} < x$ (resp. $\lambda_{\min} > x$) is equivalent to $W < xI_m$ (resp. $W > xI_m$). The assertions follow by taking $\Delta = xI_m$ in Theorem 5.1. $\square$

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# References

[1] I. Dumitriu, Eigenvalue statistics for Beta–Ensembles, *Ph D thesis*, Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA., 2003.

[2] A. Edelman and N. Raj Rao, Random matrix theory, *Acta Numerica* (2005), 233–297.

[3] K.I. Gross, Donald ST.P. Richards, Special Functions of Matrix Argument I: Algebraic Induction Zonal Polynomials and Hypergeometric Functions, *Trans. Amer. Math. Soc.*, **301** (1987), 475–501.

[4] C.S. Herz, Bessel functions of matrix argument, *Ann. of Math.*, **61**(3) (1955), 474–523.

[5] Alan T. James, Zonal Polynomials of the Real Positive Definite Symmetric Matrices, *Ann. of Math.*, **74** (1961), 456–468.

[6] A.T. James, Distributions of Matrix Variates and Latent Roots Derived from Normal Samples, *Ann. Math. Statis.*, **35** (1964), 475–501.

[7] A.T. James, Calculation of Zonal Polynomial Coefficients By Use of The Laplace Beltrami Operator, *Ann. Math. Statis.*, **39** (1968), 1711–1788.

[8] A.M. Mathai, Jacobians and functions of matrix arguments, *World Scientific*, 1997.

[9] R.J. Muirhead, Aspects of Multivariate Statistical Theory, *John Wiley & Sons*, 1982.

[10] A. Takemura, Zonal Polynomials, *Institute of Mathematical Statistics*, 1982.

[11] T. Ratnarajah, R. Vaillancourt and M. Alvo, Eigenvalues and Condition Number of Complex Random Matrices, *SIAM J.Matrix Anal.Appl.*, **26**(2) (2005), 441–456.

[12] T. Ratnarajah, R. Vaillancourt, Complex Singular Wishart Matrices and Applications, *Computers and Mathmatics with Applications*, **50** (2005), 399–411.

[13] F. Zhang, Quaternions and matrices of quaternions, *Linear Alg. Appl.*, **251** (1997), 21–57.